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NEW CUBATURE FORMULAS FOR KOROBOV SPACES

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Abstract. In this paper we present new cubature formulas for the Korobov spaces with a smoothness parameter $\alpha > 1$ and an estimate of the worst-case error of the cubature formulas.

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1 Introduction

Let $n, N \in \mathbb{N}$. We define a cubature formula as a sum $Q_N = \sum_{k=1}^N c_k f(M_k)$ that approximates the multiple integral $\int_{[0,1]^n} f(\mathbf{x}) d\mathbf{x}$ with an error R_N , where c_k are weights and M_k are nodes:

$$I(f) = \int_{[0,1]^n} f(\mathbf{x}) d\mathbf{x} = \sum_{k=1}^N c_k f(\mathbf{M}_k) - R_N.$$

Definition 1. Cubature formula's worst-case error for a normed function space $U = U[0, 1]^n$ is defined as the following quantity:

$$\varepsilon_N(U, Q_N) = \sup_{\|f\|_U=1} \left| \int_{[0,1]^n} f(\mathbf{x}) d\mathbf{x} - Q_N(f) \right|.$$

We define the Korobov space $E_n^\alpha = E^\alpha[0, 1]^n$ as the set of all measurable functions f such that

$$\|f\|_{E_n^\alpha} = \sup_{\mathbf{k} \in \mathbb{Z}^n} \bar{\mathbf{k}}^\alpha |\hat{f}(\mathbf{k})| < \infty,$$

where $\alpha > 0$, $\bar{\mathbf{k}} = \prod_{j=1}^n \bar{k}_j$, $\bar{k}_j = \max\{1, |k_j|\}$ and $\hat{f}(\mathbf{k})$ is the Fourier coefficient of the function f .

The Korobov spaces E_n^α were introduced in [6]. The following theorem shows the relation between known functional spaces and the Korobov spaces.

Theorem A. [4, Lemma 3.16] Assume $\alpha > 0$. Then

$$B_{1,1}^\alpha \hookrightarrow W_1^\alpha \hookrightarrow B_{1,\infty}^\alpha \hookrightarrow E_n^\alpha,$$

where B, W are the classical Besov and Sobolev spaces with a dominant mixed derivative, respectively.

For the historical development and the list of optimal cubature formulas for the classical Besov and Sobolev spaces see [2], [4] and the references therein.

Extensive development of the theory of cubature formulas started in the 60s after Bakhvalov [1] pointed out that the extension of univariate methods (for instance, the rectangular rule) to multivariate case provided a non-improvable error $O(N^{-\alpha/n})$, where the integrand has continuous non-mixed partial derivatives of

order up to $\alpha > 0$ and continuous mixed derivatives of order up to αn . This phenomenon called "the curse of dimensionality" (the term coined by Bellman) refers to the exponential increase in the complexity of calculating an integral as the number of dimensions grows.

Korobov [5] presented a cubature formula with weights $c_k = 1/N$ and nodes

$$\mathbf{M}_k = \left(\left\{ \frac{ka_1}{N} \right\}, \left\{ \frac{ka_2}{N} \right\}, \dots, \left\{ \frac{ka_n}{N} \right\} \right), \quad k = 1, 2, \dots, N, \quad (1.1)$$

where $\{x\}$ is the fractional part of x and $a_1(n, N), \dots, a_n(n, N)$ are some integers relatively prime with N , and the numerical integration problem was reduced to finding optimal coefficients $a_1(n, N), \dots, a_n(n, N)$ for fixed n, N . The component-by-component algorithm is used to find the optimal coefficients $a_1(n, N), \dots, a_n(n, N)$ [8]. The computational cost (the number of algebraic operations) does not exceed $O(nN^2)$, where $N > 1$ is a prime number.

It is worth to note that up to our knowledge **no concrete** cubature formula of type (1.1) for $n > 2$ has been provided. For the case $n = 2$ a cubature formula of type (1.1) is known (see [10]).

For further information on the number-theoretic methods and their development history, we refer to book [3].

Smolyak [9] considered the method of constructing cubature formulas by using tensor products of one-dimensional ones:

$$Q_m^{(n)} f = \sum_{\substack{m-n < |\mathbf{k}|_1 \leq m \\ \mathbf{k} \in \mathbb{Z}_+^n}} \left(\Delta_{k_1}^{(1)} \otimes \Delta_{k_2}^{(1)} \otimes \dots \otimes \Delta_{k_n}^{(1)} \right) f, \quad (1.2)$$

$$\Delta_{k_j}^{(1)} f = \left(Q_{k_j}^{(1)} - Q_{k_j-1}^{(1)} \right) f, \quad \Delta_0^{(1)} f = Q_0^{(1)} f, \quad j = 1, \dots, n,$$

where $|\mathbf{k}|_1 = k_1 + \dots + k_n$, $m = \lfloor \log_2 N \rfloor$, $Q_{k_j}^{(1)} f$ is a one-dimensional cubature formula and $\mathbb{Z}_+^n = \{r = (r_1, \dots, r_n) : r_j = 0, 1, 2, \dots, j = 1, \dots, n\}$.

Smolyak took $Q_k^1 f = \frac{1}{2^k} \sum_{i=0}^{2^k-1} f(\frac{i}{2^k})$ and obtained the following net, which is known now as the Smolyak grid:

$$SG(m) = \left\{ \left(\frac{r_1}{2^{k_1}}, \dots, \frac{r_n}{2^{k_n}} \right) : 0 \leq r_j \leq 2^{k_j} - 1, j = 1, \dots, n, m - n < |\mathbf{k}|_1 \leq m \right\}, \quad (1.3)$$

and proved [9] that for cubature formula with a Smolyak's grid (1.3) the upper bound of the error for Korobov spaces is $O(N^{-\alpha}(\log_2 N)^{(\alpha+1)(n-1)})$.

Using a different approach, E.D. Nursultanov and N.T. Tleukhanova [7] introduced the following cubature formula:

$$Q_m f = \frac{1}{2^m} \sum_{\substack{|\mathbf{k}|_1=m \\ \mathbf{k} \in \mathbb{Z}_+^n}} \sum_{r_1=0}^{2^{k_1}-1} \dots \sum_{r_n=0}^{2^{k_n}-1} (-1)^{\sum_{j=1}^{n-1} (r_j + \text{sgn } k_j)} f \left(\frac{r_1}{2^{k_1}}, \dots, \frac{r_n}{2^{k_n}} \right). \quad (1.4)$$

Cubature formula (1.4) is exact for polynomials with frequencies from the corresponding hyperbolic cross of order 2^m and also gives the error $O(N^{-\alpha}(\log_2 N)^{(\alpha+1)(n-1)})$ for the Korobov spaces E_n^α with $\alpha > 1$.

The purpose of this paper is to present both explicit cubature formulas of type (1.4) and estimates of the worst-case error for the cubature formulas of the form

$$\varepsilon_N(E_n^\alpha, F_m(\cdot; p)) \leq c \frac{(\log_2 N)^\beta}{N^\alpha},$$

where c, β do not depend on N and $F_m(\cdot; p)$ are the new cubature formulas (see below).

2 Main results

Let $p > 1$ be a prime number, $m \in \mathbb{N}$, $f \in C[0, 1]^n$ and f be a periodic function in each variable with period 1. We define the following functional:

$$F_m(f; p) = \frac{1}{p^m} \sum_{\substack{|\mathbf{k}|_1=m \\ \mathbf{k} \in \mathbb{Z}_+^n}} \sum_{r_1=0}^{p^{k_1}-1} \cdots \sum_{r_n=0}^{p^{k_n}-1} \prod_{j=1}^n \frac{\sum_{l=1}^{p-1} e^{-\pi i \left(\frac{2lr_j}{p} + \text{sign } k_j \right)}}{(p-1)^{1-\text{sign } k_j}} f\left(\frac{r_1}{p^{k_1}}, \dots, \frac{r_n}{p^{k_n}}\right). \quad (2.1)$$

Here $\text{sign } k = 1$ if $k \geq 1$ and $\text{sign } k = 0$ otherwise.

Remark 1. If one puts $p = 2$, then formula (2.1) coincides with cubature formula (1.4).

Theorem 1. Let $\alpha > 1$, $n, m \in \mathbb{N}$ and $p > 1$ be a prime number. Then

$$\varepsilon_m(E_n^\alpha, F_m(\cdot; p)) = \sup_{\|f\|_{E_n^\alpha}=1} \left| \int_{[0,1]^n} f(\mathbf{x}) d\mathbf{x} - F_m(f; p) \right| \leq c \frac{m^{n-1}}{p^{\alpha m}}, \quad (2.2)$$

where $c > 0$ depends only on α, p, n .

Since $c_1 m^{n-1} p^m \leq N \leq c_2 m^{n-1} p^m$, where $c_1 > 0$, $c_2 > 0$ are constants that do not depend on m and N , and N is the number of nodes, we can rewrite Theorem 1 in terms of N .

Theorem 2. Let $\alpha > 1$, $n, N \in \mathbb{N}$ and $p > 1$ be a prime number. Then

$$\varepsilon_N(E_n^\alpha, F_{m(N)}(\cdot; p)) \leq c \frac{(\log_2 N)^{(\alpha+1)(n-1)}}{N^\alpha}, \quad (2.2')$$

where $c > 0$ depends only on α, p, n .

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