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THE ESTIMATE OF ACCURACY OF THE RATIONAL APPROXIMATION OF THE MONODROMY OPERATOR

N.B. Zhuravlev, A.N. Sokolova

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Key words: monodromy operator, differential-difference equations, eigenvalue problem, delay.

AMS Mathematics Subject Classification: 34K27, 47A75, 34K13, 34E10, 47A55, 37D05, 93C73.

Abstract. The paper deals with the problem of investigation of eigenvalues of the monodromy operator for periodic solutions of nonlinear delay-differential equations. In the case the period of the solution is not commensurate with the delay time, the rational approximation is used. Thus the eigenvalues depend on the perturbation parameter. In this paper, a similar problem for a nonlinear system of ordinary differential equations is considered. Necessary and sufficient conditions for the Lipschitz behaviour of the eigenvalues are obtained.

1 Introduction

Application of the Floquet theory to differential-difference equations is described by J. Hale [2]. The Floquet theory describes the behaviour of solutions to a nonlinear equation in a neighbourhood of a T-periodic solution in terms of the eigenvalues of the monodromy operator. The monodromy operator performs a shift along solutions of the linearized equation by the time T. In the case of equations with a delay, this operator is infinite-dimensional. This problem is solved by a special change of variables. For such a replacement a commensurate period of the considered solution with the delay time in the equation is required. In the case the period is not commensurate with the delay, A.L. Skubachevskii and H.-O. Walther [4] suggest using the approximation.

The approximate operator can be used to carry out a shift along solutions to the linearized equation by time p close to T [8].

In this paper, a similar problem for a nonlinear system of ordinary differential equations is considered. Necessary and sufficient conditions for the Lipschitz behaviour of the eigenvalues in a neighbourhood of the period are obtained.

2 Definition

We consider the nonlinear autonomous delay-differential equation

$$x'(t) = f(x(t), x(t-1)).$$

Let $\tilde{x}$ be a given periodic solution to this equation with period T. Consider the family of operators $\mho_p: \ C[0,1] \to C[0,1]$, acting according to the formula

$$\mho_p \phi(t) := v_\phi(t+p),$$

where the function v_ϕ is the solution to the initial value problem for the linearized equation

$$\dot{v}(t) = \alpha_0(t) \cdot v(t) + \alpha_1(t) \cdot v(t-1), \qquad t > 0$$

where

$$\alpha_0(t) = f'_x(x, y);\ \alpha_1(t) = f'_y(x, y);\ x = \tilde{x}(t), y = \tilde{x}(t-1)$$

with the initial condition

$$v(t) = \phi(t), \qquad t \in [-1, 0].$$

The operator $\mho_T$ is called the monodromy operator. Non-zero eigenvalues of the monodromy operator are called the Floquet multipliers.

If $T = N/M \in \mathbb{Q}$ and $T > 1$ the change of variables

$$u_k(t) = v(t + \frac{k-1}{M})$$

allows us to reduce the equation $\mho_T\phi - \lambda\phi = 0$ to the following system of equations:

$$\begin{cases} u'_i(t) = \alpha_{0i}(t)u_i(t) + \alpha_{1i}(t)u_{i-M+N}(t)/\lambda, & i = 1, \dots, M, \\ \\ u'_i(t) = \alpha_{0i}(t)u_i(t) + \alpha_{1i}(t)u_{i-M}(t), & i = M+1, \dots, N. \end{cases}$$

This transition is a basis for constructing the characteristic function for the Floquet multipliers [4]. If $T \notin \mathbb{Q}$ the following theorem allows to use the above approach.

Theorem 2.1. [8] *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\sigma(\mho_T) \subset \bigcup_{\lambda \in \sigma(\mho_S)} B(\lambda, \varepsilon)$$

for all S for which $|S - T| < \delta$.

In this paper, as a special case of the equation with delay, we consider the autonomous system

$$\begin{cases} \dot{x} = f(x, y), \\ \dot{y} = -x. \end{cases} \tag{2.1}$$

The existence of a periodic solution $(\tilde{x}, \tilde{y})$ to this system is assumed.

The corresponding linearized system has the form

$$\begin{cases} \dot{u} = \alpha_0 \cdot u + \alpha_1 \cdot v, \\ \dot{v} = -u. \end{cases} \tag{2.2}$$

The coefficients of this system

$$\alpha_0(t) = f'_x(\tilde{x}(t), \tilde{y}(t)) \text{ and } \alpha_1(t) = f'_y(\tilde{x}(t), \tilde{y}(t))$$

have period T, equal to the period of the solution $(\tilde{x}, \tilde{y})$.

In this particular case, the family of operators $\mho_p : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\mho_p : (u_0, v_0) \mapsto (u_\phi(p), v_\phi(p)),$$

where the function (u_ϕ, v_ϕ) is the solution to the initial value problem for linearized system (2.2) with the initial conditions

$$u(0) = u_0, \quad v(0) = v_0.$$

Using an arbitrary fundamental matrix Φ of system (2.2), we can determine the matrix C of the monodromy operator by the formula

$$C := \Phi^{-1}(t) \cdot \Phi(t+T)$$

(see [1]). Strictly speaking, at different t we have different matrices, but all of them have the same eigenvalues (Floquet multipliers) [3, Chapter IV, § 6].

We introduce the notation for the elements of the matrix $\mho_p$ and its eigenvalues:

$$\mho_p = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix}, \qquad \lambda_\pm(p) = \frac{a(p) + d(p) \pm \sqrt{D(p)}}{2},$$

where

$$D(p) = (a(p) - d(p))^2 + 4b(p)c(p).$$

We say that the behaviour of $\lambda_\pm(p)$ is *Lipschitz* at the point T if there exist $K, \varepsilon > 0$ such that for all p satisfying $|p - T| < \varepsilon$

$$|\lambda_\pm(p) - \lambda_\pm(T)| \leqslant K|p - T|.$$

The main result of this work is the following statement.

Theorem 2.2. *Let $f \in C^2(\mathbb{R}^2)$. Let $(\tilde{x}, \tilde{y})$ be a T-periodic solution to system* (2.1). *Then behaviour of $\lambda_\pm(p)$ is Lipschitz at the point T if and only if all solutions to system* (2.2) *are periodic or*

$$\int_0^T f'_x(\tilde{x}(t), \tilde{y}(t))dt \neq 0. \tag{2.3}$$

The main idea of the proof is as follows. The derivative of $\lambda(p)$ is not bounded in a neighbourhood of T if and only if $D(T) = 0$ and $D'(T) \neq 0$. The integral in (2.3) vanishes if and only if the eigenvalue 1 of the operator $\mho_T$ has the algebraic multiplicity 2. This means that $D(T) = 0$. The existence of a non-periodic solution to system (2.2) means that the geometric multiplicity of the eigenvalue 1 equals one. If we combine this with $D(T) = 0$ we get $D'(T) \neq 0$. So the assumptions of the theorem imply that the derivative of $\lambda(p)$ is bounded in a neighbourhood of T.

Moreover, if the behaviour of the perturbed eigenvalues is non-Lipschitz, by using the Newton diagram method [5] it can be proved, that the function $\lambda(p)$ has the form

$$\lambda_\pm(p) = 1 \pm \sqrt{-\overline{u}(T)}(p - T)^{1/2} + o(|p - T|^{1/2}),$$

where the function $(\overline{u}, \overline{v})$ is the solution to the initial value problem for linearized system (2.2) with the initial conditions $u(0) = 0, \quad v(0) = 1$.

The obtained result agrees with the well-known fact [6] that non-Lipschitz behaviour of the perturbed eigenvalues may arise only in the case of a discrepancy between the algebraic and geometric multiplicities of the non-perturbed eigenvalues.

In the case $f(x, y) = y$ the assumptions of the above theorem are satisfied. The following example shows that the behaviour of the eigenvalues can be non-Lipschitz.

Example 1. For

$$f(x, y) = y + 1 - (x^2 + y^2), T = 2\pi \quad \text{and} \quad (\tilde{x}, \tilde{y}) = (\sin t, \cos t)$$

the assumptions of the above theorem are not satisfied. If $\overline{u}(0) = 0$ and $\overline{v}(0) = 1$, then $\overline{u}(2\pi) = -2.705$ and $\overline{v}(2\pi) = 1$.

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