

THE BD PROPERTY OF A CLASS OF HIGH-ORDER SINGULAR DIFFERENCE OPERATORS

R. Oinarov, D. Sarsenaly, A. Temirkhanova

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Abstract. This paper establishes a criterion for the discreteness of the spectrum and its boundedness from below for a symmetric difference operator of the $2n$ -order with two positive coefficients.

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1 Introduction

Let $n \in \mathbb{N}$, $\mathbb{N}$ be the set of all natural numbers. Let $\rho = \{\rho_k\}_{k=1}^\infty$, $v = \{v_k\}_{k=1}^\infty$ be sequences of positive real numbers.

Let us consider the difference operator

$$ly_{k+n} = (-1)^n \frac{\bar{\Delta}^n(\rho_k \Delta^n y_k)}{v_k},$$

where $\Delta y_k = y_{k+1} - y_k$, $\bar{\Delta} y_k = y_k - y_{k-1}$, $\Delta^n y_k = \Delta(\Delta^{n-1} y_k)$, $\Delta^0 y_k \equiv y_k$, $k \in \mathbb{N}$ and $\bar{\Delta}^n y_k$ is defined similarly.

Remark 1. Further, for convenience, instead of $\bar{\Delta}^n(\rho_k \Delta^n y_k)$ we will write $\Delta^n(\rho_k \Delta^n y_k)$.

Let $N \in \mathbb{N}$, and E_N be the set of all sequences of real numbers $y = \{y_k\}_{k=N}^\infty$ such that $y_N = y_{N+1} = \dots = y_{N+n-1} = 0$. Let $\mathring{E}_N \subset E_N$ and for any $y \in \mathring{E}_N$ there exists a number $M(y) \in \mathbb{N}$ such that $y_k = 0$ for all $k \geq M(y)$.

Let $\mathbb{N}(N) = \{k \in \mathbb{N} : k \geq N\}$ and $l_{2,v}(\mathbb{N}(N))$ be the Hilbert space of all sequences $f = \{f_k\}_{k=N}^\infty$ with the inner product $(f, g)_{2,v} = \sum_{k=N}^\infty f_k g_k v_k$, $f, g \in l_{2,v}(\mathbb{N}(N))$, and with the norm $\|f\|_{2,v} = \left(\sum_{i=N}^\infty v_i |f_i|^2 \right)^{\frac{1}{2}}$.

We consider the following operator L_{min} with the domain $\mathcal{D}(L_{min}) = \mathring{E}_N$

$$L_{min} y_{k+n} = ly_{k+n} = (-1)^n \frac{\Delta^n(\rho_k \Delta^n y_k)}{v_k}, \quad \forall y \in \mathcal{D}(L_{min}). \quad (1.1)$$

The operator L_{min} is symmetric on the space $l_{2,v}(\mathbb{N}(N))$, denote its closure in $l_{2,v}(\mathbb{N}(N))$ by $\bar{L}_{min}$. It is known [6] that all self-adjoint extensions L of the operator $\bar{L}_{min}$ have similar spectral properties.

In this paper, we investigate the discreteness of the spectrum of an operator L and its boundedness from below. This property of the operator is called *BD property (Bounded Discrete)* according to the terminology introduced in [9].

The BD property for self-adjoint extensions $\mathcal{L}$ of the differential analogue of the operator $\bar{L}_{min}$

$$\bar{\mathcal{L}}_{min} y(t) = (-1)^n \frac{\frac{d^n}{dt}(\rho(t) \frac{d^n}{dt} y(t))}{v(t)}$$

is well studied in books [6], [14] and in many scientific articles, for example, in [1], [7], [11], [12], [13].

As stated in [8], the study of the spectral properties of difference operators has a rather short history, which began with the work [10], although the fundamental results of the spectral theory of linear difference operators can be found in the monograph [3].

One of the first criteria for the validity of the BD property of an L operator under the assumptions $v_k \equiv 1$ and $\sum_{k=1}^{\infty} \rho_k^{-1} < \infty$ was obtained by O. Došlý [4]. In [16] for an operator L in the case $\rho_k = k^{(\alpha)}$, $\alpha \in \mathbb{R}$, $\alpha \notin \{1, 3, \dots, 2n-1\}$, $\alpha < 2n-1$ sufficient conditions for the validity of the BD property are obtained, where $k^{(\alpha)} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}$, $\Gamma(\cdot)$ is the Gamma function. For the case $v_k \equiv 1$, the necessary conditions for the validity of the BD condition for an L operator are given in [8].

In [5], by passing from the equation

$$(-1)^n \Delta^n (\rho_k \Delta^n y_k) = \lambda k^{-(\alpha)} y_{k+n} \quad (1.2)$$

to the corresponding Hamiltonian system and using the classification of solutions of a homogeneous Hamiltonian system based on the Pólya factorization [17], various conditions for non-oscillation and oscillation of solutions to equation (1.2) are found. Further, on the base of these results, various criteria are established for the validity of the BD property for an operator L for $\rho_k = k^{(\alpha)}$, $k \geq 1$ depending on the properties of polynomial and non-polynomial solutions of the homogeneous equation $\Delta^n (\rho_k \Delta^n y_k) = 0$.

The above results are based on the statement from [8], which is an analogue of Glazman's statement [6] for differential operators. We present this statement for an operator L .

Theorem 1.1. *The following statements are equivalent:*

- (i) *an operator L has the BD property;*
- (ii) *the equation $(-1)^n \Delta^n (\rho_k \Delta^n y_k) - \lambda v_k y_{k+n} = 0$ is non-oscillatory for all $\lambda > 0$;*
- (iii) *for any $\lambda > 0$, there exists $N = N(\lambda) \in \mathbb{N}$ such that*

$$\sum_{k=N}^{\infty} \left(\rho_k |\Delta^n y_k|^2 - \lambda v_k y_{k+n}^2 \right) \geq 0 \quad (1.3)$$

for all sequences $y = \{y_k\}_{k=0}^{\infty} \in \mathring{E}_N$.

In the above mentioned works, the validity of the BD property of an operator L was studied for particular cases $v_k \equiv 1$ or $\rho_k = k^{(\alpha)}$. Below, we investigate the validity of the BD property of an operator L under the assumption that sequences $\rho = \{\rho_k\}_{k=1}^{\infty}$, $v = \{v_k\}_{k=1}^{\infty}$ are arbitrary sequences of positive numbers.

2 Preliminary results

Let us consider the discrete Sobolev space $W_{2,\rho}^n(E_N)$ on the set E_N , $N \in \mathbb{N}$ with the finite norm

$$\|y\|_{W_{2,\rho}^n} = \left(\sum_{k=N}^{\infty} \rho_k |\Delta^n y_k|^2 \right)^{\frac{1}{2}}. \quad (2.1)$$

Since $\mathring{E}_N \subset W_{2,\rho}^n(E_N)$, the closure of the set $\mathring{E}_N$ in $W_{2,\rho}^n(E_N)$ is denoted by $\mathring{W}_{2,\rho}^n(E_N)$.

Let us consider (1.3) for $\lambda = 1$. Let the inequality

$$\sum_{k=N}^{\infty} \left(\rho_k |\Delta^n y_k|^2 - v_k y_{k+n}^2 \right) \geq 0, \quad \forall y \in \mathring{E}_N \quad (2.2)$$

hold for all $N \in \mathbb{N}$. The sequence $y \in \mathring{E}_N$ has only a finite number of non-zero elements and $\mathring{E}_N = \mathring{W}_{2,\rho}^n(E_N)$, therefore,

$$\sum_{k=N}^{\infty} \bar{v}_k |y_k|^2 \leq \sum_{k=N}^{\infty} \rho_k |\Delta^n y_k|^2, \quad \forall y \in \mathring{W}_{2,\rho}^n(E_N), \quad (2.3)$$

where $\bar{v}_k = v_{k-n}$.

Let us consider the inequality

$$\sum_{k=N}^{\infty} \bar{v}_k |y_k|^2 \leq C_N \sum_{k=N}^{\infty} \rho_k |\Delta^n y_k|^2, \quad \forall y \in \dot{W}_{2,\rho}^n(E_N) \quad (2.4)$$

with the least constant $C_N > 0$.

Lemma 2.1. *Let C_N be the least constant in (2.4). Relation (2.2) holds for all $N \in \mathbb{N}$ if and only if*

$$0 < C_N \leq 1. \quad (2.5)$$

Consider statement (iii) of Theorem 1.1. Let for any $\lambda > 0$ there exist $N(\lambda) \in \mathbb{N}$ such that

$$\sum_{k=N(\lambda)}^{\infty} \left(\rho_k |\Delta^n y_k|^2 - \lambda v_k y_{k+n} \right) \geq 0, \quad \forall y \in \dot{E}_{N(\lambda)}. \quad (2.6)$$

Lemma 2.2. *Let C_N be the least constant in (2.4). Statement (iii) of Theorem 1.1 holds if and only if*

$$\inf_{N \in \mathbb{N}} C_N = 0. \quad (2.7)$$

Theorem 1.1 and Lemma 2.2 directly imply the following statement.

Theorem 2.1. *Let $N \in \mathbb{N}$ and C_N be the least constant in (2.4). Any operator L generated by expression (1.1) has the BD property if and only if $\inf_{N \in \mathbb{N}} C_N = 0$.*

Thus, to establish the BD property of an operator L , we need to know the value of C_N or its estimate for all $N \in \mathbb{N}$.

3 Two-sided estimate for the least constant C_N in (2.4)

Applying the summation step by step, it is easy to verify that

$$y_k = \frac{1}{(n-1)!} \sum_{j=N}^{k-1} (k-j-1)^{(n-1)} \Delta^n y_j, \quad k > N, \quad y \in \dot{W}_{2,\rho}^n(E_N). \quad (3.1)$$

Substituting (3.1) into the left-hand side of (2.4), we obtain

$$\sum_{k=N}^{\infty} \bar{v}_{k-1} \left| \frac{1}{(n-1)!} \sum_{j=N}^k (k-j)^{(n-1)} \Delta^n y_j \right|^2 \leq C_N \sum_{k=N}^{\infty} \rho_k |\Delta^n y_k|^2, \quad y \in \dot{W}_{2,\rho}^n(E_N).$$

Suppose that $f_j = \Delta^n y_j, j \geq N$. Then $f = \{f_j\}_{j=1}^{\infty} \in l_{2,\rho}(E_N)$ and the last inequality takes the form

$$\sum_{k=N}^{\infty} \tilde{v}_k \left| \frac{1}{(n-1)!} \sum_{j=N}^k (k-j)^{(n-1)} f_j \right|^2 \leq C_N \sum_{k=N}^{\infty} \rho_k |f_k|^2, \quad f \in l_{2,\rho}(\mathbb{N}(N)), \quad (3.2)$$

where $\tilde{v}_k = v_{k-(n-1)}$. The least constants in (2.4) and (3.2) coincide.

The representation

$$(k-j)^{(n-1)} = (k-j)(k-j-1) \cdots (k-j-(n-1)+1), \quad k > j, k, j \in \mathbb{N}$$

is not convenient for investigation of inequality (3.2). It is more convenient to apply the following binomial formula for the $(k-j)^{(n-1)}$: for any $i, j, k \in \mathbb{N} : k > i > j$

$$(k-j)^{(n-1)} = \sum_{t=1}^n \frac{(n-1)!}{(t-1)!(n-t)!} (k-i)^{(t-1)} (i-j)^{(n-t)}. \quad (3.3)$$

For a non-negative matrix $(a_{k,i})$, $k, i \in \mathbb{N}$, consider the inequality

$$\sum_{k=1}^{\infty} v_k \left| \sum_{i=1}^k a_{k,i} f_i \right|^2 \leq C \sum_{j=1}^{\infty} \rho_j |f_j|^2, \quad f \in l_{2,\rho}(\mathbb{N}). \quad (3.4)$$

In [15] the classes $\mathcal{O}_n^-$, $n \in \mathbb{N} \cup \{0\}$ of matrices $(a_{k,j})$, $a_{k,j} \geq 0$, if $k \geq j \geq 1$ and $a_{k,j} = 0$, if $k < j$, were introduced.

Let us give the definitions of these classes. We assume $a_{k,j} \equiv a_{k,j}^{(n)}$, if $(a_{k,j}) \in \mathcal{O}_n^-$, $n \in \mathbb{N} \cup \{0\}$.

Definition 1. A matrix $(a_{k,j}^{(0)})$ belong to the class $\mathcal{O}_0^-$, if $a_{k,j}^{(0)} = w_i \geq 0$, $k \geq j \geq 1$.

Let the classes $\mathcal{O}_\gamma^-$, $\gamma = 0, 1, \dots, n-1$, be defined.

Definition 2. Let $(a_{k,j})$ be a matrix which is non-negative and non-increasing in the second index for all $k \geq j \geq 1$. A matrix $(a_{k,j}) \equiv (a_{k,j}^{(n)}) \in \mathcal{O}_n^-$, if there exist matrices $(a_{k,j}^{(\gamma)}) \in \mathcal{O}_\gamma^-$, $\gamma = 0, 1, \dots, n-1$, non-negative matrices $(d_{i,j}^{\gamma,n})$, $\gamma = 0, 1, \dots, n-1$ such that

$$a_{k,j}^{(n)} \approx \sum_{\gamma=0}^n a_{k,i}^{(\gamma)} d_{i,j}^{\gamma,n}, \quad n \geq 1$$

for all $k \geq i \geq j \geq 1$, where $d_{k,j}^{n,n} = 1$.

Remark 2. Here, $A \approx B$ means $A \ll B \ll A$, with $A \ll B$ denoting $A \leq cB$, where c is a constant.

By Theorem 3.1 of [15], we have the following statement.

Theorem 3.1. Let a matrix $(a_{k,j}) \in \mathcal{O}_n^-$, $n \in \mathbb{N}$. Then inequality (3.4) holds for all $f \in l_{2,\rho}(\mathbb{N})$ if and only if

$$B_n = \sup_{k \geq 1} \sum_{j=1}^k \rho_j^{-1} \sum_{i=k}^{\infty} a_{i,j}^2 v_i < \infty.$$

Moreover, there exists a constant $\beta_n > 0$, such that for the least constant C in inequality (3.4) the following estimate holds:

$$B_n \leq C \leq \beta_n B_n, \quad \forall n \in \mathbb{N}.$$

We assume $a_{k,j} \equiv (k-j)^{(n-1)} \geq 0$, if $k \geq j \geq 1$ and $a_{k,j} \equiv (k-j)^{(n-1)} = 0$, if $k < j$.

We establish the following statement.

Proposition 3.1. The matrix $(a_{k,j}) \equiv ((k-j)^{(n-1)}) \in \mathcal{O}_{n-1}^-$.

By Theorem 3.1 and Proposition 3.1, we obtain the following statement.

Theorem 3.2. Inequality (3.2)(or (2.4)) holds for all $y \in \dot{W}_{2,\rho}^n(E_N)$ if and only if

$$B_{n-1}(N) = \sup_{s \geq N} \sum_{j=N}^s \rho_j^{-1} \sum_{k=s}^{\infty} \tilde{v}_k [(k-j)^{(n-1)}]^2 < \infty. \quad (3.5)$$

Moreover, there exists a constant $\beta_{n-1} > 0$, such that for the least constant C_N in inequality (3.2)(or (2.4)) the following estimate holds:

$$\frac{B_{n-1}(N)}{[(n-1)!]^2} \leq C_N \leq \beta_{n-1} \frac{B_{n-1}(N)}{[(n-1)!]^2}. \quad (3.6)$$

4 BD property of an operator L

On the basis of Theorems 2.1 and 3.1, by virtue of (3.6) it follows that $\inf_{N \in \mathbb{N}} C_N = 0$ if and only if $\inf_{N \in \mathbb{N}} B_{n-1}(N) = 0$, where C_N is the least constant in inequality (3.2) ((2.4)).

From expression (3.5), it follows that $B_{n-1}(N)$ is decreasing with respect to $N \in \mathbb{N}$, hence $\inf_{N \in \mathbb{N}} C_N = 0$ if and only if $\lim_{N \rightarrow \infty} B_{n-1}(N) = 0$.

Next, we prove that $\frac{B_{n-1}(N)}{[(n-1)!]^2} \approx \max_{1 \leq t \leq n} \left[\frac{1}{(t-1)!(n-t)!} \right]^2 B_{n-1}(N, t)$, where

$$B_{n-1}(N, t) = \sup_{s \geq N} \sum_{j=N}^s \left[\rho_j^{-1}(s-j)^{(n-t)} \right]^2 \sum_{k=s}^{\infty} \tilde{v}_k \left[(k-s)^{(t-1)} \right]^2.$$

Hence, we obtain the following statement.

Theorem 4.1. *An operator L has the BD property if and only if*

$$\lim_{N \rightarrow \infty} \sup_{s \geq N} \sum_{j=N}^s \rho_j^{-1} [(s-j)^{(n-t)}]^2 \sum_{k=s}^{\infty} \tilde{v}_k [(k-s)^{(t-1)}]^2 = 0 \quad (4.1)$$

for all $t = 1, 2, \dots, n$.

Below, we present an alternative criterion that is equivalent to (4.1).

Theorem 4.2. *An operator L has the BD property if and only if*

$$\lim_{s \rightarrow \infty} \sup \sum_{j=1}^s \rho_j^{-1} [(s-j)^{(n-t)}]^2 \sum_{k=s}^{\infty} \tilde{v}_k [(k-s)^{(t-1)}]^2 = 0$$

for all $t = 1, 2, \dots, n$.

Theorem 4.2 implies the following corollaries.

Corollary 4.1. *If $B_{n-1}(N, 1) \geq \max_{1 \leq t \leq n} B_{n-1}(N, t)$, $\forall N \in \mathbb{N}$, then an operator L has the BD property if and only if*

$$\lim_{s \rightarrow \infty} \sup \sum_{z=0}^{s-1} \rho_{s-z}^{-1} [z^{(n-1)}]^2 \sum_{k=s}^{\infty} \tilde{v}_k = 0.$$

Corollary 4.2. *If $B_{n-1}(N, n) \geq \max_{1 \leq t \leq n} B_{n-1}(N, t)$, $\forall N \in \mathbb{N}$, then an operator L has the BD property if and only if*

$$\lim_{s \rightarrow \infty} \sup \sum_{j=1}^s \rho_s^{-1} \sum_{z=0}^j \tilde{v}_{z+s} [z^{(n-1)}]^2 = 0.$$

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Ryskul Oinarov
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St,
010008 Astana, Republic of Kazakhstan
E-mail: o_ryskul@mail.ru
and
Samarkand State University
15 Univ Blvd, Samarkand 140104, Republic of Uzbekistan

Dinara Sarsenaly
School of Artificial Intelligence and Data Science
Astana IT University
55/11 Mangilik El Avenue,
010000 Astana, Republic of Kazakhstan
E-mail: dinara.sarsenali@mail.ru

Ainur Temirkhanova
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St,
010008 Astana, Republic of Kazakhstan
E-mail: ainura-t@yandex.kz

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