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- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator* **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony* *Memory photo*
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

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ON SOME CONSTRUCTIONS OF A NON-PERIODIC MODULUS OF SMOOTHNESS RELATED TO THE RIESZ DERIVATIVE

S.Yu. Artamonov

Communicated by E.D. Nursultanov

Key words: modulus of smoothness, Riesz derivative, K-functional, Bernstein space

AMS Mathematics Subject Classification: 42A10, 42A05, 42A45, 42A50

Abstract. A new non-periodic modulus of smoothness related to the Riesz derivative is constructed. Its properties are studied in the spaces $L_p(\mathbb{R})$ of non-periodic functions with $1 \leq p \leq +\infty$. The direct Jackson type estimate is proved. It is shown that the introduced modulus is equivalent to the K-functional related to the Riesz derivative and to the approximation error of the convolution integrals generated by the Fejér kernel.

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1 Introduction, notations and preliminaries

In [13] a periodic modulus of smoothness related to the Riesz derivative was introduced. The Jackson type estimate and equivalence to the approximation error of the Fejér means were proved. In [1] it was shown that the construction of a modulus of smoothness, equivalent to K-functionals related to the Riesz derivative can be modified.

In [5] it was shown that the approximation error of the convolution integrals $\mathcal{M}_\sigma(f)$ generated by the Fejér kernel can be completely described in terms of the K-functional related to the Riesz derivative

$$\| f - \mathcal{M}_\sigma(f) \|_p \asymp K_{\langle \prime \rangle} \left(f, \sigma^{-1}\right)_p , \quad \sigma > 0 . \tag{1.1}$$

In the present paper we continue chain of equivalences (1.1) by adding modulus of smoothness (Theorem 4.2). Some of the assertions were announced in [2]. In this paper we give complete proofs of these results. For the generalized moduli of smoothness in non-periodic case we refer to [3].

We consider $L_p(\mathbb{R})$ spaces with $1 \leq p \leq +\infty$. If $p = +\infty$ we consider the space $\mathcal{C}(\mathbb{R})$ of bounded uniformly continuous functions equipped with the Chebyshev norm

$$\| f \|_{\mathcal{C}} = \sup_{x \in \mathbb{R}} |f(x)| < +\infty .$$

We denote unimportant positive constants by c (with subscripts and superscripts). They may be different in different formulas (but not in the same formula). The relation $A(f,\sigma) \asymp B(f,\sigma)$ indicates the equivalence. It means that there exist positive constants c_1 and c_2 that do not depend on f and σ, such that $c_1 A(f,\sigma) \leq B(f,\sigma) \leq c_2 A(f,\sigma)$.

By $\mathcal{S}$ and $\mathcal{S}'$ we denote the Schwartz space of infinitely differentiable rapidly decreasing functions and its dual space respectively. The Fourier transform and its inverse for $g \in \mathcal{S}'$ are given by

$$\langle Fg, \varphi \rangle = \langle g, F\varphi \rangle , \qquad \varphi \in \mathcal{S},$$

$$\left\langle F^{-1}g,\varphi\right\rangle = \left\langle g, F^{-1}\varphi\right\rangle, \quad \varphi\in\mathcal{S}.$$

Following [5], we denote by B^p_σ the Bernstein space, which is, by definition, the space of all functions in $L_p(\mathbb{R})$, which are restrictions to the $\mathbb{R}$ of analytic functions of exponential type σ defined on $\mathbb{C}$, that is,

$$B^p_\sigma = \{f = F|_{\mathbb{R}} \in L_p(\mathbb{R}) : \text{F is analytic on}\, \mathbb{C}, |F(x+iy)| \le A\cdot e^{\sigma|y|}\},$$

where $A > 0$ does not depend on $z = x + iy$. Henceforth, we use the following notation $B^p = \bigcup\limits_{\sigma>0} B^p_\sigma$. As is well known (see, for instance, [10, p. 181]), by the Paley-Wiener-Schwartz theorem the Bernstein space can be characterized also in terms of the Fourier transform:

$$B^p_\sigma = \{f\in L_p(\mathbb{R}), \operatorname{supp} F^{-1}f \subset [-\sigma,\sigma]\}, \quad 1\le p<+\infty,$$

$$B^p_\sigma = \{f\in \mathcal{C}(\mathbb{R}), \operatorname{supp} F^{-1}f \subset [-\sigma,\sigma]\}, \quad p=+\infty.$$

The space B^p_σ is Banach space, equipped with the norm in $L_p(\mathbb{R})$. Based on the Nikolski inequality (see for example [14]) we have the continuous embeddings

$$B^p_\sigma \subset B^q_\sigma, \quad 1\le p < q \le \infty.$$

In particular, any function belonging to a Bernstein space is bounded. Because of the above properties entire functions of exponential type are often called bandlimited functions.

A number of monographs and papers are devoted to the theory of approximation by bandlimited functions (see, e.g., [4], [5], [6], [7], [14], [17]). As usual, the best approximation of $f\in L_p(\mathbb{R})$ of order $\sigma > 0$ is given by

$$E_\sigma(f)_p = \inf_{g\in B^p_\sigma} \|f-g\|_p.$$

The convolution integrals $\mathcal{M}^{(\phi)}_\sigma$, that are non-periodic counterparts of the Fourier means in the trigonometric case (see e.g. [5], [12], [8]), are given by the convolution of $f\in L_p(\mathbb{R})$ with the kernel $K^\phi_\sigma = F\left[\phi\left(\frac{\cdot}{\sigma}\right)\right](x) \in B^1_{r(\phi)\sigma}, \sigma > 0$, where the generator of method ϕ is a continuous complex-valued function, having compact support ($r(\phi) = \sup\{\,|\xi|: \xi\in\operatorname{supp}\phi\,\} < +\infty$), such that $\phi(-\xi) = \overline{\phi(\xi)}$ for all $\xi\in\mathbb{R}$ and $\phi(0)=1$. The class of all functions satisfying the above conditions for ϕ we denote by $\mathcal{K}$. We note that for $f\in B^p$ one has

$$\mathcal{M}^{(\phi)}_\sigma(f;x) = F\left[\phi\left(\frac{\cdot}{\sigma}\right)F^{-1}[f](\cdot)\right](x). \tag{1.2}$$

The following theorem holds [5].

Theorem 1.1. *Let $1\le p\le+\infty$, $\phi\in\mathcal{K}$ and $F[\phi](x)\in L_1(\mathbb{R})$, $r = r(\phi)$. Then*
1) $\mathcal{M}^{(\phi)}_\sigma$, $\sigma>0$ are linear bounded operators mapping $L_p(\mathbb{R})$ to $B^p_{r\sigma}$ and

$$\begin{aligned}\|\mathcal{M}^{(\phi)}_\sigma\|_{(p)} &\le \|\mathcal{M}^{(\phi)}_\sigma\|_{(1)} = \|\mathcal{M}^{(\phi)}_\sigma\|_{(\infty)} \\ &= (2\pi)^{-1}\|K^\phi_\sigma\|_1 = (2\pi)^{-1}\|F[\phi]\|_1;\end{aligned} \tag{1.3}$$

2) the method $\mathcal{M}^{(\phi)}_\sigma$ converges in $L_p(\mathbb{R})$, i.e.,

$$\lim_{\sigma\to+\infty}\|f-\mathcal{M}^{(\phi)}_\sigma(f)\|_p = 0, \; f\in L_p(\mathbb{R}); \tag{1.4}$$

3) if, in addition, $\phi(\xi) = 1$ for $\xi\in[-\rho;\rho]$, $\rho > r(\phi)$, then the inequality

$$\|f-\mathcal{M}^{(\phi)}_\sigma(f)\|_p \le cE_\sigma(f)_p, \; f\in L_p(\mathbb{R}), \sigma>0, \tag{1.5}$$

holds, where the positive constant c does not depend on f and σ.

Convolution integrals generated by the Fejér kernel $\mathcal{M}_\sigma$, where $\phi(\xi) = (1 - |\xi|)_+$ and $a_+ = \max\{a, 0\}$.

It follows from (1.4) that B^p is dense in $L_p(\mathbb{R})$, if $1 \le p < +\infty$; also B^∞ is dense in $\mathcal{C}(\mathbb{R})$.

Norm of a linear bounded operator $\mathcal{A}$ in $L_p(\mathbb{R})$ is given by $\|\mathcal{A}\|_{(p)} = \sup_{\|f\|_p \le 1} \|\mathcal{A}(f)\|_p$.

We note that spaces $\mathcal{S}_\sigma = B^p_\sigma \cap \mathcal{S}$ are dense in B^p_σ only for $1 \le p < +\infty$ [18, p. 22-23]. We apply the following approximation procedure [14] which works for all $1 \le p \le +\infty$:

Theorem 1.2. *Let $\varphi \in \mathcal{S}$, such that $F[\varphi](x) \ge 0, \operatorname{supp} F[\varphi] \subset [-1; 1]$ and $\varphi(0) = 1$.*

(i) If $f \in B^p_\sigma$, $1 \le p < +\infty$ and $\epsilon > 0$, then $\varphi(\epsilon t) f(t) \in B^p_{\sigma+\epsilon} \cap \mathcal{S}$ and $(\varphi(\epsilon \cdot) f) \to f$ in $L_p(\mathbb{R})$ for $\epsilon \to 0$.

(ii) If $f \in B^\infty_\sigma$ and $\epsilon > 0$, then $\varphi(\epsilon t) f(t) \in B^\infty_{\sigma+\epsilon} \cap \mathcal{S}$ and $\|\varphi(\epsilon t) f(t)\|_\infty \to \|f\|_\infty$ for $\epsilon \to 0$.

In our application it will be convenient to work with equivalent definition of the Riesz derivative of $g \in B^\infty$ (see [5] for details)

$$g^{\langle\prime\rangle}(x) = (2\pi)^{-1} \left(g * F\left[|\cdot| \eta(\cdot)\right]\right)(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where η is an infinitely differentiable function, which has compact support and $\eta(\xi) = 1$ for $\xi \in [-\rho(g), \rho(g)]$, where $\rho(g) = \sup\{\,|\xi| : \xi \in \operatorname{supp} F^{-1} g\,\} < +\infty$.

The K-functional, related to the Riesz derivative is given [5] by

$$K_{\langle\prime\rangle}(f, \delta)_p = \inf_{g \in B^p} \left\{\|f - g\|_p + \delta \|g^{\langle\prime\rangle}\|_p\right\}, \ \delta \ge 0. \tag{1.7}$$

We list some elementary properties of (1.7).

Lemma 1.1. *Let $1 \le p \le +\infty$ and $f \in L_p(\mathbb{R})$. Then*

(i) the function $K_{\langle\prime\rangle}(f, \delta)_p$ increases on $[0; +\infty)$ and $K_{\langle\prime\rangle}(f, 0)_p = 0$;

(ii) for $\delta_1, \delta_2 \ge 0$ one has

$$K_{\langle\prime\rangle}(f, \delta_1 + \delta_2)_p \le K_{\langle\prime\rangle}(f, \delta_1)_p + K_{\langle\prime\rangle}(f, \delta_2)_p;$$

(iii) for $\delta, t \ge 0$ one has

$$K_{\langle\prime\rangle}(f, t\delta)_p \le \max(1, t)\, K_{\langle\prime\rangle}(f, \delta)_p; \tag{1.8}$$

(iv) for $f_1, f_2 \in L_p(\mathbb{R})$ and $\delta \ge 0$ it holds

$$K_{\langle\prime\rangle}(f_1 + f_2, \delta)_p \le K_{\langle\prime\rangle}(f_1, \delta)_p + K_{\langle\prime\rangle}(f_2, \delta)_p. \tag{1.9}$$

Proof.

(i) The proof is based on the following inequality for $0 \le \delta_1 \le \delta_2$

$$\|f - g\|_p + \delta_1 \|g^{\langle\prime\rangle}\|_p \le \|f - g\|_p + \delta_2 \|g^{\langle\prime\rangle}\|_p.$$

Taking into account that B^p is dense in $L_p(\mathbb{R})$ for $1 \le p \le +\infty$, one has

$$K_{\langle ' \rangle}(f,0) \;=\; \inf_{g\in B^p} \|f \,-\, g\|_p \,+\, 0 \;=\; 0\,.$$

Part (ii) follows from definition of K-functional (1.7).
(iii) For $0 \le t \le 1$ (1.8) follows from (i). Let $t > 1$ and $g \in B^p$. Then

$$\begin{aligned} &\|f \,-\, g\|_p \,+\, (t\delta)\|g^{\langle ' \rangle}\|_p \,= \\ &=\; t\left(t^{-1}\|f \,-\, g\|_p \,+\, \delta\|g^{\langle ' \rangle}\|_p\right) \\ &\le\; t\left(\|f \,-\, g\|_p \,+\, \delta\|g^{\langle ' \rangle}\|_p\right)\,. \end{aligned} \tag{1.10}$$

Now (1.8) follows from (1.10).
(iv) Let $g_1\,, g_2 \in B^p$. Then

$$\begin{aligned} K_{\langle ' \rangle}(f_1 \,+\, f_2\,,\delta)_p \;\le\;& \|f_1 \,-\, g_1\|_p \,+\, \delta\|g_1^{\langle ' \rangle}\|_p \\ +\;& \|f_2 \,-\, g_2\|_p \,+\, \delta\|g_2^{\langle ' \rangle}\|_p \end{aligned} \tag{1.11}$$

Now (1.9) is a corollary of (1.11). □

2 The construction of the modulus

We consider the operators:

$$\widetilde{T}_h f(x) \;=\; \frac{3}{\pi^2}\sum_{\nu\neq 0}\frac{f(x+\nu h)}{\nu^2}\,,\;\; f\in L_p(\mathbb{R})\,,\;\; h\in\mathbb{R}\,, \tag{2.1}$$

$$\widetilde{\Delta}_h \;=\; \widetilde{T}_h \,-\, I\,, \tag{2.2}$$

where I is the identity operator. We introduce the modulus of smoothness as

$$\omega_{\langle ' \rangle}(f,\delta)_p \;=\; \sup_{0\le h\le\delta} \|\,\widetilde{\Delta}_h f(x)\,\|_p\,,\;\; \delta\ge 0\,,\; f\in L_p(\mathbb{R})\,. \tag{2.3}$$

Following [13], we call (2.3) modulus of smoothness related to the Riesz derivative. We note that the difference operator $\widetilde{\Delta}_h$ is associated with the Fourier coefficients $\theta^\wedge(\nu)\,, \nu\in\mathbb{Z}$ of the 2π-periodic function θ, given on $x\in[0,2\pi)$ by

$$\theta(x) = \frac{3}{2\pi^2}x^2 - \frac{3}{\pi}x\,, \tag{2.4}$$

by the following relation

$$\widetilde{\Delta}_h f(x) \;=\; \sum_{\nu\in\mathbb{Z}} \theta^\wedge(\nu) f(x \,+\, \nu h)\,. \tag{2.5}$$

Formulas (2.1) and (2.5) are understood in the sense of convergence in $L_p(\mathbb{R})$.
We list some elementary properties of the operators $\widetilde{T}_h$ and $\widetilde{\Delta}_h$.

Lemma 2.1. *Let $1 \le p \le \infty$. Then $\widetilde{T}_h$ and $\widetilde{\Delta}_h$, $h \in \mathbb{R}$ are linear bounded operators in $L_p(\mathbb{R})$:*

$$\| \widetilde{T}_h \|_{(p)} \ \le \ 1 \,, \ \ h \in \mathbb{R} \,, \tag{2.6}$$

$$\| \widetilde{\Delta}_h \|_{(p)} \ \le \ 2 \,, \ \ h \in \mathbb{R} \,, \tag{2.7}$$

$$\omega_{\langle ' \rangle}(f, \delta)_p \ \le \ 2 \| f \|_p \,, \ \ f \in L_p(\mathbb{R}) \,, \ \ \delta \ge 0 \,. \tag{2.8}$$

Proof. First we prove (2.6). Applying (2.1) and a relation $\sum\limits_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6}$, we get for $f \in L_p(\mathbb{R})$

$$\| \widetilde{T}_h f \|_p \ \le \ \frac{3}{\pi^2} \sum_{\nu \ne 0} \frac{\| f(x + \nu h) \|_p}{\nu^2} \ \le \ \| f \|_p \,.$$

Estimate (2.7) follows from (2.2). Estimate (2.8) is a corollary of (2.3) and (2.7). □

3 Jackson type estimate

We note that the difference operator is of multiplier type, more precisely, for $f \in L_2(\mathbb{R})$ one has

$$\frac{3}{\pi^2} \sum_{\nu \in \mathbb{Z}/\{0\}} \frac{f(x + \nu h)}{\nu^2} \ = \ F \left[\left(\frac{3}{2\pi^2} (h \cdot)^2 \ - \ \frac{3}{\pi} (h \cdot) \right) F^{-1}[f](\cdot) \right] (x) \,, \ h \in \mathbb{R} \tag{3.1}$$

almost everywhere.

Indeed, both sides of (3.1) make sense in $L_2(\mathbb{R})$. Let $\varphi \in \mathcal{S}$. Then by virtue of (2.5) we have

$$F \left[\theta \left(h \cdot \right) F^{-1}[f](\cdot) \right] (\varphi) \ = \ \int\limits_{-\infty}^{+\infty} \theta(h\xi) (F^{-1}[f])(\xi) (F[\varphi])(\xi) d\xi$$

$$= \ \int\limits_{-\infty}^{+\infty} (F^{-1}[f])(\xi) \sum_{\nu \in \mathbb{Z}} \theta^\wedge(\nu) e^{i\nu h\xi} (F[\varphi])(\xi) d\xi$$

$$= \ \int\limits_{-\infty}^{+\infty} (F^{-1}[f])(\xi) \sum_{\nu \in \mathbb{Z}} \theta^\wedge(\nu) F[\varphi(\cdot - \nu h)](\xi) d\xi$$

$$= \ \int\limits_{-\infty}^{+\infty} (F^{-1}[f])(\xi) F \left[\sum_{\nu \in \mathbb{Z}} \theta^\wedge(\nu) \varphi(\cdot - \nu h) \right] (\xi) d\xi$$

$$= \ \int\limits_{-\infty}^{+\infty} (F^{-1}[f])(\xi) 2\pi \overline{F^{-1} \left[\overline{\sum_{\nu \in \mathbb{Z}} \theta^\wedge(\nu) \varphi(\cdot - \nu h)} \right]} (\xi) d\xi \,, \tag{3.2}$$

where $\theta(\cdot)$ is defined by (2.4). By applying (3.2) and the Plancherel's theorem we get

$$\begin{aligned} F\left[\theta(h\cdot)F^{-1}[f](\cdot)\right](\varphi) &= \int_{-\infty}^{+\infty} f(x)\sum_{\nu\in\mathbb{Z}}\theta^\wedge(\nu)\varphi(x-\nu h)dx \\ &= \sum_{\nu\in\mathbb{Z}}\theta^\wedge(\nu)\int_{-\infty}^{+\infty} f(x)\varphi(x-\nu h)dx \\ &= \sum_{\nu\in\mathbb{Z}}\theta^\wedge(\nu)\int_{-\infty}^{+\infty} f(x+\nu h)\varphi(x)dx \\ &= \int_{-\infty}^{+\infty}\left[\sum_{\nu\in\mathbb{Z}}\theta^\wedge(\nu)f(x+\nu h)\right]\varphi(x)dx\,. \end{aligned}$$

□

The next theorem establishes the possibility of moving the constants out of the modulus of smoothness.

Theorem 3.1. *For $1\le p\le+\infty$ and $t\ge 0$*

$$\omega_{\langle\prime\rangle}(f,t\delta)_p \le c\,(1+t)^3\,\omega_{\langle\prime\rangle}(f,\delta)_p\,,\ f\in L_p(\mathbb{R})\,,\ \delta,t\ge 0\,, \tag{3.3}$$

where the positive constant c does not depend on δ, t and f.

Proof. It is clear that (3.3) follows from the inequality

$$\|\,\widetilde{\Delta}_{mh}f(x)\,\|_p \le Cm^3\,\|\,\widetilde{\Delta}_h f(x)\,\|_p\,,\ f\in L_p(\mathbb{R})\,,\ h,t\ge 0\,. \tag{3.4}$$

The proof of (3.4) is splitted into two steps. On the first step we obtain (3.4) for $f\in B^p\cap\mathcal{S}$. Consider the function $\theta_m(\xi) = \frac{\theta(m\xi)}{\theta(\xi)}$, where $m\in\mathbb{N}\,, m\ge 2$. We note that by virtue of (3.1) it holds for $f\in\mathcal{S}$

$$\begin{aligned} \widetilde{\Delta}_{mh}f(x) &= F\left[\frac{\theta(mh\cdot)}{\theta(h\cdot)}\cdot\theta(h\cdot)F^{-1}[f](\cdot)\right](x) \\ &= F\left[\theta_m(h\cdot)F^{-1}[\Delta_h^{(\theta)}f](\cdot)\right](x) \\ &= \sum_{\nu\in\mathbb{Z}}\theta_m^\wedge(\nu)g(x+\nu h)\,, \end{aligned} \tag{3.5}$$

where

$$g(x) = \widetilde{\Delta}_h f(x)\in L_2(\mathbb{R})\,. \tag{3.6}$$

Applying (3.5), (3.6) and the relation [1, Lemma 2]

$$\sum_{k\in\mathbb{Z}}|\,\theta_m^\wedge(k)\,| \le Cm^3\,,$$

where the positive constant C does not depend on m, we get

$$\begin{aligned}
\| \widetilde{\Delta}_{mh} f \|_p &= \left\| \sum_{\nu\in\mathbb{Z}} \theta_m^{\wedge}(\nu)\, g(x+\nu h) \right\|_p \\
&\le \sum_{\nu\in\mathbb{Z}} |\, \theta_m^{\wedge}(\nu) \,|\, \|\, g(x+\nu h) \,\|_p \\
&\le \|\, g \,\|_p \sum_{\nu\in\mathbb{Z}} |\, \theta_m^{\wedge}(\nu) \,| \\
&\le C_1 m^3 \,\|\, g \,\|_p \,.
\end{aligned}$$

On the second step, by applying Theorem 1.2 and the density argument, we obtain (3.3) for $f \in L_p(\mathbb{R})$. □

The proof of Jackson type estimate is based on the development of the ideas of D. Jackson [9] and S.B. Stechkin [15], [16] and follows the scheme elaborated by K.V. Runovski and H.-J. Schmeisser in [13] and [11] for the periodic case. More precisely, we obtain a representation for $\mathcal{M}_\sigma^{(\phi)}$, containing the shift operator $\widetilde{T}_h$ in its structure, for a suitable choice of the generator.

We consider for $\phi \in \mathcal{K}$ the function

$$\Phi(\xi) \;=\; \frac{3}{\pi^2} \sum_{\nu\neq 0} \frac{\phi(\nu\xi)}{\nu^2} \,.$$

One can show, that $\Phi \in \mathcal{K}$, $r(\Phi) \;=\; r(\phi)$ and

$$\mathcal{M}_\sigma^{(\Phi)}(f;x) = (2\pi)^{-1} \int\limits_{\mathbb{R}} \widetilde{T}_{-\sigma^{-1}h} f(x)\, \overline{F[\phi](h)}\, dh \,. \tag{3.7}$$

Theorem 3.2. *(Jackson type estimate) Let $1 \le p \le +\infty$. Then*

$$E_\sigma(f)_p \;\le\; c\,\omega_{\langle\prime\rangle}\left(f, \sigma^{-1}\right)_p \,,\; f \in L_p(\mathbb{R}) \,,\; \sigma \;>\; 0 \,, \tag{3.8}$$

where the positive constant c does not depend on f and σ.

Proof. We consider real-valued even infinitely differentiable function ϕ with support concentrated in $[-1,1]$. It is clear that

$$|\, F[\phi](h) \,| \;\le\; c\,(1 + |\, h \,|)^{-5} \,,\;\; h \in \mathbb{R} \,. \tag{3.9}$$

By applying (2.1), (3.9), Theorem 3.1 and (3.7), we obtain for $f \in L_p(\mathbb{R})$ and $\sigma > 0$

$$\begin{aligned}
E_\sigma(f)_p &\le \| f - \mathcal{M}_\sigma^{(\Phi)}(f) \|_p \le (2\pi)^{-1} \int_{\mathbb{R}} \| T^{(\theta)}_{-\sigma^{-1}h} f(x) - f(x) \|_p \, | \overline{F[\phi](h)} | \, dh \le \\
&\le \int_{\mathbb{R}} \omega_\theta \left(f, \sigma^{-1} | \, h \, | \right)_p | \, F[\phi](h) | \, dh \\
&\le c \, \omega_\theta \left(f, \sigma^{-1} \right)_p \int_{\mathbb{R}} (\, 1 + | \, h \, | \,)^3 \, | \, F[\phi](h) | \, dh \\
&\le c \, \omega_\theta \left(f, \sigma^{-1} \right)_p \int_{\mathbb{R}} (\, 1 + | \, h \, | \,)^{-2} \, dh \\
&\le c' \, \omega_\theta \left(f, \sigma^{-1} \right)_p \, ,
\end{aligned}$$

that completes the proof of (3.8). □

4 Equivalence

It is obvious that for $\phi \in \mathcal{K}$ and $f \in S$ the following statement holds (in the sense of S')

$$F \left[\phi \left(\frac{\cdot}{\sigma} \right) F^{-1}[f](\cdot) \right](x) \; = \; F^{-1} \left[\phi \left(\frac{\cdot}{\sigma} \right) F[f](\cdot) \right](x) \, . \tag{4.1}$$

Theorem 4.1. *Let $1 \le p \le +\infty$. Then for $f \in L_p(\mathbb{R})$, $\sigma > 0$*

$$K_{\langle ' \rangle} \left(f, \sigma^{-1} \right)_p \asymp \omega_{\langle ' \rangle} \left(f, \sigma^{-1} \right)_p \, . \tag{4.2}$$

Proof.

Step 1. First we show that

$$\omega_{\langle ' \rangle}(f, \delta)_p \le c \, K_{\langle ' \rangle}(f, \delta)_p \, , \;\; f \in L_p(\mathbb{R}) \, , \;\; \delta \ge 0 \, . \tag{4.3}$$

Taking into account that $\phi(\xi) = (1 - | \, \xi \, |)_+ \, , \, \theta(\xi) \; = \; \frac{3}{2\pi^2} \xi^2 - \frac{3}{\pi} \xi$, we have

$$\begin{aligned}
\frac{2}{3} \, \theta(\pi \xi) &= (1 - \phi^2(\xi)) \left[\frac{2}{3} \theta(\pi \xi) - \phi^2(\xi) \right] = \\
&= \left[\frac{2}{3} \theta(\pi \xi) \; - \; 1 \right] (1 - \phi^2(\xi)) + \left(1 - \phi^2(\xi) \right)^2 , \;\; \xi \in \mathbb{R} \, .
\end{aligned}$$

Then we get for all $h > 0$ ($\mathcal{M}_\infty \; = \; I$) on $B^p \cap \mathcal{S}$

$$\frac{2}{3} \widetilde{\Delta}_{\pi h} \; = \; \left(\frac{2}{3} \widetilde{\Delta}_{\pi h} - I \right) \circ (I + \mathcal{M}_{\frac{1}{h}}) \circ (I - \mathcal{M}_{\frac{1}{h}}) \; + \; (I + \mathcal{M}_{\frac{1}{h}})^2 \circ (I - \mathcal{M}_{\frac{1}{h}})^2 . \tag{4.4}$$

By applying properties of the operators $\widetilde{\Delta}_h$ and $\mathcal{M}_\sigma$, (1.3) and (2.7) we get for $g \in B^p \cap \mathcal{S}$ by virtue of (4.4)

$$\begin{aligned}
\tfrac{2}{3}\|\widetilde{\Delta}_{\pi h} g\|_p &\le \left\| \tfrac{2}{3}\widetilde{\Delta}_{\pi h} - I \right\|_{(p)} \left\| I + \mathcal{M}_{1/h} \right\|_{(p)} \| g - \mathcal{M}_{1/h}(g) \|_p \\
&+ \left\| I + \mathcal{M}_{1/h} \right\|_{(p)}^2 \left\| \left(I - \mathcal{M}_{1/h}\right)\left(g - \mathcal{M}_{1/h}(g)\right) \right\|_p \\
&\le c_1 \| (I - \mathcal{M}_{1/h}) g \|_p + c_2 \| (I - \mathcal{M}_{1/h}) g \|_p \\
&+ \| \mathcal{M}_{1/h} \|_{(p)} \| (I - \mathcal{M}_{1/h}) g \|_p \\
&\le c_3 \| (I - \mathcal{M}_{1/h}) g \|_p\,.
\end{aligned}$$

By using the approximation procedure (Theorem 1.2) we get

$$\|\widetilde{\Delta}_{\pi h} g\|_p \le C \| (I - \mathcal{M}_{1/h}) g \|_p \tag{4.5}$$

for $g \in B^p$. Hence, inequality (4.5) is valid also for $g \in L_p(\mathbb{R})$. By applying (1.1), we have

$$\begin{aligned}
\omega_{\langle ' \rangle}(f,\delta)_p &\le \omega_{\langle ' \rangle}(f,\pi\delta)_p = \sup_{0 \le h \le \delta} \| \widetilde{\Delta}_{\pi h} f(x) \|_p \\
&\le C \sup_{\lambda \ge \delta^{-1}} \| (I - \mathcal{M}_\lambda) f \|_p \\
&\le c_3 \sup_{\lambda \ge \delta^{-1}} K_{\langle ' \rangle}\left(f, 1/\lambda\right)_p \le c\, K_{\langle ' \rangle}(f,\delta)_p\,,
\end{aligned}$$

that completes the proof of (4.3).

Step 2. Now we prove the inverse estimate

$$K_{\langle ' \rangle}(f,\delta)_p \le c\,\omega_{\langle ' \rangle}(f,\delta)_p\,,\ \ f \in L_p(\mathbb{R})\,,\ \ \delta \ge 0\,. \tag{4.6}$$

For $\delta = 0$ relation (4.6) immediately follows from the definition of the modulus of smoothness and the K-functional. We consider $\mathcal{M}^{(\varphi)}_{1/\delta}(f;x)$, $\delta > 0$, where real-valued even infinitely differentiable function φ has a compact support on $[-\pi,\pi]$ and $\varphi(\xi) = 1$ for $|\,\xi\,| \le 1$. By virtue of (1.5) one has

$$\| f - \mathcal{M}^{(\varphi)}_\sigma(f) \|_p \le c\, E_\sigma(f)_p\,,\ \ f \in L_p(\mathbb{R})\,,\ \ \sigma \ge 0\,. \tag{4.7}$$

By applying (4.7) and Theorem 3.2, we have

$$\| f - \mathcal{M}^{(\varphi)}_\sigma(f) \|_p \le c\, E_{1/\delta}(f)_p \le c'\, \omega_{\langle ' \rangle}(f,\delta)_p\,. \tag{4.8}$$

We consider $\phi(x) = \frac{-3}{\pi}\frac{1}{Ax-B}\varphi(x)$, where $A = \frac{3}{2\pi^2}$, $B = \frac{3}{\pi}$. It is obvious that $\phi \in \mathcal{K}$ and

$$\theta(\xi)\,\phi(\xi) = -(3/\pi)\,|\,\xi\,|\,\varphi(\xi)\,,\ \ \xi \in \mathbb{R}\,.$$

Hence, by virtue of (1.6), (1.2), (4.1) and (3.1), we get on $B^p \cap \mathcal{S}$

$$\widetilde{\Delta}_\delta \circ \mathcal{M}^{(\phi)}_{1/\delta} = -(3/\pi)\,\delta\,(\,\mathcal{M}^{(\varphi)}_{1/\delta}\,)^{\langle ' \rangle}\,. \tag{4.9}$$

Combining (4.9) and (1.6) we obtain

$$\widetilde{\Delta}_\delta\, \mathcal{M}^{(\phi)}_{1/\delta}(g;x) \;=\; -(3/\pi)\,\delta\,(\,\mathcal{M}^{(\varphi)}_{1/\delta}(g;x)\,)^{\langle\prime\rangle}\,,\;\; g\in B^p\cap\mathcal{S}\,. \tag{4.10}$$

By using (1.3) and (4.10), we get

$$\begin{aligned}\delta\,\|\,(\,\mathcal{M}^{(\varphi)}_{1/\delta}(g)\,)^{\langle\prime\rangle}\,\|_p \;&=\; (\pi/3)\,\left\|\,\widetilde{\Delta}_\delta\,\left(\mathcal{M}^{(\phi)}_{1/\delta}(g;x)\right)\,\right\|_p \;=\; (\pi/3)\,\left\|\,\mathcal{M}^{(\phi)}_{1/\delta}\left(\,\widetilde{\Delta}_\delta g(x)\right)\,\right\|_p \;\le\\ &\le\; (\pi/3)\|\,\mathcal{M}^{(\phi)}_{1/\delta}\,\|_{(p)}\,\|\,\widetilde{\Delta}_\delta g(x)\,\|_p \;\le\; C\,\|\,\widetilde{\Delta}_\delta g(x)\,\|_p\,.\end{aligned} \tag{4.11}$$

By applying the approximation procedure (Theorem 1.2), we obtain that (4.11) is valid also for $g\in B^p$. Now (4.6) follows from (4.8) and (4.11). □

Combining (1.1) and Theorem 4.1, we obtain the following chain of equivalences.

Theorem 4.2. *Let $1\le p\le +\infty$. Then for $f\in L_p(\mathbb{R})$, $\sigma>0$ it holds*

$$\|\,f-\mathcal{M}_\sigma(f)\,\|_p \asymp K_{\langle\prime\rangle}\left(f,\sigma^{-1}\right)_p \asymp \omega_{\langle\prime\rangle}\left(f,\sigma^{-1}\right)_p\,.$$

As a corollary of (4.2) we show that estimate (3.3) can be strengthened. Indeed, for $m\in\mathbb{N}$ one has

$$\omega_{\langle\prime\rangle}(f,m\delta)_p \le cK_{\langle\prime\rangle}(f,m\delta)_p \le c'mK_{\langle\prime\rangle}(f,\delta)_p \le c''m\omega_{\langle\prime\rangle}(f,\delta)_p\,,$$

that implies

$$\omega_{\langle\prime\rangle}(f,t\delta)_p \;\le\; c\,(\,1+\,t\,)\,\omega_{\langle\prime\rangle}(f,\delta)_p\,,\; f\in L_p(\mathbb{R})\,,\;\delta,t\ge 0\,.$$

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