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ASYMPTOTICS OF THE SOLUTION OF PARABOLIC PROBLEMS WITH NONSMOOTH BOUNDARY FUNCTIONS

A. Omuraliev, E. Abylaeva

Communicated by M.A. Ragusa

Key words: singularity, perturbation, parabolic equation, nonsmooth boundary function, regularized asymptotics.

AMS Mathematics Subject Classification: 39A14, 34E10.

Abstract. In this paper, we construct the asymptotics of the solution to a singularly perturbed parabolic problem with a nonsmooth boundary layer function. In contrast to works devoted to this direction, our asymptotics contains only one boundary layer function, which is the product of parabolic and exponential boundary layer functions. Our approach allows us to construct a classical solution without applying smoothing procedures.

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1 Introduction

Singularly perturbed problems with nonsmooth regular boundary functions were studied in [1]-[8], [10]-[11]. To construct the asymptotics of the solution of such problems, the method of matching asymptotic expansions was used in [7]. In [4], [5] the asymptotics is constructed by using the smoothing procedure. Using the methodology of [2], in [3] the asymptotics of a solution of any order was constructed without the use of matching and smoothing procedures. In this paper, using the method of [10], a regularized asymptotics of the problem posed is constructed, applying for regularization the method of [9], regularizing functions are introduced, which are determined from partial differential equations of the first order and an ordinary differential equation. This choice of regularizing functions made it possible to pass the difficulties related to the non-smoothness of the boundary functions. The asymptotics of the solution constructed by us, in contrast to [2], [5]-[4], [7]-[10], contains only angular boundary layer functions represented as a product of parabolic and exponential boundary layer functions. The parabolic boundary layer function describes the boundary layer along the characteristic $t + B(x)/\sqrt{\varepsilon} = 0$, and the exponential boundary layer function describes the boundary layer along $x = 0$.

2 Statement of the problem

We consider the following singularly perturbed parabolic equation with nonsmooth boundary functions:

$$L_\varepsilon u(x,t,\varepsilon) \equiv -\partial_t u + \varepsilon^2 a(x)\partial_x^2 u + \sqrt{\varepsilon} b(x)\partial_x u - c(x,t)u = f(x,t) \tag{2.1}$$
$$(x,t) \in \Omega, u|_{t=0} = u|_{x=0} = 0,$$

which is studied in [3], where $\varepsilon > 0$ is a small parameter, $a(x), b(x), c(x,t), f(x,t)$ are continuously differentiable and bounded together with their derivatives in Ω, moreover $a(x) > 0, b(x) > 0, \Omega = (0 < x < \infty) \times (0 < t < T]$.

3 Regularization of the problem

Following [9], [10] we introduce the regularizing variables:

$$\xi = \varphi(x,t,\varepsilon), \eta = \psi(x,\varepsilon) \tag{3.1}$$

and extended function $u(M,\varepsilon), M = (x,t,\xi,\eta)$ such that:

$$\tilde{u}(M,\varepsilon)|_{\theta=\gamma(x,t,\varepsilon)} \equiv u(x,t,\varepsilon), \theta = (\xi,\eta), \gamma(x,t,\varepsilon) = (\varphi(x,t,\varepsilon), \psi(x,\varepsilon)). \tag{3.2}$$

Based on (3.1) we find the derivatives:

$$\partial_t u \equiv (\partial_t \tilde{u} + \partial_t \varphi(x,t,\varepsilon)\partial_\xi \tilde{u})_{\theta=\gamma(x,t,\varepsilon)},$$

$$\partial_x u \equiv \left(\partial_x \tilde{u} + \partial_x \varphi(x,t,\varepsilon)\partial_\xi \tilde{u} + \psi^{'}(x,\varepsilon)\partial_\eta \tilde{u}\right)_{\theta=\gamma(x,t,\varepsilon)},$$

$$\partial_x^2 \equiv \left(\partial_x^2 \tilde{u} + (\partial_x \varphi(x,t,\varepsilon))^2 \partial_\xi^2 \tilde{u} + (\psi^{'}(x,\varepsilon))^2 \partial_\eta^2 \tilde{u} + L_\xi \tilde{u} + L_\eta \tilde{u}\right)_{\theta=\gamma(x,t,\varepsilon)},$$

$$L_\xi \equiv 2\partial_x \varphi \partial^2_{x,\xi} + \partial_x^2 \varphi \partial_\xi,$$

$$L_\eta \equiv 2\psi^{'} \partial^2_{x,\eta} + \psi^{''} \partial_\eta,$$

then, instead of problem (2.1), we pose the extended problem:

$$\tilde{L}_\varepsilon \tilde{u}(M,\varepsilon) \equiv -(\partial_t \tilde{u} + \partial_t \varphi(x,t,\varepsilon)\partial_\xi \tilde{u}) + \tag{3.3}$$

$$\varepsilon^2 a(x)\left(\partial_x^2 \tilde{u} + (\partial_x \varphi(x,t,\varepsilon))^2 \partial_\xi^2 \tilde{u} + (\psi^{'}(x,\varepsilon))^2 \partial_\eta^2 \tilde{u} + L_\xi \tilde{u} + L_\eta \tilde{u}\right) +$$

$$\sqrt{\varepsilon} b(x)\left(\partial_x \tilde{u} + \partial_x \varphi(x,t,\varepsilon)\partial_\xi \tilde{u} + \psi^{'}(x,\varepsilon)\partial_\eta \tilde{u}\right) -$$

$$c(x,t)\tilde{u} = f(x,t), M \in Q,$$

$$\tilde{u}|_{t=0} = \tilde{u}|_{x=\xi=\eta=0} = 0.$$

Let us choose the regularizing functions $\varphi(x,t,\varepsilon), \psi(x,\varepsilon)$ as solutions of the problems:

$$-\partial_t \varphi + \sqrt{\varepsilon} b(x)\partial_x \varphi = 0, \varphi(0,t,\varepsilon) = 0,$$

$$\sqrt{\varepsilon} b(x)\psi^{'} = 1, \psi(0,\varepsilon) = 0.$$

The solution to these problems will be:

$$\varphi(x,t,\varepsilon) = \Phi\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right), \psi(x,\varepsilon) = \frac{1}{\sqrt{\varepsilon}}B(x),$$

$$B(x) = \int_0^x \frac{dt}{b(t)},$$

where $\Phi\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right)$ is an arbitrary function such that $\Phi(0) = 0$. Taking into account the found functions, extended equation (3.3) can be rewritten as:

$$\tilde{L}_\varepsilon \tilde{u}(M,\varepsilon) \equiv -(\partial_t \tilde{u}) + \varepsilon^2 a(x)\left[\partial_x^2 \tilde{u} + \left(\Phi_x^{'}\left(t + \frac{1}{\sqrt{\varepsilon}}B(x)\right)\right)^2 \partial_\xi^2 \tilde{u} + L_\xi \tilde{u}\right] + \tag{3.4}$$

$$\varepsilon^2 a(x)\left[(\psi'(x,\varepsilon))^2\partial_\eta^2\tilde{u}+L_\eta\tilde{u}\right]+$$

$$\sqrt{\varepsilon}b(x)\left(\partial_x\tilde{u}+\psi'(x,\varepsilon)\partial_\eta\tilde{u}\right)-$$

$$c(x,t)\tilde{u}=f(x,t),M\in Q,$$

$$\tilde{u}|_{t=0}=\tilde{u}|_{x=\xi=\eta=0}=0,Q=\Omega\times(0,\infty)^2.$$

Let us choose the function $\Phi'_x\left(t+\frac{1}{\sqrt{\varepsilon}}B(x)\right)$ as a solution to the equation $\varepsilon^2 a(x)\left(\Phi'\left(t+\frac{1}{\sqrt{\varepsilon}}B(x)\right)\frac{1}{\sqrt{\varepsilon}b(x)}\right)^2=1$ with the initial condition $\Phi(0)=0$, then:

$$\Phi\left(t+\frac{1}{\sqrt{\varepsilon}}B(x)\right)=\frac{1}{\sqrt{\varepsilon}}\int_0^{t+\frac{1}{\sqrt{\varepsilon}}B(x)}\frac{b\left(B^{-1}(\sqrt{\varepsilon}(u-s))\right)}{\sqrt{a\left(B^{-1}(\sqrt{\varepsilon}(u-s))\right)}}du.$$

After this choice of the regularizing functions, equation (3.4) takes the form:

$$\tilde{L}_\varepsilon\tilde{u}(M,\varepsilon)\equiv-\partial_t\tilde{u}+\partial_\xi^2\tilde{u}+\partial_\eta\tilde{u}-c(x,t)\tilde{u}+\sqrt{\varepsilon}b(x)\partial_x\tilde{u}+ \tag{3.5}$$

$$\varepsilon\frac{a(x)}{b^2(x)}\partial_\eta^2\tilde{u}+\varepsilon a(x)L_\xi\tilde{u}+\sqrt{\varepsilon^3}a(x)L_\eta\tilde{u}+\varepsilon^2 a(x)\partial_x^2\tilde{u}=f(x,t),M\in Q,$$

$$\tilde{u}|_{t=0}=\tilde{u}|_{x=\xi=\eta=0}=0.$$

Equation (3.5) is regular on ε as ε tends to zero. The solution to problem (3.5) will be defined as:

$$\tilde{u}(M,\varepsilon)=\sum_{k=1}^{\infty}\varepsilon^{k/2}u_k(M).$$

For the coefficients of this series, we obtain the following iterative problems:

$$T_0u_0(M)\equiv-\partial_t u_0+\partial_\xi^2 u_0+\partial_\eta u_0-c(x,t)u_0=f(x,t), \tag{3.6}$$

$$T_0u_k=H_k(M),u_k|_{t=0}=u_k|_{x=\xi=\eta=0}=0.$$

4 Solvability of iterative problems

We introduce the class of functions in which iterative equations will be solved:

$$U=\left\{f(M)=f_1(x,t)+f_2(x,t)erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)):f_1,f_2\in C^\infty(\Omega)\right\},$$

where $erfc\left(\frac{\xi}{2\sqrt{t}}\right)$ describes the parabolic boundary layer along $x=0$, the function $\exp(-(t+\eta))$ is the boundary layer along $t=0$.

Theorem 4.1. *Suppose that $H_k(M)\in U$, then the equation*

$$T_0u_k=H_k(M) \tag{4.1}$$

has a solution $u_k(M)\in U$.

Proof. Let $H_k(M) \in U$, namely

$$H_k(M) = h_1(x,t) + h_2(x,t) erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)), \quad \text{where } h_1, h_2 \in C^\infty(\Omega).$$

Notice that functions $u_k(M)$ satisfy equation (4.1) if the functions $v_k(x,t), d_k(x,t)$ are solutions to the equations

$$\partial_t v_k(x,t) - c(x,t)v_k(x,t) = h_1(x,t), \quad \partial_t d_k(x,t) - c(x,t)d_k(x,t) = h_2(x,t). \tag{4.2}$$

By our assumptions on the functions $c(x,t), h_1(x,t), h_2(x,t)$, these equations have smooth solutions. □

Theorem 4.2. *Equation* (4.1) *under the following additional conditions:*

1. $u_k|_{t=0} = u_k|_{x=\xi=\eta=0} = 0$;
2. $L_\xi u_k = 0$

has a unique solution.

Proof. Let the function $u_k(M)$ satisfy boundary conditions 1), then:

$$v_k(x,0) = 0, d_k(x,0) = d_k^0(x), d_k(0,t) = -v_k(0,t)\exp(t), \tag{4.3}$$

where $d_k^0(x)$ is an arbitrary function. Solving equation (4.2) with respect to $d_k(x,0)$, for an arbitrary initial condition, we find:

$$d_k(x,t) = d_k^0(x)p_1(x,t) + p_2(x,t),$$

where $p_l(x,t), l = 1,2$ are known functions. Condition 2) of the theorem, taking into account the found value, is equivalent to the equation:

$$\left(d_k^0(x)\right)' + q_1(x,t)d_k^0(x) = q_2(x,t), \tag{4.4}$$

which we solve under the initial condition $d_k^0(0) = q_3(t)$, is determined from (4.3), here $q_l(.), l = 1,2,3$ are the known functions. This uniquely determines $d_k(x,t)$, also the function $v_k(x,t)$ is uniquely determined from equation (4.2) under the initial condition from (4.3). □

5 Solution of iterative problems

Consider equation (3.6) for $k = 0$, with the right-hand side $f(x,y) \in U$. By Theorem 4.1, this equation has a solution $u_0(M) \in U$, i.e.

$$u_0(M) = v_0(x,t) + d_0(x,t) erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)),$$

where the arbitrary functions $v_0(x,t)$ and $d_0(x,t)$ are defined as in Theorem 4.2. The right-hand sides of the iterative equations will have the form:

$$H_k(M) = -b(x)\left[v_{k-1}(x,t) + a(x)\partial_x^2\partial_x^2 v_{k-4}(x,t)\right] erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)) -$$

$$\left\{b(x)d_{k-1}(x,t) + a(x)\left[\partial_x^2 d_{k-4}(x,t) + \left(\frac{1}{b(x)}\right)'\right]\right\} erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta)) +$$

$$a(x)\left[\frac{2}{b(x)}\partial_x d_{k-3}(x,t)+\left(\frac{1}{b(x)}\right)' d_{k-3}(x,t)\right] erfc\left(\frac{\xi}{2\sqrt{t}}\right)\exp(-(t+\eta))\in U.$$

Further, using Theorems 4.1 and 4.2, we successively determine the coefficients of the partial sum:

$$u_{\varepsilon,n}=\sum_{k=0}^{n}\varepsilon^{k/2}u(M). \tag{5.1}$$

6 Estimate of the remainder term

Considering that

$$\left(\tilde{L}_\varepsilon\tilde{u}(M,\varepsilon)\right)_{\theta=\gamma(x,t,\varepsilon)}\equiv L_\varepsilon u(x,t,\varepsilon) \tag{6.1}$$

and substituting

$$R_{\varepsilon,n}(M)=\tilde{u}(M,\varepsilon)-u_{\varepsilon,n}(M)$$

into equation (3.5), then, taking into account (3.6) and making the restriction by means of regularizing functions, based on (6.1), we obtain the following problem for the remainder:

$$L_\varepsilon R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)=\varepsilon^{\frac{n+1}{2}}g_n(x,t,\varepsilon),$$

$$R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)_{t=0}=R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)_{x=0}=0.$$

Using the maximum principle [8], we establish the estimate:

$$\|R_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)\|<c\varepsilon^{\frac{n+1}{2}}. \tag{6.2}$$

Theorem 6.1. *The function* $u_{\varepsilon,n}\left(x,t,\gamma(x,t,\varepsilon)\right)$ *is the asymptotic solution of problem* (2.1) *and is such that in the region* $\bar{\Omega}=(0\le x\le\infty)\times(0\le t\le T)$ *estimate* (6.2) *holds, where* c *is independent of* ε.

The solution constructed above is asymptotic, namely the difference between the exact and asymptotic solutions satisfies (6.2).

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