

RIEMANN–LIOUVILLE INTEGRALS IN BESOV SPACES

E.P. Ushakova

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Abstract. Criteria for the validity of inequalities in weighted smoothness function spaces of Besov type with Riemann–Liouville operators of natural orders on the real axis and semi–axes are found. The obtained estimates are refined under additional conditions on weights.

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1 Introduction

Consider the left and right Riemann–Liouville operators of natural orders α of the forms

$$I_{c+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_c^x (x-y)^{\alpha-1} f(y) dy \quad (x > c) \quad (1.1)$$

and

$$I_{c-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^c (x-y)^{\alpha-1} f(y) dy \quad (x < c) \quad (1.2)$$

(see e.g. [19]). Here $c \in [-\infty, +\infty]$. If $c = \mp\infty$ we denote (1.1) and (1.2) by I_+^{α} and I_-^{α} , respectively.

It is assumed that images $I_{c\pm}^{\alpha} f$ and pre–images f of our operators $I_{c\pm}^{\alpha}$ are identified with density functions of regular distributions (see e.g. [23]) from spaces of Besov type $B_{pq}^{s,w}(\mathbb{R})$ with summation parameters $p > 0$ and $0 < q \leq \infty$, smoothness indices $s \in \mathbb{R}$ and weights w of the Muckenhoupt type $\mathcal{A}_{\infty}^{\text{loc}}$ (see related definitions in Section 2). By a weight w we mean a locally integrable almost everywhere positive function on $\mathbb{R}$. The main problem studied in this paper is characterisation of inequalities connecting the norms of images and pre–images of $I_{c\pm}^{\alpha}$ in $B_{pq}^{s,w}(\mathbb{R})$.

Similar inequalities for some classes of operators acting in Besov type spaces were studied in [25, Theorem 2.3.8], [5, [18, Theorem 2.20], [10, § 4], [27, p. 23], [22]. For $I_{c\pm}^{\alpha}$ the problem was considered in [18, Theorem 2.20], [10, Proposition 4.2], as well as in [30, 34, 31] and [32]. In particular, the following results were obtained in [32].

Theorem 1.1 ([32, Theorem 5.5]). *Let $p > 1$, $0 < q \leq \infty$, $s \in \mathbb{R}$, weights $u, v, w \in \mathcal{A}_{\infty}^{\text{loc}}$, $\alpha \in \mathbb{N}$ and $f \in L^1(\mathbb{R})$.*

(i) *Then $I_{\pm}^{\alpha} f \in B_{pq}^{s,u}(\mathbb{R})$, if $f \in B_{pq}^{s+\kappa^*-\alpha,v}(\mathbb{R})$ with some $\kappa^* \in \mathbb{R}$ such that*

$$\mathbf{C}_{\pm}^{\alpha}(\kappa^*) = \mathbf{M}_{\pm}^{\alpha}(1) + \mathbf{M}_{\pm}^{\alpha}(0) + \sup_{d \in \mathbb{N}} \left[\mathcal{M}_{\pm}^{\alpha}(d, \kappa^*, 1) + \mathcal{M}_{\pm}^{\alpha}(d, \kappa^*, 0) \right] < \infty, \quad (1.3)$$

where

$$\begin{aligned} \mathbf{M}_+^\theta(\varepsilon) &:= \sup_{\tau \in \mathbb{Z}} \left(\sum_{r \geq \tau} \frac{u(Q_{0r})}{(\tau - r + 1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r \leq \tau} \frac{(\tau - r + 1)^{p'(\theta-1)(1-\varepsilon)}}{[v(Q_{0r})]^{p'-1}} \right)^{\frac{1}{p'}}, \\ \mathbf{M}_-^\theta(\varepsilon) &:= \sup_{\tau \in \mathbb{Z}} \left(\sum_{r \leq \tau} \frac{u(Q_{0r})}{(\tau - r + 1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r \geq \tau} \frac{(\tau - r + 1)^{p'(\theta-1)(1-\varepsilon)}}{[v(Q_{0r})]^{p'-1}} \right)^{\frac{1}{p'}}, \\ \mathcal{M}_+^\theta(d, \kappa, \varepsilon) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in \mathbb{Z}} \left(\sum_{r \geq \tau} \frac{u(Q_{(d-1)r})}{(\tau - r + 1)^{-p(2\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r \leq \tau} \frac{(\tau - r + 1)^{p'(2\theta-1)(1-\varepsilon)}}{[v(Q_{(d-1)r})]^{p'-1}} \right)^{\frac{1}{p'}}, \\ \mathcal{M}_-^\theta(d, \kappa, \varepsilon) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in \mathbb{Z}} \left(\sum_{r \leq \tau} \frac{u(Q_{(d-1)r})}{(\tau - r + 1)^{-p(2\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r \geq \tau} \frac{(\tau - r + 1)^{p'(2\theta-1)(1-\varepsilon)}}{[v(Q_{(d-1)r})]^{p'-1}} \right)^{\frac{1}{p'}}. \end{aligned}$$

Here $Q_{d\tau} := [2^{-d}\tau, 2^{-d}(\tau+1)]$ and $h(\Omega) := \int_\Omega h(x) dx$. Moreover,

$$\|I_\pm^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})} \lesssim \mathbf{C}_\pm^\alpha(\kappa^*) \|f\|_{B_{pq}^{s+\kappa^*-\alpha,v}(\mathbb{R})}. \quad (1.4)$$

(ii) If $I_\pm^\alpha f \in B_{pq}^{s,u}(\mathbb{R})$, then $f \in B_{pq}^{s-\kappa_*-\alpha,w}(\mathbb{R})$ with $\kappa_* \in \mathbb{R}$ such that there exist $0 \leq \varepsilon_M, \varepsilon_{\mathbb{M}} \leq 1$, satisfying the condition

$$\mathbb{C}_\pm^\alpha(\kappa_*) = M_\pm^\alpha(\varepsilon_M) + \sup_{d \in \mathbb{N}} \mathbb{M}_\pm^\alpha(d, \kappa_*, \varepsilon_{\mathbb{M}}) < \infty, \quad (1.5)$$

where

$$\begin{aligned} M_+^\theta(\varepsilon) &:= \sup_{\tau \in \mathbb{Z}} \left(\sum_{r=\tau-\theta}^{\tau} \frac{w(Q_{0r})}{(\tau - r + 1)^{p(\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau}^{\tau+\theta} \frac{(\tau - r + 1)^{-p'(\theta+1)(1-\varepsilon)}}{[u(Q_{0r})]^{p'-1}} \right)^{\frac{1}{p'}}, \\ M_-^\theta(\varepsilon) &:= \sup_{\tau \in \mathbb{Z}} \left(\sum_{r=\tau}^{\tau+\theta} \frac{w(Q_{0r})}{(\tau - r + 1)^{p(\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau-\theta}^{\tau} \frac{(\tau - r + 1)^{-p'(\theta+1)(1-\varepsilon)}}{[u(Q_{0r})]^{p'-1}} \right)^{\frac{1}{p'}}, \\ \mathbb{M}_+^\theta(d, \kappa, \varepsilon) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in \mathbb{Z}} \left(\sum_{r=\tau-\theta}^{\tau} \frac{w(Q_{(d-1)r})}{(\tau - r + 1)^{p(2\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau}^{\tau+\theta} \frac{(\tau - r + 1)^{-p'(2\theta+1)(1-\varepsilon)}}{[u(Q_{(d-1)r})]^{p'-1}} \right)^{\frac{1}{p'}}, \\ \mathbb{M}_-^\theta(d, \kappa, \varepsilon) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in \mathbb{Z}} \left(\sum_{r=\tau}^{\tau+\theta} \frac{w(Q_{(d-1)r})}{(\tau - r + 1)^{p(2\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau-\theta}^{\tau} \frac{(\tau - r + 1)^{-p'(2\theta+1)(1-\varepsilon)}}{[u(Q_{(d-1)r})]^{p'-1}} \right)^{\frac{1}{p'}}. \end{aligned}$$

Moreover,

$$\|f\|_{B_{pq}^{s-\kappa_*-\alpha,w}(\mathbb{R})} \lesssim \mathbb{C}_\pm^\alpha(\kappa_*) \|I_\pm^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})}. \quad (1.6)$$

(iii) If $I_\pm^\alpha f \in B_{pq}^{s,u}(\mathbb{R})$, then $f \in B_{pq}^{s-\alpha,u}(\mathbb{R})$, and

$$\|f\|_{B_{pq}^{s-\alpha,u}(\mathbb{R})} \lesssim \|I_\pm^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})}. \quad (1.7)$$

To further list the results in [32], we introduce the following notations:

$$\begin{aligned} \widetilde{\mathbf{C}}_\pm^\alpha(\kappa^*, c) &= \widetilde{\mathbf{M}}_\pm^\alpha(1, c) + \widetilde{\mathbf{M}}_\pm^\alpha(0, c) + \sup_{d \in \mathbb{N}} [\widetilde{\mathcal{M}}_\pm^\alpha(d, \kappa^*, 1, c) + \widetilde{\mathcal{M}}_\pm^\alpha(d, \kappa^*, 0, c)], \\ \widetilde{\mathbb{C}}_\pm^\alpha(\kappa_*, c) &= \widetilde{M}_\pm^\alpha(\varepsilon_{\widetilde{M}}, c) + \sup_{d \in \mathbb{N}} \widetilde{\mathbb{M}}_\pm^\alpha(d, \kappa_*, \varepsilon_{\widetilde{\mathbb{M}}}, c), \end{aligned}$$

where with $Q_{dr}^{(c)} := [2^{-d}(r+c), 2^{-d}(r+c+1)]$ for some $c \in \mathbb{R}$

$$\begin{aligned}
\widetilde{M}_+^\theta(\varepsilon, c) &:= \sup_{\tau \in \mathbb{N}_0} \left(\sum_{r \geq \tau} \frac{u(Q_{0r}^{(c)})}{(r-\tau+1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{0 \leq r \leq \tau} \frac{(\tau-r+1)^{p'(\theta-1)(1-\varepsilon)}}{[v(Q_{0r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{M}_-^\theta(\varepsilon, c) &:= \sup_{\tau \in -\mathbb{N}_0} \left(\sum_{r \leq \tau} \frac{u(Q_{0r}^{(c)})}{(\tau-r+1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{0 \geq r \geq \tau} \frac{(r-\tau+1)^{p'(\theta-1)(1-\varepsilon)}}{[v(Q_{0r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{\mathcal{M}}_+^\theta(d, \kappa, \varepsilon, c) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in \mathbb{N}_0} \left(\sum_{r \geq \tau} \frac{u(Q_{(d-1)r}^{(c)})}{(r-\tau+1)^{-p(2\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{0 \leq r \leq \tau} \frac{(\tau-r+1)^{p'(2\theta-1)(1-\varepsilon)}}{[v(Q_{(d-1)r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{\mathcal{M}}_-^\theta(d, \kappa, \varepsilon, c) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in -\mathbb{N}_0} \left(\sum_{r \leq \tau} \frac{u(Q_{(d-1)r}^{(c)})}{(\tau-r+1)^{-p(2\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{0 \geq r \geq \tau} \frac{(r-\tau+1)^{p'(2\theta-1)(1-\varepsilon)}}{[v(Q_{(d-1)r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{M}_+^\theta(\varepsilon, c) &:= \sup_{\tau \in \mathbb{N}_0} \left(\sum_{r=\tau-\theta}^{\tau} \frac{w(Q_{0r}^{(c)})}{(\tau-r+1)^{p(\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau}^{\tau+\theta} \frac{(r-\tau+1)^{-p'(\theta+1)(1-\varepsilon)}}{[u(Q_{0r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{M}_-^\theta(\varepsilon, c) &:= \sup_{\tau \in -\mathbb{N}_0} \left(\sum_{r=\tau}^{\tau+\theta} \frac{w(Q_{0r}^{(c)})}{(r-\tau+1)^{p(\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau-\theta}^{\tau} \frac{(\tau-r+1)^{-p'(\theta+1)(1-\varepsilon)}}{[u(Q_{0r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{\mathcal{M}}_+^\theta(d, \kappa, \varepsilon, c) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in \mathbb{N}_0} \left(\sum_{r=\tau-\theta}^{\tau} \frac{w(Q_{(d-1)r}^{(c)})}{(\tau-r+1)^{p(2\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau}^{\tau+\theta} \frac{(r-\tau+1)^{-p'(2\theta+1)(1-\varepsilon)}}{[u(Q_{(d-1)r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}, \\
\widetilde{\mathcal{M}}_-^\theta(d, \kappa, \varepsilon, c) &:= \frac{1}{2^{d\kappa}} \sup_{\tau \in -\mathbb{N}_0} \left(\sum_{r=\tau}^{\tau+\theta} \frac{w(Q_{(d-1)r}^{(c)})}{(r-\tau+1)^{p(2\theta+1)\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{r=\tau-\theta}^{\tau} \frac{(\tau-r+1)^{-p'(2\theta+1)(1-\varepsilon)}}{[u(Q_{(d-1)r}^{(c)})]^{p'-1}} \right)^{\frac{1}{p'}}.
\end{aligned}$$

Theorem 1.2 ([32, Theorem 6.1]). *Let $p > 1$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $u, v, w \in \mathcal{A}_\infty^{\text{loc}}$, $\alpha \in \mathbb{N}$ and $f \in L_{\text{loc}}(\mathbb{R})$. Suppose $f \equiv 0$ on $(-\infty, c)$ in the case of I_{c+}^α and, similarly, $f \equiv 0$ on (c, ∞) for I_{c-}^α .*

(i) *Then $I_{c\pm}^\alpha f \in B_{pq}^{s,u}(\mathbb{R})$, if $f \in B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ with some $\kappa^* \in \mathbb{R}$ such that $\widetilde{\mathbf{C}}_\pm^\alpha(\kappa^*, c) < \infty$. Moreover,*

$$\|I_{c\pm}^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})} \lesssim \widetilde{\mathbf{C}}_\pm^\alpha(\kappa^*, c) \|f\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})}. \quad (1.8)$$

(ii) *If $I_{c\pm}^\alpha f \in B_{pq}^{s,u}(\mathbb{R})$, then $f \in B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})$ with such $\kappa_* \in \mathbb{R}$ that with some $0 \leq \widetilde{\varepsilon}_M, \widetilde{\varepsilon}_\mathbb{M} \leq 1$ $\widetilde{\mathbf{C}}_\pm^\alpha(\kappa_*, a) < \infty$. Moreover,*

$$\|f\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})} \lesssim \widetilde{\mathbf{C}}_\pm^\alpha(\kappa_*, a) \|I_{c\pm}^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})}. \quad (1.9)$$

(iii) *If $I_{c\pm}^\alpha f \in B_{pq}^{s,u}(\mathbb{R})$, then $f \in B_{pq}^{s-\alpha, u}(\mathbb{R})$, and (1.7) is true with $I_{c\pm}^\alpha$ instead of $I_\pm^\alpha$.*

In Theorems 1.1 or 1.2, sufficient conditions of form (1.3) (or $\widetilde{\mathbf{C}}_\pm^\alpha(\kappa^*, c) < \infty$) and (1.5) (or $\widetilde{\mathbf{C}}_\pm^\alpha(\kappa_*, a) < \infty$) for the validity of inequalities (1.4) (or (1.8)) and (1.6) (or (1.9)), respectively, are obtained. In this paper, we improve these results up to criteria (see Theorems 3.4 and 3.5) by suitably modifying the second component (the mother wavelet) of the spline wavelet systems used for deriving upper bounds in the decomposition theorems (see Theorem 3.3).

We write $h(\Omega) := \int_\Omega h(x) dx$, where $\Omega \subset \mathbb{R}$ is a bounded measurable set. Symbol $\mathbb{R}$ denotes the real axis, $\mathbb{N}$ and $\mathbb{Z}$ — the sets of all natural and integer numbers, respectively. Notation $\mathbb{N}_0$ is used for the set $\mathbb{N} \cup \{0\}$, $[s]$ — for the integer part of $s \in \mathbb{R}$. Relation $A \lesssim B$ means that $A \leq cB$ with some constant $c > 0$, depending only on fixed numerical parameters, for any A, B under consideration. Similarly, $A \simeq B$ is used instead of $A = cB$. We write $A \approx B$ in the case of $A \lesssim B \lesssim A$. For $r > 1$, the conjugate (or dual) exponent r' is defined as $r/(r-1)$, with $r' = \infty$ if $r = 1$. The symbols $:=$ and $=:$ are used to denote new quantities. By $L_{\text{loc}}(\mathbb{R})$ we denote the collection of all locally integrable functions on $\mathbb{R}$. The notation $L^p(\mathbb{R})$, $0 < p \leq \infty$, stands for the Lebesgue spaces of all measurable functions on $\mathbb{R}$ with the (quasi-)norm $\|f\|_{L^p(\mathbb{R})} := (\int_\mathbb{R} |f(x)|^p dx)^{1/p}$ with standard modification for the case $p = \infty$.

2 Weighted Besov type spaces

2.1 Weights of the Muckenhoupt class A_∞

Let Q be a segment in $\mathbb{R}$ and $|Q|$ be its length. Denote

$$A_\rho[w(Q)] := \frac{w(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q w^{1-\rho'} \right)^{\frac{\rho}{\rho'}}, \quad A_1[w(Q)] := \frac{w(Q)}{|Q|} \|1/w\|_{L^\infty(Q)}. \quad (2.1)$$

Definition 1. (i) A weight w belongs to the *Muckenhoupt class* A_ρ , $1 < \rho < \infty$, if

$$A_\rho(w) := \sup_{Q \subset \mathbb{R}} A_\rho[w(Q)] < \infty;$$

(ii) $w \in A_1$ if $A_1(w) := \sup_{Q \subset \mathbb{R}} A_1[w(Q)] < \infty$;

(iii) the *Muckenhoupt class* A_∞ is $\bigcup_{\rho \geq 1} A_\rho$.

Below, we mention some properties of weights from the Muckenhoupt class [13, 24]:

- if $w \in A_\rho$, $1 < \rho < \infty$, then $w^{-\rho'/\rho} \in A_{\rho'}$;
- if $1 \leq \rho_1 < \rho_2 \leq \infty$, then $A_{\rho_1} \subset A_{\rho_2}$;
- if $w \in A_\rho$, $\rho > 1$, then there exists $r < \rho$ such that $w \in A_r$.

The last property implies the existence of the number

$$r_0 := r_0(w) := \inf\{r \geq 1 : w \in A_r\} < \infty, \quad w \in A_\infty.$$

Functions $w \in A_\infty$ satisfy two important generalised doubling-type conditions [6].

(I) For $w \in A_\rho$, $\rho \geq 1$, there exists a constant $c_I > 0$ such that for all segments B and F satisfying $F \subset B$, the following inequality holds:

$$\left(\frac{|F|}{|B|} \right)^\rho \leq c_I \frac{w(F)}{w(B)}. \quad (2.2)$$

(II) If $w \in A_\infty$, then there are $\rho^* > 0$ and $c_{II} > 0$ such that for all B and E , such that $E \subset B$,

$$\frac{w(E)}{w(B)} \leq c_{II} \left(\frac{|E|}{|B|} \right)^{\rho^*}. \quad (2.3)$$

For a more detailed study of the class A_∞ , one can refer to [24, Chapter V] or to [9, Lemma 1.3], [8, Lemma 2.3] and [9, Lemma 1.4] (and the literature therein).

2.2 Weights of the local Muckenhoupt class A_∞^{loc}

The next class of weights covers admissible [7], locally regular [12, 20, 21] as well as A_∞ -weights.

Definition 2. (i) A weight w is of *local Muckenhoupt type* A_ρ^{loc} , $1 < \rho < \infty$, if (see. (2.1))

$$A_\rho^{\text{loc}}(w) := \sup_{|Q| \leq 1} A_\rho[w(Q)] < \infty;$$

(ii) $w \in A_1^{\text{loc}}$ if $A_1^{\text{loc}}(w) := \sup_{|Q| \leq 1} A_1[w(Q)] < \infty$;

(iii) we say that $w \in A_\infty^{\text{loc}}$ if $w \in A_\rho^{\text{loc}}$ for some $1 \leq \rho < \infty$, that is

$$A_\infty^{\text{loc}} := \bigcup_{\rho \geq 1} A_\rho^{\text{loc}}.$$

The weights of the class A_∞^{loc} mostly have the properties that coincide with the properties of the weights of the class A_∞ [18]. In particular, by analogy with r_0 in the case of $w \in A_\infty$, there exists a number

$$r_w := \inf\{r \geq 1 : w \in A_r^{\text{loc}}\} < \infty, \quad w \in A_\infty^{\text{loc}}. \quad (2.4)$$

The doubling type property for $w \in A_\infty^{\text{loc}}$ can be formulated as follows: if $w \in A_\rho^{\text{loc}}$, $1 < \rho \leq \infty$, then there exists a constant $c_w > 0$ such that the estimate

$$w(Q_t) \leq \exp(c_w t/s) w(Q_s)$$

holds for all Q_s with $|Q_s| = s$, $0 < s \leq 1$ and Q_t such that $Q_s \subset Q_t$, where $|Q_t| = t$, $t \geq 1$.

Let us present the *basic property* of A_ρ^{loc} -weights with respect to A_ρ -functions: any $w \in A_\rho^{\text{loc}}$ can be continued outside a given cube Q in such a way that a new extended weight $\bar{w}$ belongs to A_ρ .

Lemma 2.1 ([18, Lemma 1.1]). *Let $1 \leq \rho < \infty$, $w \in A_\rho^{\text{loc}}$ and Q be a cube with $|Q| = 1$. Then there is $\bar{w} \in A_\rho$ such that $\bar{w} = w$ on Q and with a constant c independent of Q the following holds:*

$$A_\rho(\bar{w}) \leq c A_\rho^{\text{loc}}(w).$$

More information about local Muckenhoupt weights can be found in [18, 12, 35, 36].

2.3 Weighted Besov type spaces

Let $p > 0$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Definitions of unweighted Besov type spaces $B_{pq}^s(\mathbb{R})$ are given in [25, 26]. For a given $w \neq 1$, their weighted generalisations can be specified in several ways depending on a class of which the function w belongs [9, 17, 20, 18].

The space $\mathcal{S}'(\mathbb{R})$, which is topologically dual to the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions, is used in unweighted case, as well as in situations when w belongs to the Muckenhoupt class A_∞ . For A_∞^{loc} -weights, a more general space of distributions $\mathcal{S}'_e(\mathbb{R})$ is required than $\mathcal{S}'(\mathbb{R})$.

Definition 3. The space $\mathcal{S}_e(\mathbb{R})$ consists of all functions $\phi \in C^\infty(\mathbb{R})$ such that

$$q_N(\phi) := \sup_{x \in \mathbb{R}} e^{N|x|} \sum_{0 \leq \gamma \leq N} |D^\gamma \phi(x)| < \infty \quad \text{for any } N \in \mathbb{N}_0.$$

The set $\mathcal{S}_e(\mathbb{R})$ is equipped with a locally convex topology defined by the system of seminorms q_N . More details on the properties of $\mathcal{S}_e(\mathbb{R})$ can be found in [20, 18] and [12, § 3].

By $\mathcal{S}'_e(\mathbb{R})$ we denote the space of all linear continuous functionals on $\mathcal{S}_e(\mathbb{R})$. The class $\mathcal{S}'_e(\mathbb{R})$ is sometimes conveniently identified with the subset in $\mathcal{D}'(\mathbb{R})$ of all distributions f on $\mathcal{D}(\mathbb{R})$ for which the estimate

$$|\langle f, \phi \rangle| \leq C \sup \left\{ |D^\gamma \phi(x)| e^{N|x|} : x \in \mathbb{R}, 0 \leq \gamma \leq N \right\}$$

holds for any $\phi \in \mathcal{D}(\mathbb{R})$ with some constants C and N depending on f . Each such f can be extended to a continuous functional on $\mathcal{S}_e(\mathbb{R})$.

For $\phi, \psi \in \mathcal{S}_e(\mathbb{R})$, the convolution $\phi * \psi$ belongs to $\mathcal{S}_e(\mathbb{R})$. If $f \in \mathcal{S}'_e(\mathbb{R})$ and $\phi \in \mathcal{S}_e(\mathbb{R})$, then the convolution

$$f * \phi(x) =: \langle f(\cdot), \phi(x - \cdot) \rangle, \quad x \in \mathbb{R},$$

is a function in $C^\infty(\mathbb{R})$ of at most exponential growth.

To define Besov spaces with Muckenhoupt weights of local type, we take $\phi_0 \in \mathcal{D}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \phi_0(x) dx \neq 0, \quad (2.5)$$

further, we set

$$\phi(x) = \phi_0(x) - 2^{-1} \phi_0(x/2) \quad (2.6)$$

and define $\phi_d(x) := 2^{(d-1)} \phi(2^{d-1}x)$ for $d \in \mathbb{N}$. One can choose ϕ_0 satisfying

$$\int_{\mathbb{R}} x^\gamma \phi(x) dx = 0 \quad (2.7)$$

for all $\gamma \in \mathbb{N}_0$, $\gamma \leq \Gamma$, with $\Gamma \in \mathbb{N}$. We write $\Gamma = -1$ if (2.7) does not hold.

Definition 4. Let (2.5) hold for some $\phi_0 \in \mathcal{D}(\mathbb{R})$, and let ϕ of form (2.6) satisfy (2.7) with $0 \leq \gamma \leq \Gamma$, where $\Gamma \geq [s]$. For $w \in A_\infty^{\text{loc}}$, we define the weighted Besov type space $B_{pq}^{s,w}(\mathbb{R})$ of distributions as the set of all $f \in \mathcal{S}'_e(\mathbb{R})$ such that the (quasi-)norm

$$\|f\|_{B_{pq}^{s,w}(\mathbb{R})} := \left(\sum_{d=0}^{\infty} 2^{dsq} \|\phi_d * f\|_{L^p(\mathbb{R},w)}^q \right)^{\frac{1}{q}}$$

(with standard modification for $q = \infty$) is finite. Here for $p > 0$ and a weight w the symbol $L^p(\mathbb{R}, w)$ denotes the weighted Lebesgue space with the norm $\|f\|_{L^p(\mathbb{R},w)} := \|w^{1/p} f\|_{L^p(\mathbb{R})}$.

The definition of $B_{pq}^{s,w}(\mathbb{R})$ does not depend on the choice of the function ϕ_0 . It generalises the notion of Besov type smoothness spaces with $w \in A_\infty$ [18, § 2.2 and Remark 2.23].

The spaces $B_{pq}^{s,w}(\mathbb{R})$ are similar in their properties to the unweighted spaces $B_{pq}^s(\mathbb{R})$, as well as to the classes $B_{pq}^s(\mathbb{R}, w)$ with $w \in A_\infty$ (for more details, see [10, 12, 18, 35, 36]).

Next, for $\tau \in \mathbb{Z}$ and $d \in \mathbb{N}_0$ we define the dyadic segments $Q_{d\tau} := [2^{-d}\tau, 2^{-d}(\tau+1)]$ with the origin at the point $x_{Q_{d\tau}} := 2^{-d}\tau$. In what follows, we will also need the Besov type spaces $b_{pq}^s(w)$ of sequences $\lambda = \{\lambda_{d\tau}\}_{d \in \mathbb{N}_0, \tau \in \mathbb{Z}}$, $\lambda_{d\tau} \in \mathbb{C}$, such that

$$\|\lambda\|_{b_{pq}^s(w)} := \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{0\tau}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R},w)} + \left(\sum_{d \in \mathbb{N}_0} 2^{qds} \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{d\tau}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R},w)}^q \right)^{1/q} < \infty.$$

In the definition of $\|\lambda\|_{b_{pq}^s(w)}$, we can take $Q_{(d-1)(\tau \pm 1/2)}$ instead of $Q_{(d-1)\tau}$ in the second term. Due to property (2.2) of the Muckenhoupt weights (see also Lemma 2.1), this is possible in this situation.

3 Criteria for the validity of the norm inequalities

The results obtained in this section are based on decomposition theorems which relate elements f of the Besov spaces $B_{pq}^{s,w}(\mathbb{R})$ with the sequential spaces $b_{pq}^s(w)$, as well as their corresponding norms. To prove them, we decompose f with respect to spline wavelet systems of natural orders.

3.1 Decompositions in terms of spline wavelets

Characterization of the spaces $B_{pq}^{s,w}(\mathbb{R})$ in terms of spline wavelets of natural orders were given in [29, 32] (see also [28]). In particular, it was established that

$$\|f\|_{B_{pq}^{s,w}(\mathbb{R})} \approx \|\tilde{\lambda}^{(\mathbb{k},0)}\|_{b_{pq}^s(w)}, \quad (3.1)$$

where elements

$$\tilde{\lambda}_{0\tau} = \langle f, \tilde{\Phi}_\tau \rangle \quad (\tau \in \mathbb{Z}), \quad \tilde{\lambda}_{d\tau}^{(\mathbb{k},0)} = 2^{d/2} \langle f, \tilde{\Psi}_{(d-1)\tau}^{(\mathbb{k},0)} \rangle \quad (d \in \mathbb{N}, \tau \in \mathbb{Z}) \quad (3.2)$$

of the sequences $\tilde{\lambda}^{(\mathbb{k},0)} = \{\tilde{\lambda}_{0,\tau}\}_{\tau \in \mathbb{Z}} \cup \{\tilde{\lambda}_{d\tau}^{(\mathbb{k},0)}\}_{d \in \mathbb{N}, \tau \in \mathbb{Z}}$ with $\mathbb{k} \in \{0, 1\}$ are defined by $\tilde{\Phi}_\tau$ and $\tilde{\Psi}_{(d-1)\tau}^{(\mathbb{k},0)}$ of form (5.15) with possible "starting" points $\mathbf{a} \in \{0, \pm 1/2\}$. In [29], the case $\mathbf{a} = 0$ was considered, while in [28] — a simpler situation $\mathbf{a} = \mathbb{k} = 0$.

Result (3.1) allows a generalisation (see Theorem 3.3). Let us start, however, with the basic case $\zeta = \mathbb{k} = 0$. For this, taking into account (2.4), we denote

$$\sigma_p(w) := r_w / \min\{p, r_w\} - 2 + r_w.$$

We will assume that $\mathbf{s} = 0$ in the definitions of $\Psi^{(\mathbf{a})} \in (5.14)$ and $\tilde{\Psi}^{(\mathbb{k},\zeta)} \in (5.13)$, $\mathbb{k}, \zeta \in \{0, 1\}$.

Theorem 3.1 ([32, 29]). Let $p > 0$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in A_\infty^{\text{loc}}$. Let $\Phi^{(a)}$ and $\Psi^{(a)}$, where $a \in \{0, \pm 1/2\}$, be functions satisfying (5.14) with some $n \in \mathbb{N}$ such that

$$n \geq \max\left\{0, [s] + 1, [(r_w - 1)/p - s] + 1, [\sigma_p(w) - s]\right\} + 1. \quad (3.3)$$

Then $f \in \mathcal{S}'_e(\mathbb{R})$ belongs to $B_{pq}^{s,w}(\mathbb{R})$ if and only if the following representation holds:

$$f = \sum_{\tau \in \mathbb{Z}} \lambda_{0\tau}^{(a)} \Phi_\tau^{(a)} + \sum_{d \in \mathbb{N}} \sum_{\tau \in \mathbb{Z}} \lambda_{d\tau}^{(a)} 2^{-d/2} \Psi_{(d-1)\tau}^{(a)}, \quad (3.4)$$

where $\{\lambda_{d\tau}^{(a)}\}_{d \in \mathbb{N}_0, \tau \in \mathbb{Z}} =: \lambda^{(a)} \in b_{pq}^s(w)$ and the series converges in $\mathcal{S}'_e(\mathbb{R})$. Representation (3.4) is unique with

$$\lambda_{0\tau}^{(a)} = \langle f, \Phi_\tau^{(a)} \rangle \quad (\tau \in \mathbb{Z}), \quad \lambda_{d\tau}^{(a)} = 2^{d/2} \langle f, \Psi_{(d-1)\tau}^{(a)} \rangle \quad (d \in \mathbb{N}, \tau \in \mathbb{Z}) \quad (3.5)$$

and $I: f \mapsto \lambda^{(a)}$ is a linear isomorphism of $B_{pq}^{s,w}(\mathbb{R})$ onto $b_{pq}^s(w)$. Moreover,

$$\|f\|_{B_{pq}^{s,w}(\mathbb{R})} \approx \|\lambda^{(a)}\|_{b_{pq}^s(w)}. \quad (3.6)$$

Below there is a formulation of an analogue of Theorem 3.1, which uses a dictionary of functions $\tilde{\Phi}$ and $\tilde{\Psi}^{(k,0)}$ of form (5.13) with $k = 1$ and $\zeta = 0$. It also asserts (3.1) (see [32, Theorem 4.8]), but with different representations for f in its sufficiency and necessity parts.

Theorem 3.2. Let $p > 0$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in A_\infty^{\text{loc}}$. Suppose that for some $n \in \mathbb{N}$ condition (3.3) is satisfied, and there are functions $\Phi^{(a)}$ and $\Psi^{(a)}$ with $a \in \{0, \pm 1/2\}$, satisfying (5.14), as well as functions $\tilde{\Phi}$ and $\tilde{\Psi}^{(1,0)}$ of form (5.13), both the pairs are of smoothness n .

(i) A distribution $f \in \mathcal{S}'_e(\mathbb{R})$ belongs to $B_{pq}^{s,w}(\mathbb{R})$, if there is a representation

$$f = \sum_{\tau \in \mathbb{Z}} \tilde{\lambda}_{0\tau} \tilde{\Phi}_\tau + \sum_{d \in \mathbb{N}} \sum_{\tau \in \mathbb{Z}} \tilde{\lambda}_{d\tau}^{(1,0)} 2^{-d/2} \tilde{\Psi}_{(d-1)\tau}^{(1,0)}, \quad (3.7)$$

where $\tilde{\lambda}^{(1,0)}$ of form (3.2) is from $b_{pq}^s(w)$ and series (3.7) converges in $\mathcal{S}'_e(\mathbb{R})$. Moreover,

$$\|f\|_{B_{pq}^{s,w}(\mathbb{R})} \lesssim \|\tilde{\lambda}^{(1,0)}\|_{b_{pq}^s(w)} \lesssim \sum_{a=0, \pm 1/2} \|\lambda^{(a)}\|_{b_{pq}^s(w)}$$

with $\lambda^{(a)}$ of form (3.5).

(ii) Conversely, if $f \in \mathcal{S}'_e(\mathbb{R})$ belongs to $B_{pq}^{s,w}(\mathbb{R})$, then for $a \in \{0, \pm 1/2\}$ we have (3.4) in the sense of convergence in $\mathcal{S}'_e(\mathbb{R})$, where $\lambda^{(a)} \in b_{pq}^s(w)$ is given by (3.5). Moreover,

$$\|\tilde{\lambda}^{(1,0)}\|_{b_{pq}^s(w)} \leq \sum_{a=0, 1/2} \|\lambda^{(a)}\|_{b_{pq}^s(w)} \lesssim \|f\|_{B_{pq}^{s,w}(\mathbb{R})}.$$

A generalisation of Theorem 3.2 in terms of $\tilde{\Phi}$ and $\tilde{\Psi}^{(1,1)}$ of form (5.13) is given below.

Theorem 3.3. Let $p > 0$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in A_\infty^{\text{loc}}$. Suppose that for some $n \in \mathbb{N}$ condition (3.3) is satisfied, and there are functions $\Phi^{(a)}$ and $\Psi^{(a)}$ with $a \in \{0, \pm 1/2\}$, satisfying (5.14), as well as functions $\tilde{\Phi}$ and $\tilde{\Psi}^{(1,1)}$ of form (5.13), the both pairs are of smoothness n .

(i) A distribution $f \in \mathcal{S}'_e(\mathbb{R})$ belongs to $B_{pq}^{s,w}(\mathbb{R})$ if there is a representation

$$f = \sum_{\tau \in \mathbb{Z}} \tilde{\lambda}_{0\tau} \tilde{\Phi}_\tau + \sum_{d \in \mathbb{N}} \sum_{\tau \in \mathbb{Z}} \tilde{\lambda}_{d\tau}^{(1,1)} 2^{-d/2} \tilde{\Psi}_{(d-1)\tau}^{(1,1)},$$

where the sequence $\tilde{\lambda}^{(1,1)} = \{\tilde{\lambda}_{0,\tau}\}_{\tau \in \mathbb{Z}} \cup \{\tilde{\lambda}_{d\tau}^{(1,1)}\}_{d \in \mathbb{N}, \tau \in \mathbb{Z}}$ of the form

$$\tilde{\lambda}_{0\tau} = \langle f, \tilde{\Phi}_\tau \rangle \quad (\tau \in \mathbb{Z}), \quad \tilde{\lambda}_{d\tau}^{(1,1)} = 2^{d/2} \langle f, \tilde{\Psi}_{(d-1)\tau}^{(1,1)} \rangle \quad (d \in \mathbb{N}, \tau \in \mathbb{Z}) \quad (3.8)$$

is from $b_{pq}^s(w)$ with $\tilde{\Phi}_\tau$ and $\tilde{\Psi}_{(d-1)\tau}^{(1,1)}$ given by (5.15) with $\mathbf{a} = \{0, \pm 1/2\}$. In this case, series (3.7) converges in $\mathcal{S}'_e(\mathbb{R})$. In addition, with $\lambda^{(a)}$ of form (3.5), we have

$$\|f\|_{B_{pq}^{s,w}(\mathbb{R})} \lesssim \|\tilde{\lambda}^{(1,1)}\|_{b_{pq}^s(w)} \lesssim \sum_{\mathbf{a}=0, \pm 1/2} \|\lambda^{(a)}\|_{b_{pq}^s(w)}. \quad (3.9)$$

(ii) Conversely, if $f \in \mathcal{S}'_e(\mathbb{R})$ belongs to $B_{pq}^{s,w}(\mathbb{R})$, then representation (3.4) holds for $\mathbf{a} \in \{0, \pm 1/2\}$ in the sense of convergence in $\mathcal{S}'_e(\mathbb{R})$, where $\lambda^{(a)} \in b_{pq}^s(w)$ is given by (3.5). Moreover,

$$\|\tilde{\lambda}^{(1,1)}\|_{b_{pq}^s(w)} \leq \sum_{\mathbf{a}=0, \pm 1/2} \|\lambda^{(a)}\|_{b_{pq}^s(w)} \lesssim \|f\|_{B_{pq}^{s,w}(\mathbb{R})}.$$

Observe that in the original decomposition theorem using spline wavelet systems of natural orders (see [29, § 4.3]) the case $\mathbf{a} = 0$ was considered. Later in [32, Theorem 4.8] the cases $\mathbf{a} = \pm 1/2$ were added, they are justified by property (2.2) of the Muckenhoupt weights and Lemma 2.1. The proof of Theorem 3.3 is similar to the proof of Theorem 3.2 (see [32, Theorem 4.8], as well as [29, Theorem 4.3]). It is based on the fact that the elements of the sequence $\tilde{\lambda}^{(1,1)}$ form a family of atoms [35, 10] (see also [29, Definition 4.1]).

3.2 Main result

Let us introduce into consideration the functional

$$\mathfrak{M}(d, \kappa; \sigma_1, \sigma_2) := \sup_{\tau \in \mathbb{Z}} \begin{cases} [\sigma_1(Q_{0\tau})]^{-\frac{1}{p}} [\sigma_2(Q_{0\tau})]^{\frac{1}{p}}, & d = 0, \\ 2^{-d\kappa} [\sigma_1(Q_{(d-1)\tau})]^{-\frac{1}{p}} [\sigma_2(Q_{(d-1)\tau})]^{\frac{1}{p}}, & d \in \mathbb{N}. \end{cases}$$

Theorem 3.4. Let $p > 1$, $0 < q \leq \infty$, $s \in \mathbb{R}$, weights $u, v, w \in \mathcal{A}_\infty^{\text{loc}}$ and $\alpha \in \mathbb{N}$. Suppose that $f \in L^1(-\infty, x)$ for any $x \in \mathbb{R}$ in the case of I_+^α and $f \in L^1(x, +\infty)$ for all $x \in \mathbb{R}$ if we consider I_-^α .

(i) The operator $I_\pm^\alpha$ is bounded from $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ to $B_{pq}^{s, u}(\mathbb{R})$ with some $\kappa^* \in \mathbb{R}$ if and only if

$$\mathfrak{C}_\pm^\alpha(\kappa^*) = \mathbf{M}_\pm^\alpha(1) + \mathbf{M}_\pm^\alpha(0) + \sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u) < \infty. \quad (3.10)$$

Moreover, the norm $\|I_\pm^\alpha\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R}) \rightarrow B_{pq}^{s, u}(\mathbb{R})}$ of the operator $I_\pm^\alpha: B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R}) \rightarrow B_{pq}^{s, u}(\mathbb{R})$ is equivalent to $\mathfrak{C}_\pm^\alpha(\kappa^*)$ with constants depending only on the numerical parameters.

(ii) Let $\kappa_* \in \mathbb{R}$ and suppose that $0 < C$ is the best constant in the inequality

$$\|f\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})} \leq C \|I_\pm^\alpha f\|_{B_{pq}^{s, u}(\mathbb{R})} \quad (3.11)$$

independent of f , where $I_\pm^\alpha f \in B_{pq}^{s, u}(\mathbb{R})$. Inequality (3.11) holds if and only if

$$C(\kappa_*) := \sup_{d \in \mathbb{N}_0} \mathfrak{M}(d, \kappa_*; u, w) < \infty,$$

and $C \approx C(\kappa_*)$. In particular, if $w = u$, then $C \approx C(0) = 1$.

Proof. (i) *Sufficiency.* Let us turn to the beginning of the proof of assertion (i) from [32, Theorem 5.5] in the sufficiency part and show the validity of the inequality

$$\|I_+^\alpha\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R}) \rightarrow B_{pq}^{s, u}(\mathbb{R})} \lesssim \mathfrak{C}_+^\alpha(\kappa^*).$$

To do this, we choose $n^* \in \mathbb{N}$ satisfying condition (3.3) with respect to parameters of the space $B_{pq}^{s, u}(\mathbb{R})$ including r_u (see (2.4)). Moreover, we set $m^* := n^* + \alpha$, which must also satisfy (3.3) with respect to parameters of the space $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ and r_v . Then, by (3.9) (see Theorem 3.3(i)),

$$\|I_+^\alpha f\|_{B_{pq}^{s, u}(\mathbb{R})} \approx \|\tilde{\lambda}^{(1,1)}\|_{b_{pq}^s(u)},$$

where $\tilde{\lambda}^{(1,1)}$ is defined by (3.8) with $I_+^\alpha f$ instead of f . Moreover, $\tilde{\Phi}$ is given by B_{n*} , and $\tilde{\Psi}^{(1,1)}$ (see (5.13)) is specified by $\Psi_{n*,0,0;m^*(1),\alpha(1)}$. This means that we can write:

$$\|I_+^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})} \approx \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{0\tau}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R},u)} + \left(\sum_{d \in \mathbb{N}} 2^{qds} \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{d\tau}^{(1,1)}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R},u)}^q \right)^{\frac{1}{q}}, \quad (3.12)$$

where

$$\tilde{\lambda}_{0\tau} = \langle I_+^\alpha f, \tilde{\Phi}_\tau \rangle \quad (\tau \in \mathbb{Z}), \quad \tilde{\lambda}_{d\tau}^{(1,1)} = 2^{d/2} \langle I_+^\alpha f, \tilde{\Psi}_{(d-1)\tau}^{(1,1)} \rangle \quad (d \in \mathbb{N}, \tau \in \mathbb{Z}).$$

The first term on the right-hand side of (3.12) can be estimated by repeating a similar stage of the proof of assertion (i) from [32, Theorem 5.5]. We present it here in a slightly modified form, by relying on differential property (5.1) of basic splines B_n .

As in the proof of Theorem 5.5 of [32], we denote $\tilde{\Phi}_\tau(\cdot) = B_{n*}(\cdot - \tau) =: P(\tau)$ and set

$$R(\tau - r) := \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - r - j).$$

Note that

$$P(\tau - r) = R(\tau - r) - \sum_{j=1}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - r - j). \quad (3.13)$$

Then,

$$P(\tau) = R(\tau) - \sum_{j=1}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j) = R(\tau) + \binom{\alpha}{1} P(\tau - 1) - \sum_{j=2}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j).$$

Moreover, by (3.13)

$$\begin{aligned} P(\tau) - R(\tau) &= \binom{\alpha}{1} \left[R(\tau - 1) - \sum_{j=1}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 1 - j) \right] - \sum_{j=2}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j) \\ &= \binom{\alpha}{1} R(\tau - 1) - \binom{\alpha}{1} \sum_{j=1}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 1 - j) - \sum_{j=2}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j) = \binom{\alpha}{1} R(\tau - 1) \\ &\quad + \left[\binom{\alpha}{1}^2 - \binom{\alpha}{2} \right] P(\tau - 2) - \binom{\alpha}{1} \sum_{j=2}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 1 - j) - \sum_{j=3}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j). \end{aligned}$$

Setting $A_0 := \binom{\alpha}{0} = 1$, $A_1 := A_0 \binom{\alpha}{1}$, $A_2 := A_1 \binom{\alpha}{1} - A_0 \binom{\alpha}{2} = \binom{\alpha}{1}^2 - \binom{\alpha}{2}$, we continue the procedure for forming finite iterated differences as shown below

$$\begin{aligned} P(\tau) &= A_0 R(\tau) + A_1 R(\tau - 1) + A_2 \left[R(\tau - 2) - \sum_{j=1}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 2 - j) \right] \\ &\quad - \binom{\alpha}{1} \sum_{j=2}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 1 - j) - \sum_{j=3}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j) \\ &= A_0 R(\tau) + A_1 R(\tau - 1) + A_2 R(\tau - 2) - \underbrace{\left[A_2 \binom{\alpha}{1} - A_1 \binom{\alpha}{2} + A_0 \binom{\alpha}{3} \right]}_{:=A_3} P(\tau - 3) \\ &\quad - A_2 \sum_{j=2}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 2 - j) - \binom{\alpha}{1} \sum_{j=3}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - 1 - j) - \sum_{j=4}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau - j), \end{aligned}$$

which results in obtaining the sum of the form

$$P(\tau) = \sum_{r \geq 0} A_r R(\tau - r), \quad \text{where} \quad A_r = A_r(\alpha) = \sum_{j=1}^r (-1)^{j-1} A_{r-j} \binom{\alpha}{j}.$$

In [32], it was established that $A_r(\alpha) \leq (1+r)^{\alpha-1}$ (see the proof of (5.18) in [32]).

A similar procedure of forming differences can be done by "moving to the right-hand side" and denoting

$$R^*(\tau + r) := \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} P(\tau + r + j).$$

Then,

$$P(\tau) = \sum_{r \geq 0} A_r R^*(\tau + r), \quad \text{where} \quad R(\tau + r)^* = B_{m^*}^{(\alpha)}(\cdot - \tau - r).$$

Since $f \in L^1(-\infty, x)$ for any $x \in \mathbb{R}$, also

$$\text{supp } B_{m^*}^{(\alpha)}(\cdot - \tau + r + \alpha) = \text{supp } B_{m^*, \tau-r-\alpha}^{(\alpha)}(\cdot) = [\tau - r - \alpha, \tau - r + n^* + 1]$$

and $B_{m^*, \tau-r-\alpha}^{(k)} = 0$ at the points $\tau - r - \alpha$ and $\tau - r + n^* + 1$ for all $k = 0, \dots, \alpha$ (see §5), we have

$$\int_{\mathbb{R}} I_+^{\alpha} f(x) B_{m^*}^{(\alpha)}(x - \tau - r) dx = (-1)^{\alpha} \int_{\mathbb{R}} f(y) B_{m^*}(y - \tau + r + \alpha) dy. \quad (3.14)$$

Hence, taking into account the equality $R(\tau - r) = (-1)^{\alpha} B_{m^*}^{(\alpha)}(\cdot - \tau + r + \alpha)$ (see (5.1)), the first term in (3.12) can be estimated as follows:

$$\begin{aligned} \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{0\tau}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, u)}^p &= \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) |\langle I_+^{\alpha} f, \tilde{\Phi}_{\tau} \rangle|^p \\ &= \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left[\left| \sum_{r \geq 0} A_r \int_{\mathbb{R}} I_+^{\alpha} f(x) B_{m^*}^{(\alpha)}(x - \tau + r + \alpha) dx \right| \right]^p \\ &= \Gamma(\alpha) \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left[\left| \sum_{r \geq 0} A_r \int_{\mathbb{R}} f(y) B_{m^*}(y - \tau + r + \alpha) dy \right| \right]^p \\ &\lesssim \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left[\sum_{r \geq 0} (r+1)^{\alpha-1} \left| \int_{\mathbb{R}} f(y) B_{m^*}(y - \tau + r + \alpha) dy \right| \right]^p \\ &= \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left[\sum_{l \leq \tau} (\tau - l + 1)^{\alpha-1} \left| \int_{\mathbb{R}} f(y) B_{m^*}(y + \alpha - l) dy \right| \right]^p. \end{aligned}$$

Further, by virtue of (3.10),

$$\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) |\langle I_+^{\alpha} f, \tilde{\Phi}_{\tau} \rangle|^p \lesssim [\mathbf{M}_+^{\alpha}(1) + \mathbf{M}_+^{\alpha}(0)]^p \sum_{\tau \in \mathbb{Z}} v(Q_{0\tau}) \left| \int_{\mathbb{R}} f(y) B_{m^*}(y + \alpha - \tau) dy \right|^p \quad (3.15)$$

(see [32, Theorem 5.1] or [16, Theorems 1.6 & 1.11]). Using (2.2) and relying on Lemma 2.1, we derive an estimate of the form

$$v(Q_{0\tau}) = \int_{Q_{0\tau}} v \leq \int_{\cup_{i=\tau-\alpha}^{\tau} Q_{0i}} v \leq \alpha^{r_v} \int_{Q_{0(\tau-\alpha)}} v.$$

Then, by substituting $r = \tau - \alpha$, we can transform (3.15) to

$$\begin{aligned} \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{0\tau}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, u)} &= \left(\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) |\langle I_+^{\alpha} f, \tilde{\Phi}_{\tau} \rangle|^p \right)^{\frac{1}{p}} \\ &\lesssim \mathfrak{C}_+^{\alpha}(\kappa^*) \left(\sum_{r \in \mathbb{Z}} v(Q_{0r}) \left| \int_{\mathbb{R}} f(y) B_{m^*}(y - r) dy \right|^p \right)^{\frac{1}{p}}. \end{aligned}$$

We set $\int_{\mathbb{R}} f(y) B_{m^*}(y - \tau) dy =: \langle f, \bar{\Phi}_\tau \rangle$ and set $\bar{\Phi}$ to be the first component of a new wavelet system forming spline basis of order m^* (see (5.13) with $\bar{\Phi}$ instead of $\tilde{\Phi}$ and $n = m^*$). The second component of this system, as well as the elements it generates, will be denoted by $\bar{\Psi}^{(1,0)}$ and $\bar{\Psi}_{(d-1)\tau}^{(1,0)}$, $d \in \mathbb{N}$, respectively (see (5.13) and (5.15) with $\bar{\Psi}^{(1,0)}$ instead of $\tilde{\Psi}^{(1,0)}$, $m(1) = n^*$ and $n = m^*$).

The next step is to estimate $\tilde{\lambda}_{d\tau}^{(1,1)}$ for $d > 0$ on the right-hand side (3.12) via $\langle f, \bar{\Psi}_{(d-1)\tau}^{(1,0)} \rangle$. To do this, we write out an exact expression for the element $\tilde{\Psi}_{(d-1)\tau}^{(1,1)}$, based on (5.10) and (5.12):

$$\begin{aligned} 2^{-d/2} \gamma_{n^*} \Lambda_{n^*} \tilde{\Psi}_{(d-1)\tau}^{(1,1)}(\cdot) &= \gamma_{n^*} \Psi_{n^*, 0, 0; m^*(1), \alpha(1)}(2^{d-1} \cdot -\tau) \\ &= \sum_{|l| \leq m^*} \lambda_{|l|}(m^*) (-1)^{|l|} \sum_{|s| \leq n^*} \lambda_{|s|}(n^*) (-1)^{|s|} \\ &\quad \times \sum_{k=0}^{2\alpha+n^*+1} (-1)^k \binom{1+m^*+2\alpha}{k} B_{n^*}(2^d \cdot -2\tau + (n^* - k) - s - l). \end{aligned}$$

We will also need the Zhu–Vandermonde identity [19, page 15]

$$\binom{r+s}{k} = \sum_{n=0}^k \binom{r}{n} \binom{s}{k-n}, \quad \text{where } r, s \in \mathbb{R} \text{ and } k \in \mathbb{N}, \quad (3.16)$$

from which it follows for $r = 1 + m^*$ and $s = \alpha$ that

$$\begin{aligned} \sum_{k=0}^{2\alpha+n^*+1} (-1)^k \binom{1+n^*+2\alpha}{k} B_{n^*}(2^d x - 2\tau + (m^* - k) - s - l) \\ = \sum_{k=0}^{2\alpha+n^*+1} (-1)^k \sum_{r=0}^k \binom{1+m^*}{r} \binom{\alpha}{k-r} B_{n^*}(2^d x - 2\tau + (m^* - k) - s - l) \\ = \sum_{r=0}^{m^*+1} (-1)^r \binom{1+m^*}{r} \sum_{m=0}^{\alpha} (-1)^m \binom{\alpha}{m} B_{n^*}(2^d x - 2\tau + (m^* - r - m) - s - l). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{m=0}^{\alpha} (-1)^m \binom{\alpha}{m} B_{n^*}(2^d x - 2\tau + (m^* - r - m) - s - l) \\ = 2^{-d\alpha} B_{m^*}^{(\alpha)}(2^d x - 2\tau + (m^* - r) - s - l). \end{aligned} \quad (3.17)$$

Also,

$$\begin{aligned} \text{supp } B_{m^*}^{(\alpha)}(2^d x - 2\tau + (m^* - r) - s - l) &=: \text{supp } B_{m^*, 2\tau+s+l+r-m^*}^{(\alpha)}(2^d \cdot) \\ &= \left[\frac{2\tau + s + l + r - m^*}{2^d}, \frac{2\tau + s + l + r + 1}{2^d} \right] \end{aligned}$$

and $B_{m^*, 2\tau+s+l+r-m^*}^{(\alpha)}(2^d \cdot) = 0$ at the points $2^{-d}(2\tau + s + l + r - m^*)$ and $2^{-d}(2\tau + s + l + r + 1)$ for all

$k = 0, \dots, \alpha$. Therefore, and in view of integrability of f on the semiaxes $(-\infty, x)$ for any x ,

$$\begin{aligned}
& \int_{\mathbb{R}} \left(\int_{-\infty}^x \frac{f(y) dy}{(x-y)^{1-\alpha}} \right) \sum_{m=0}^{\alpha} (-1)^m \binom{\alpha}{m} B_{n^*}(2^d x - 2\tau + (m^* - r - m) - s - l) \\
&= \int_{\frac{2\tau+s+l+r+m+1}{2^d}}^{\frac{2\tau+s+l+r+m+1}{2^d}} \left(\int_{-\infty}^x \frac{f(y) dy}{(x-y)^{1-\alpha}} \right) B_{m^*}^{(\alpha)}(2^d x - 2\tau + (m^* - r) - s - l) \\
&= \int_{-\infty}^{\frac{2\tau+s+l+r+m+1}{2^d}} f(y) \left(\int_{\frac{2\tau+s+l+r+m-m^*}{2^d}}^{\frac{2\tau+s+l+r+m+1}{2^d}} \frac{B_{m^*}^{(\alpha)}(2^d x - 2\tau + (m^* - r) - s - l) dx}{(x-y)^{1-\alpha}} \right) dy \\
&+ \int_{\frac{2\tau+s+l+r+m-m^*}{2^d}}^{\frac{2\tau+s+l+r+m+1}{2^d}} f(y) \left(\int_{\frac{2\tau+s+l+r+m-m^*}{2^d}}^y \frac{B_{m^*}^{(\alpha)}(2^d x - 2\tau + (m^* - r) - s - l) dx}{(x-y)^{1-\alpha}} \right) dy \\
&= \frac{\Gamma(\alpha)}{2^d \alpha} \int_{\frac{2\tau+s+l+r-m^*}{2^d}}^{\frac{2\tau+s+l+r+m+1}{2^d}} f(y) B_{m^*}(2^d y - 2\tau + (m^* - r) - s - l) dy.
\end{aligned}$$

From this we obtain that

$$\begin{aligned}
\frac{2^{d\alpha} \Lambda_{n^*}}{2^{d/2} \Gamma(\alpha)} \langle I_+^\alpha f, \tilde{\Psi}_{(d-1)\tau}^{(1,1)} \rangle &= \frac{2^{d\alpha} \gamma_{n^*}}{\Gamma(\alpha)} \langle I_+^\alpha f, \Psi_{n^*, 0, 0; m^*(1), \alpha(1)}(2^{d-1} \cdot -\tau) \rangle \\
&= \sum_{|l| \leq m^*} \lambda_{|l|}(m^*) (-1)^{|l|} \sum_{|s| \leq n^*} \lambda_{|s|}(n^*) (-1)^{|s|} \sum_{r=0}^{m^*+1} (-1)^r \binom{1+m^*}{r} \\
&\quad \times \int_{\mathbb{R}} f(y) B_{m^*}(2^d y - 2\tau + (m^* - r) - s - l) dy = \frac{\gamma_{m^*} \Lambda_{m^*}}{2^{d/2}} \langle f, \bar{\Psi}_{(d-1)\tau}^{(1,0)} \rangle,
\end{aligned}$$

where $\bar{\Psi}^{(1,0)}$ is specified by $\Psi_{m^*, 0, 0; n^*(1), \alpha(0)}$ (see (5.13)), and we can write that

$$2^{pd\alpha} \sum_{\tau \in \mathbb{Z}} u(Q_{(d-1)\tau}) |\langle I_+^\alpha f, \tilde{\Psi}_{(d-1)\tau}^{(1,1)} \rangle|^p = \sum_{\tau \in \mathbb{Z}} u(Q_{(d-1)\tau}) \left| \int_{\mathbb{R}} f(y) \bar{\Psi}_{(d-1)\tau}^{(1,0)}(y) dy \right|^p.$$

Applying condition (3.10), we derive the following estimate:

$$\begin{aligned}
2^{-d/2} 2^{d(-\kappa^* + \alpha)} \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\chi}_{d\tau}^{(1,1)}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R}, u)} &= 2^{d(-\kappa^* + \alpha)} \left(\sum_{\tau \in \mathbb{Z}} u(Q_{(d-1)\tau}) |\langle I_+^\alpha f, \tilde{\Psi}_{(d-1)\tau}^{(1,1)} \rangle|^p \right)^{\frac{1}{p}} \\
&\simeq 2^{-d\kappa^*} \left(\sum_{\tau \in \mathbb{Z}} u(Q_{(d-1)\tau}) |\langle f, \bar{\Psi}_{(d-1)\tau}^{(1,0)} \rangle|^p \right)^{\frac{1}{p}} \\
&\lesssim \sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u) \left(\sum_{\tau \in \mathbb{Z}} v(Q_{(d-1)\tau}) |\langle f, \bar{\Psi}_{(d-1)\tau}^{(1,0)} \rangle|^p \right)^{\frac{1}{p}} \\
&\lesssim \mathfrak{C}_+^\alpha(\kappa^*) \left(\sum_{\tau \in \mathbb{Z}} v(Q_{(d-1)\tau}) |\langle f, \bar{\Psi}_{(d-1)\tau}^{(1,0)} \rangle|^p \right)^{1/p} = \mathfrak{C}_+^\alpha(\kappa^*) \left\| \sum_{\tau \in \mathbb{Z}} \chi_{Q_{(d-1)\tau}} |\langle f, \bar{\Psi}_{(d-1)\tau}^{(1,0)} \rangle| \right\|_{L^p(\mathbb{R}, v)}.
\end{aligned}$$

Thus, $\bar{\Psi}_{(d-1)\tau}^{(1,0)}$ together with $\bar{\Phi}_\nu$ (see page 64) forms a spline wavelet system of order m^* . The sequential norms corresponding to this system, by Theorem 3.2, are equivalent to the norm of f in the space $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$. This means that assertion (i) is proved.

Necessity. Suppose that the operator I_+^α is bounded from $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ with some $\kappa^* \in \mathbb{R}$ to $B_{pq}^{s, u}(\mathbb{R})$. Accordingly, the following inequality holds

$$\|I_+^\alpha f\|_{B_{pq}^{s, u}(\mathbb{R})} \leq \mathfrak{C} \|f\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})}, \quad (3.18)$$

where $\mathfrak{C} > 0$ is the best constant, i.e. $\mathfrak{C} = \|I_{\pm}^{\alpha}\|_{B_{pq}^{s+\kappa^*-\alpha,v}(\mathbb{R}) \rightarrow B_{pq}^{s,u}(\mathbb{R})}$. We will show that

$$\mathfrak{C} \gtrsim \mathfrak{C}_+(\kappa^*) \geq \max\{\mathbf{M}_+^{\alpha}(1), \mathbf{M}_+^{\alpha}(0), \sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u)\}. \quad (3.19)$$

Let $n^* \in \mathbb{N}$ and $m^* \in \mathbb{N}$ satisfy condition (3.3) with respect to the parameters of the spaces $B_{pq}^{s,u}(\mathbb{R})$ and $B_{pq}^{s+\kappa^*-\alpha,v}(\mathbb{R})$, respectively. By Theorem 3.1,

$$\|I_+^{\alpha} f\|_{B_{pq}^{s,u}(\mathbb{R})} \approx \|\tilde{\lambda}_{(I_+^{\alpha} f)}^{(\tilde{\mathbf{a}})}\|_{b_{pq}^s(u)}, \quad (3.20)$$

where the sequence $\tilde{\lambda}_{(I_+^{\alpha} f)}^{(\tilde{\mathbf{a}})}$ is determined by the elements

$$\tilde{\lambda}_{0\tau}^{(\tilde{\mathbf{a}})} = \langle I_+^{\alpha} f, \tilde{\Phi}_{\tau}^{(\tilde{\mathbf{a}})} \rangle \quad (\tau \in \mathbb{Z}), \quad \tilde{\lambda}_{d\tau}^{(\tilde{\mathbf{a}})} = 2^{d/2} \langle I_+^{\alpha} f, \tilde{\Psi}_{(d-1)\tau}^{(\tilde{\mathbf{a}})} \rangle \quad (d \in \mathbb{N}, \tau \in \mathbb{Z}). \quad (3.21)$$

Here, $\tilde{\Phi}^{(\tilde{\mathbf{a}})}$ is defined by $B_{n^*, \tilde{\mathbf{a}}}$, and $\tilde{\Psi}^{(\tilde{\mathbf{a}})}$ (see (5.13)) is specified via $\Psi_{n^*, \tilde{\mathbf{a}}, 0}$, where $\tilde{\mathbf{a}}$ can take values 0 or $\pm 1/2$. Based on the above, we can write:

$$\|I_+^{\alpha} f\|_{B_{pq}^{s,u}(\mathbb{R})} \approx \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{0\tau}^{(\tilde{\mathbf{a}})}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, u)} + \left(\sum_{d \in \mathbb{N}} 2^{qd} \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{d\tau}^{(\tilde{\mathbf{a}})}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R}, u)}^q \right)^{\frac{1}{q}}. \quad (3.22)$$

Moreover, also basing on Theorem 3.1,

$$\|f\|_{B_{pq}^{s+\kappa^*-\alpha,v}(\mathbb{R})} \approx \|\lambda_f^{(\mathbf{a})}\|_{b_{pq}^{s+\kappa^*-\alpha}(v)},$$

where the sequence $\lambda_f^{(\mathbf{a})}$ with $\mathbf{a} \in \{0, \pm 1/2\}$ is determined by the elements

$$\lambda_{0\tau}^{(\mathbf{a})} = \langle f, \Phi_{\tau}^{(\mathbf{a})} \rangle \quad (\tau \in \mathbb{Z}), \quad \lambda_{d\tau}^{(\mathbf{a})} = 2^{d/2} \langle f, \Psi_{(d-1)\tau}^{(\mathbf{a})} \rangle \quad (d \in \mathbb{N}, \tau \in \mathbb{Z}). \quad (3.23)$$

In this case, $\Phi^{(\mathbf{a})}$ is specified by $B_{m^*, \mathbf{a}}$, and $\Psi^{(\mathbf{a})}$ (see (5.14)) is defined by $\Psi_{m^*, \mathbf{a}, 0}$. Thus, for the right-hand side of (3.18) we can write that

$$\|f\|_{B_{pq}^{s+\kappa^*-\alpha,v}(\mathbb{R})} \approx \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{0\tau}^{(\mathbf{a})}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, v)} + \left(\sum_{d \in \mathbb{N}} 2^{qd(s+\kappa^*-\alpha)} \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{d\tau}^{(\mathbf{a})}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R}, v)}^q \right)^{\frac{1}{q}}. \quad (3.24)$$

To prove (3.19) we first suppose that the maximum on the right side of (3.19) is equal to

$$\mathbf{M}_+^{\alpha}(0) = \sup_{\tau \in \mathbb{Z}} \left(\sum_{r \geq \tau} u(Q_{0r}) \right)^{\frac{1}{p}} \left(\sum_{r \leq \tau} (\tau - r + 1)^{p'(\alpha-1)} [v(Q_{0r})]^{1-p'} \right)^{\frac{1}{p'}}.$$

Then, we fix n^* with $m^* = n^* + \alpha$ and leave only the first of the two terms on the right side of (3.22) under consideration. Due to the procedure of forming iterative differences (to the left side) of order α with coefficients

$$A_r(\alpha) = \sum_{j=1}^r (-1)^{j-1} \binom{\alpha}{j} A_{r-j}(\alpha), \quad \text{where } A_0(\alpha) = 1,$$

(see e.g. (3.14) and (5.1)) the following equalities hold:

$$\begin{aligned} \frac{m^*!}{n^*!} \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{0\tau}^{(\tilde{\mathbf{a}})}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, u)}^p &= \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) |\langle I_+^{\alpha} f, \tilde{\Phi}_{\tau}^{(\tilde{\mathbf{a}})} \rangle|^p \\ &= \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{r \geq 0} A_r(\alpha) \int_{\mathbb{R}} f(y) B_{m^*}(y - \tau + \alpha - \tilde{\mathbf{a}} + r) dy \right|^p \\ &= \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{r \geq 0} A_r(\alpha) \langle f, B_{m^*, \tau - \alpha + \tilde{\mathbf{a}} - r} \rangle \right|^p = \sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{l \leq \tau} A_{\tau-l}(\alpha) \langle f, B_{m^*, l - \alpha + \tilde{\mathbf{a}}} \rangle \right|^p. \end{aligned}$$

Then, we obtain from (3.18), (3.22) and (3.24):

$$\begin{aligned}
& \left(\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{l \leq \tau} A_{\tau-l}(\alpha) \langle f, B_{m^*, l-\alpha+\tilde{a}} \rangle \right|^p \right)^{\frac{1}{p}} \simeq \left\| \sum_{\tau \in \mathbb{Z}} |\tilde{\lambda}_{0\tau}^{(\tilde{a})}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, u)} \\
& \lesssim \|I_+^\alpha f\|_{B_{pq}^{s,u}(\mathbb{R})} \lesssim \mathfrak{C} \|f\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})} \\
& \approx \mathfrak{C} \left[\left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{0\tau}^{(a)}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, v)} + \left(\sum_{d \in \mathbb{N}} 2^{qd(s+\kappa^*-\alpha)} \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{d\tau}^{(a)}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R}, v)}^q \right)^{\frac{1}{q}} \right] \\
& = \mathfrak{C} \left[\left(\sum_{\tau \in \mathbb{Z}} v(Q_{0\tau}) |\langle f, B_{m^*, \tau+a} \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{d \in \mathbb{N}} 2^{qd(s+\kappa^*-\alpha)} \left(\sum_{\tau \in \mathbb{Z}} v(Q_{(d-1)\tau}) |\langle f, \Psi_{(d-1)\tau}^{(a)} \rangle|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

By Lemma 2.1 and property (2.2),

$$\sum_{\tau \in \mathbb{Z}} v(Q_{0\tau}) |\langle f, B_{m^*, \tau+a} \rangle|^p \lesssim \sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f, B_{m^*, l-\alpha+a} \rangle|^p.$$

Thus,

$$\begin{aligned}
& \left(\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{l \leq \tau} A_{\tau-l}(\alpha) \langle f, B_{m^*, l-\alpha+\tilde{a}} \rangle \right|^p \right)^{\frac{1}{p}} \lesssim \left[\left(\sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f, B_{m^*, l-\alpha+a} \rangle|^p \right)^{\frac{1}{p}} \right. \\
& \quad \left. + \left(\sum_{d \in \mathbb{N}} 2^{qd(s+\kappa^*-\alpha)} \left(\sum_{\tau \in \mathbb{Z}} v(Q_{(d-1)\tau}) |\langle f, \Psi_{(d-1)\tau}^{(a)} \rangle|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right]. \quad (3.25)
\end{aligned}$$

For $A_r(m)$ with fixed $m \in \mathbb{N}$, the following lower bound holds for all $r \in \mathbb{N}$:

$$(m-1)! A_r(m) \geq (1+r)^{m-1} = \sum_{s=0}^{m-1} \frac{(m-1)! r^s}{s!(m-1-s)!}. \quad (3.26)$$

We will show that this is true, by taking into account (see [32] (5.20)) that $A_0(m) = 1$, $A_1(m) = m$ and

$$A_r(m) = \sum_{1 \leq l \leq m} A_{r-1}(l), \quad (m \in \mathbb{N}). \quad (3.27)$$

Then, (3.26) holds for $r = 1$ by (3.27) for all $m \in \mathbb{N}$, in particular, for $1 \leq m \leq m_0$ with some $m_0 \geq 1$. Suppose that (3.26) holds for $r = k-1$ with some $k > 1$ and any $m = l$, where $1 \leq l \leq m_0$. Then it follows from (3.27) that

$$\begin{aligned}
A_k(m_0) &= \sum_{1 \leq l \leq m_0} A_{k-1}(l) \geq \sum_{1 \leq l \leq m_0} \frac{k^{l-1}}{(l-1)!} \\
&= \sum_{s=0}^{m_0-1} \frac{k^s}{s!} \geq \sum_{s=0}^{m_0-1} \frac{k^s}{s!(m_0-1-s)!} = \frac{(1+k)^{m_0-1}}{(m_0-1)!}.
\end{aligned}$$

Next, we set $\mathbf{a} = \tilde{\mathbf{a}} = 0$ and denote by $\mathfrak{B}_{m^*}$ the class of all functions belonging to $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ which can be represented as a finite linear combination of integer shifts of B -splines of order m^* . Due to the semi-orthogonality of the basis elements in the right-hand side of (3.24) (see (5.16)), the restriction of (3.25) to the sequences $\lambda_f^{(0)}$ with f belonging to the class $\mathfrak{B}_{m^*}$ has the form

$$\left(\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{l \leq \tau} A_{\tau-l}(\alpha) \langle f, B_{m^*, l-\alpha} \rangle \right|^p \right)^{\frac{1}{p}} \lesssim \mathfrak{C}_{\mathfrak{B}_{m^*}} \left(\sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f, B_{m^*, l-\alpha} \rangle|^p \right)^{\frac{1}{p}}. \quad (3.28)$$

Moreover, the best constant $\mathfrak{C}_{\mathfrak{B}_{m^*}}$ in this inequality does not exceed $\mathfrak{C}$.

For a sufficiently large $R \in \mathbb{N}$, we substitute into (3.28) a function f_R^* of the form

$$\begin{aligned} f_R^*(y) &= \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)(p'-1)} [v(Q_{0\tau})]^{1-p'} B_{m^*, \tau-\alpha}(y) \\ &= \sum_{\tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)(p'-1)} [v(Q_{0\tau})]^{1-p'} B_{m^*}(y - \tau + \alpha) \quad (\nu \in \mathbb{Z}). \end{aligned} \quad (3.29)$$

This will lead us to the inequality

$$\left(\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{l \leq \tau} A_{\tau-l}(\alpha) \langle f_R^*, B_{m^*, l-\alpha} \rangle \right|^p \right)^{\frac{1}{p}} \lesssim \mathfrak{C}_{\mathfrak{B}_{m^*}} \left(\sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f_R^*, B_{m^*, l-\alpha} \rangle|^p \right)^{\frac{1}{p}}. \quad (3.30)$$

Due to non-negativity of f_R^* and B_{m^*} and on the basis of (3.26),

$$\sum_{l \leq \tau} A_{\tau-l}(\alpha) \langle f_R^*, B_{m^*, l-\alpha} \rangle \gtrsim \sum_{l \leq \tau} (\tau - l + 1)^{\alpha-1} \langle f_R^*, B_{m^*, l-\alpha} \rangle.$$

Moreover,

$$\begin{aligned} \left(\sum_{\tau \in \mathbb{Z}} u(Q_{0\tau}) \left| \sum_{l \leq \tau} (\tau - l + 1)^{\alpha-1} \langle f_R^*, B_{m^*, l-\alpha} \rangle \right|^p \right)^{\frac{1}{p}} \\ \geq \left(\sum_{\tau \geq \nu} u(Q_{0\tau}) \right)^{1/p} \sum_{l \leq \nu} (\nu - l + 1)^{\alpha-1} \langle f_R^*, B_{m^*, l-\alpha} \rangle. \end{aligned}$$

Thus, (3.30) is transformable into

$$\left(\sum_{\tau \geq \nu} u(Q_{0\tau}) \right)^{1/p} \sum_{l \leq \nu} (\nu - l + 1)^{\alpha-1} \langle f_R^*, B_{m^*, l-\alpha} \rangle \lesssim \mathfrak{C}_{\mathfrak{B}_{m^*}} \left(\sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f_R^*, B_{m^*, l-\alpha} \rangle|^p \right)^{\frac{1}{p}}. \quad (3.31)$$

Further, we write for the sum depending on f_R^* in the left-hand side of (3.31):

$$\begin{aligned} \sum_{l \leq \nu} (\nu - l + 1)^{\alpha-1} \langle f_R^*, B_{m^*, l-\alpha} \rangle \\ &= \sum_{l \leq \nu} (\nu - l + 1)^{\alpha-1} \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)(p'-1)} [v(Q_{0\tau})]^{1-p'} \langle B_{m^*, \tau-\alpha}, B_{m^*, l-\alpha} \rangle \\ &= \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)(p'-1)} [v(Q_{0\tau})]^{1-p'} \sum_{l \leq \nu} (\nu - l + 1)^{\alpha-1} \langle B_{m^*, \tau-\alpha}, B_{m^*, l-\alpha} \rangle \\ &\geq \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)(p'-1)} [v(Q_{0\tau})]^{1-p'} \sum_{l \leq \tau} (\nu - l + 1)^{\alpha-1} \langle B_{m^*, \tau-\alpha}, B_{m^*, l-\alpha} \rangle \\ &\geq \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{1-p'} \sum_{l \leq \tau} \langle B_{m^*, \tau-\alpha}, B_{m^*, l-\alpha} \rangle. \end{aligned}$$

Due to (5.5) and in view of the orthonormality of $\{\phi_{m^*, \tau}\}_{\tau \in \mathbb{Z}}$,

$$\begin{aligned} \beta_{m^*}^2 \sum_{l \leq \tau} \langle B_{m^*, \tau-\alpha}, B_{m^*, l-\alpha} \rangle \\ &= \sum_{\mathcal{K}'=0}^{m^*} \alpha'_{\mathcal{K}'} \sum_{\mathcal{K}''=0}^{m^*} \alpha'_{\mathcal{K}''} \sum_{l \leq \tau} \int \phi_{m^*}(y + \mathcal{K}' - \tau + \alpha) \phi_{m^*}(y + \mathcal{K}'' - l + \alpha) \\ &= \sum_{\mathcal{K}=0}^{m^*} (\alpha'_{\mathcal{K}})^2 + \sum_{\mathcal{K}''-\mathcal{K}'=1} \alpha'_{\mathcal{K}'} \alpha'_{\mathcal{K}''} + \dots + \sum_{\mathcal{K}''-\mathcal{K}'=m^*} \alpha'_{\mathcal{K}'} \alpha'_{\mathcal{K}''} \\ &\gtrsim \sum_{\mathcal{K}=0}^{m^*} (\alpha'_{\mathcal{K}})^2 + \sum_{|\mathcal{K}'-\mathcal{K}''|=1} \alpha'_{\mathcal{K}'} \alpha'_{\mathcal{K}''} + \dots + \sum_{|\mathcal{K}'-\mathcal{K}''|=m^*} \alpha'_{\mathcal{K}'} \alpha'_{\mathcal{K}''} = \left(\sum_{\mathcal{K}=0}^{m^*} \alpha'_{\mathcal{K}} \right)^2 = \beta_{m^*}^2. \end{aligned} \quad (3.32)$$

Thus, the left-hand side of (3.31) is estimated from below, up to some constant, by

$$\left(\sum_{\tau \geq \nu} u(Q_{0\tau})\right)^{\frac{1}{p}} \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{1-p'}. \quad (3.33)$$

To estimate the right-hand side of (3.31) from above, we write by using the Hölder inequality:

$$\begin{aligned} |\langle f_R^*, B_{m^*, l-\alpha} \rangle|^p &= \left[\sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)(p'-1)} [v(Q_{0\tau})]^{1-p'} \langle B_{m^*}(\cdot - \tau + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \right]^p \\ &\leq \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{-p'} \langle B_{m^*}(\cdot - \tau + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \\ &\quad \times \left[\sum_{-R \leq r \leq \nu} \langle B_{m^*}(\cdot - r + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \right]^{p-1}. \end{aligned}$$

Hence, due to the boundedness of the supports of B_{m^*} ,

$$\begin{aligned} \sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f_R^*, B_{m^*, l-\alpha} \rangle|^p &\leq \sum_{l \in \mathbb{Z}} v(Q_{0l}) \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{-p'} \langle B_{m^*}(\cdot - \tau + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \\ &\quad \times \left[\sum_{-R \leq r \leq \nu} \langle B_{m^*}(\cdot - r + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \right]^{p-1} \\ &= \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{-p'} \sum_{l \in \mathbb{Z}} v(Q_{0l}) \langle B_{m^*}(\cdot - \tau + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \\ &\quad \times \left[\sum_{-R \leq r \leq \nu} \langle B_{m^*}(\cdot - r + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \right]^{p-1} \\ &\leq \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{-p'} \sum_{|l-\tau| \leq m^*} v(Q_{0l}) \langle B_{m^*}(\cdot - \tau + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \\ &\quad \times \left[\sum_{r \leq \nu} \langle B_{m^*}(\cdot - r + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \right]^{p-1}. \end{aligned}$$

For a fixed l , due to (5.5) and in view of the orthonormality of $\{\phi_{m^*, \tau}\}_{\tau \in \mathbb{Z}}$ (see (3.32)),

$$\sum_{r \leq \nu} \langle B_{m^*}(\cdot - r + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \leq \sum_{|r-l| \leq m^*} \langle B_{m^*}(\cdot - r + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \leq 1.$$

By virtue of Lemma 2.1 and property (2.2) we have $v(Q_{0l}) \lesssim v(Q_{0\tau})$ for any $|l - \tau| \leq m^*$. Therefore, for the right-hand side of (3.31) it holds that

$$\begin{aligned} \sum_{l \in \mathbb{Z}} v(Q_{0l}) |\langle f_R^*, B_{m^*, l-\alpha} \rangle|^p &\lesssim \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{1-p'} \sum_{|l-\tau| \leq m^*} \langle B_{m^*}(\cdot - \tau + \alpha), B_{m^*}(\cdot - l + \alpha) \rangle \\ &\leq \sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{(\alpha-1)p'} [v(Q_{0\tau})]^{1-p'}. \end{aligned}$$

Combining this with (3.31) and (3.33), we obtain from (3.30) that

$$\mathfrak{C}_{\mathfrak{B}_{m^*}} \gtrsim \sup_{\nu \in \mathbb{Z}} \left(\sum_{\tau \geq \nu} u(Q_{0\tau}) \right)^{\frac{1}{p}} \left(\sum_{-R \leq \tau \leq \nu} (\nu - \tau + 1)^{p'(\alpha-1)} [v(Q_{0\tau})]^{1-p'} \right)^{\frac{1}{p'}},$$

from which, by tending $R \rightarrow \infty$, we get the estimate $\mathfrak{C} \geq \mathfrak{C}_{\mathfrak{B}_{m^*}} \gtrsim \mathbf{M}_+^\alpha(0)$ for the best constant in inequality (3.18). Similarly, it can be shown that $\mathfrak{C} \gtrsim \mathbf{M}_+^\alpha(1)$, where

$$\mathbf{M}_+^\alpha(1) = \sup_{\tau \in \mathbb{Z}} \left(\sum_{r \geq \tau} (r - \tau + 1)^{p(\alpha-1)} u(Q_{0r}) \right)^{\frac{1}{p}} \left(\sum_{r \leq \tau} [v(Q_{0r})]^{1-p'} \right)^{\frac{1}{p'}}.$$

To do this, one should define the function

$$g_R^*(y) = \sum_{\nu \leq \tau \leq R} (\tau - \nu + 1)^{(\alpha-1)(p-1)} [u(Q_{0\tau})] B_{m^*, \tau-\alpha}(y)$$

and substitute it into the dual to (3.28) inequality of the form

$$\left(\sum_{\tau \in \mathbb{Z}} [v(Q_{0\tau})]^{1-p'} \left| \sum_{l \geq \tau} A_{l-\tau}(\alpha) \langle g, B_{m^*, l-\alpha} \rangle \right|^{p'} \right)^{\frac{1}{p'}} \lesssim \mathfrak{C}_{\mathfrak{B}_{m^*}} \left(\sum_{l \in \mathbb{Z}} [u(Q_{0l})]^{1-p'} \left| \langle g, B_{m^*, l-\alpha} \rangle \right|^{p'} \right)^{\frac{1}{p'}}.$$

Now, we suppose that the maximum in the right-hand side of (3.19) is achieved at $\sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u)$. Let us show the necessity of such a characteristic. To this end, for some d_0 and τ_0 we define

$$\begin{aligned} h_{\tau_0}^*(y) &= \Lambda_{m^*}^{-1} \Psi_{m^*, \bar{a}, \bar{s}; n^*(1), \alpha(1)}(2^{d_0-1}y - \tau_0) \\ &= \sum_{k=0}^{2\alpha} (-1)^k \binom{2\alpha}{k} h_{\tau_0; k}(y) = \sum_{|\kappa| \leq n^*} \lambda_{|\kappa|}(n^*) (-1)^{|\kappa|} \sum_{k=0}^{2\alpha} (-1)^k \binom{2\alpha}{k} h_{\tau_0; k, \kappa}(y) \end{aligned} \quad (3.34)$$

with $\bar{s} \in \mathbb{R}$, which we will specify later, and

$$\begin{aligned} h_{\tau_0; k}(y) &= \Psi_{m^*, \bar{a}, \bar{s}; n^*(1), \alpha(0)}(2^{d_0-1}y - \tau_0 - k/2 + \alpha/2), \\ h_{\tau_0; k, \kappa}(y) &= \Psi_{m^*, \bar{a}, \bar{s}}(2^{d_0-1}y - \tau_0 - k/2 - \kappa/2 + \alpha/2). \end{aligned}$$

We will need representation (3.22) for $f = h_{\tau_0}^*$ and (3.24) for $f = h_{\tau_0; k, \kappa}$ with $k = 0, \dots, 2\alpha$ and $\kappa = -n^*, \dots, 0, \dots, n^*$. In this situation, we fix n^* and m^* so that $n^* = m^* + \alpha$.

Then (see (5.1) and (3.17)),

$$\begin{aligned} \gamma_{m^*} I_+^\alpha h_{\tau_0}^*(x) &= 2^{-\alpha d_0} \gamma_{n^*} \Psi_{n^*, \bar{a}, \bar{s}; m^*(1), \alpha(0)}(2^{d_0-1} \cdot -\tau_0) = 2^{-\alpha d_0} \sum_{|r| \leq m^*} \lambda_{|r|}(m^*) (-1)^{|r|} \\ &\times \sum_{|\kappa| \leq n^*} \lambda_{|\kappa|}(n^*) (-1)^{|\kappa|} \sum_{j=0}^{n^*+1} (-1)^j \binom{1+n^*}{j} B_{n^*}(2^{d_0}(x - \bar{s} - \bar{a}) - 2\tau_0 + n^* - j - r - \kappa). \end{aligned} \quad (3.35)$$

Note that

$$\text{supp}(I_+^\alpha h_{\tau_0}^*) = \left[\frac{2(\tau_0 + \bar{s} + \bar{a}) - 2n^* - m^*}{2^{d_0}}, \frac{2(\tau_0 + \bar{s} + \bar{a}) + 2n^* + m^* + 2}{2^{d_0}} \right]$$

(see §5). On the other hand, the supports of the functions $\Psi_{n^*, \bar{a}, 0}(2^{d-1} \cdot -\tau)$ forming the coefficients $\tilde{\lambda}_{d\tau}^{(\tilde{a})}$ in (3.22) coincide with

$$\left[\frac{2\tau + 2\tilde{a} - 2n^*}{2^d}, \frac{2\tau + 2\tilde{a} + 2n^* + 2}{2^d} \right]$$

(see §5). Moreover,

$$\begin{aligned} \gamma_{n^*} \Psi_{n^*, \bar{a}, 0}(2^{d-1} \cdot -\tau) &= \sum_{|l| \leq n^*} \lambda_{|l|}(n^*) (-1)^{|l|} \\ &\times \sum_{s=0}^{n^*+1} (-1)^s \binom{1+n^*}{s} B_{n^*}(2^{d_0}(y - \tilde{a}) - 2\tau + n^* - s - l). \end{aligned} \quad (3.36)$$

Denote $\int_{\mathbb{R}} B_n(t)B_n(t-n)dt =: \mathbf{B}_n$ and observe that $\mathbf{B}_n > 0$ for any $n \in \mathbb{N}$. We call $\mathbf{B}_n$ as $\mathbf{B}_n$ -intersection, which is minimal in some sense. The following relations hold (see (3.22)):

$$\begin{aligned} \|I_+^\alpha h_{\tau_0}^*\|_{B_{pq}^{s,u}(\mathbb{R})}^p &\gtrsim 2^{d_0 sp} |\tilde{\lambda}_{d_0 \tau_0}^{\langle \tilde{\mathbf{a}} \rangle}|^p [u(Q_{(d_0-1)\tau_0})] = 2^{d_0 sp} 2^{d_0 p/2} |\langle I_+^\alpha h_{\tau_0}^*, \tilde{\Psi}_{(d_0-1)\tau_0}^{\langle \tilde{\mathbf{a}} \rangle} \rangle|^p [u(Q_{(d_0-1)\tau_0})] \\ &= 2^{d_0(s-\alpha)p} 2^{d_0 p} |\langle \Psi_{n^*, \bar{\mathbf{a}}, \bar{\mathbf{s}}; m^*(1), \alpha(0)}(2^{d_0-1} \cdot -\tau_0), \Psi_{n^*, \tilde{\mathbf{a}}, 0}(2^{d_0-1} \cdot -\tau_0) \rangle|^p [u(Q_{(d_0-1)\tau_0})] \\ &= 2^{d_0(s-\alpha)p} |\langle \Psi_{n^*, \bar{\mathbf{a}}, \bar{\mathbf{s}}; m^*(1), \alpha(0)}(\cdot/2 - \tau_0), \Psi_{n^*, \tilde{\mathbf{a}}, 0}(\cdot/2 - \tau_0) \rangle|^p [u(Q_{(d_0-1)\tau_0})]. \end{aligned} \quad (3.37)$$

If we choose $\bar{\mathbf{s}} = -1 - m^* - 4n^*$ and set $\bar{\mathbf{a}} = \tilde{\mathbf{a}} = 0$, then the supports of $\Psi_{n^*, \bar{\mathbf{a}}, \bar{\mathbf{s}}; m^*(1), \alpha(0)}$ and $\Psi_{n^*, \tilde{\mathbf{a}}, 0}$ in (3.37) intersect only at an intersection set of the supports of the leftmost B_{n^*} in representation (3.36), which is $B_{n^*}(\cdot - 2\tau_0 + 2n^*)$, and the rightmost B_{n^*} in representation (3.35), which corresponds to $B_{n^*}(\cdot - 2\tau_0 + 3n^*)$. So, it will be exactly the intersection of the type $\mathbf{B}_{n^*}$. The supports of other B_{n^*} in the specified representations will not intersect each other if $\bar{\mathbf{s}} = -1 - m^* - 4n^*$. Therefore, taking into account all the coefficients in the definitions of $\Psi_{n^*, \bar{\mathbf{a}}, \bar{\mathbf{s}}; m^*(1), \alpha(0)}$ and $\Psi_{n^*, \tilde{\mathbf{a}}, 0}$, we conclude that

$$\langle \Psi_{n^*, \bar{\mathbf{a}}, \bar{\mathbf{s}}; m^*(1), \alpha(0)}(\cdot/2 - \tau_0), \Psi_{n^*, \tilde{\mathbf{a}}, 0}(\cdot/2 - \tau_0) \rangle = \frac{16\mathbf{B}_{n^*}}{\gamma_{m^*}\gamma_{n^*}} =: \Theta(n^*). \quad (3.38)$$

From here and (3.37) we obtain that

$$\|I_+^\alpha h_{\tau_0}^*\|_{B_{pq}^{s,u}(\mathbb{R})} \gtrsim 2^{d_0(s-\alpha)} [u(Q_{(d_0-1)\tau_0})]^{\frac{1}{p}} \Theta(n^*).$$

On the other hand,

$$\|h_{\tau_0}^*\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})} \lesssim \sum_{|\kappa| \leq n^*} \lambda_{|\kappa|}(n^*) \sum_{k=1}^{2\alpha} \binom{2\alpha}{k} \|h_{\tau_0; k, \kappa}\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})},$$

where $h_{\tau_0; k, \kappa}$ are finite linear combinations of mutually orthogonal integer shifts of $\psi_{m^*, \bar{\mathbf{a}}, \bar{\mathbf{s}}}(2^{d_0-1} \cdot -\tau_0 - k/2 - \kappa/2 + \alpha/2)$ with $\bar{\mathbf{a}}$ and $\bar{\mathbf{s}}$ as above. For fixed k and κ , this corresponds to $\psi_{m^*, \mathbf{a}, \mathbf{s}}(2^{d_0-1} \cdot -\tau_0)$ with $\mathbf{a} = (k + \kappa - \alpha)/2 - [(k + \kappa - \alpha)/2]$ and $\mathbf{s} = \bar{\mathbf{s}} + [(k + \kappa - \alpha)/2]$ (see (5.7)).

From representation (3.24) we can write for each $k + \kappa$:

$$\|h_{\tau_0; k, \kappa}\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})} \approx \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{0\tau}^{\langle \mathbf{a} \rangle}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, v)} + \left(\sum_{d \in \mathbb{N}} 2^{qd(s+\kappa^*-\alpha)} \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{d\tau}^{\langle \mathbf{a} \rangle}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R}, v)}^q \right)^{\frac{1}{q}}, \quad (3.39)$$

where the elements $\lambda_{h_{\tau_0; k, \kappa}}^{\langle \mathbf{a} \rangle}$ are given in (3.23) by $B_{m^*, \mathbf{a}}$ and $\Psi_{m^*, \mathbf{a}, 0}$ with $\mathbf{a}$ as above. The basis generated by $B_{m^*, \mathbf{a}}$ and $\Psi_{m^*, \mathbf{a}, 0}$ is semi-orthogonal (see detailed comments at the end of Section 5). Within the frame of $KMA_{B_{m^*, \mathbf{a}}}$, this implies that the basis elements $\Phi_\tau^{\langle \mathbf{a} \rangle}$, $\tau \in \mathbb{Z}$, of the subspace $V_0 \subset L^2(\mathbb{R})$ are orthogonal to the elements $\Psi_{(d-1)\tau}^{\langle \mathbf{a} \rangle}$, $\tau \in \mathbb{Z}$, forming the basis in the subspace $W_{d-1} \subset L^2(\mathbb{R})$ for any $d \in \mathbb{N}$. Moreover, $W_{i-1} \perp W_{j-1}$, $i \neq j$. Due to $h_{\tau_0; k, \kappa} \in W_{d_0-1}$, we have

$$\begin{aligned} \|h_{\tau_0; k, \kappa}\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})}^p &\approx 2^{d_0(s-\alpha+\kappa^*+1)p} \left\| \sum_{\tau \in \mathbb{Z}} |\langle h_{\tau_0; k, \kappa}(\cdot), \Psi_{m^*, \mathbf{a}, 0}(2^{d_0-1} \cdot -\tau) \rangle| \chi_{Q_{(d_0-1)\tau}} \right\|_{L^p(\mathbb{R}, v)}^p \\ &= \Lambda_{m^*}^{-2} 2^{d_0(s-\alpha+\kappa^*+1)p} \sum_{\tau \in \mathbb{Z}} |\langle \Psi_{m^*, \mathbf{a}, \mathbf{s}}(2^{d_0-1} \cdot -\tau_0), \Psi_{m^*, \mathbf{a}, 0}(2^{d_0-1} \cdot -\tau) \rangle|^p [v(Q_{(d_0-1)\tau})]. \end{aligned} \quad (3.40)$$

Since the support of $\Psi_{m^*, \mathbf{a}, 0}$ is compact, the sum over τ is bounded: $|\tau_0 + \mathbf{s} - \tau| \leq 2m^*$. In this case, $v(Q_{(d_0-1)\tau}) \lesssim v(Q_{(d_0-1)\tau_0})$ with a constant depending on n^* and α . In view of the equality

$$\gamma_{m^*} \Psi_{m^*, \mathbf{a}, 0}(2^{d_0-1} \cdot -\tau) = \sum_{|\rho| \leq m^*} \alpha_\rho'' \cdot \psi_{m^*, \mathbf{a}, 0}(2^{d_0-1} \cdot -\rho - \tau)$$

(see (5.7)), we have

$$\sum_{|\tau_0 + \mathbf{s} - \tau| \leq 2m^*} |\langle \Psi_{m^*, \mathbf{a}, \mathbf{s}}(2^{d_0-1} \cdot -\tau_0), \Psi_{m^*, \mathbf{a}, 0}(2^{d_0-1} \cdot -\tau) \rangle| \lesssim \Lambda_{m^*}^2$$

(see [28, § 4.1]). From this we obtain that

$$\mathfrak{C} \gtrsim 2^{-d_0 \kappa^*} [v(Q_{(d_0-1)\tau_0})]^{-\frac{1}{p}} [u(Q_{(d_0-1)\tau_0})]^{\frac{1}{p}}$$

for arbitrary d_0 and τ_0 . Hence,

$$\mathfrak{C} \gtrsim \sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u). \quad (3.41)$$

A proof of statement (i) for I^α is similar. In this case, it is required to apply the procedure of forming iterative differences to the right side.

(ii) *Sufficiency* of the conditions in this part follows from Theorem 1.1 (iii) and the criteria for continuous embeddings of the Besov weighted spaces (see [9, Proposition 2.1] and [12, Proposition 5.1]).

Necessity can be proved similarly to the proof of the estimate $\mathfrak{C} \gtrsim \sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u)$. Namely, it is should be assumed first that inequality (3.11) is valid. Next, one should fix $n_* \in \mathbb{N}$ and $m_* \in \mathbb{N}$ satisfying $n_* = m_* + \alpha$ and condition (3.3) with respect to the parameters of the spaces $B_{pq}^{s, u}(\mathbb{R})$ and $B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})$, respectively. Then, by Theorem 3.1, representation (3.20) is valid with $\tilde{\lambda}_{(I_+^\alpha f)}^{(\tilde{a})}$ of form (3.21), where $\tilde{\Phi}^{(\tilde{a})}$ are defined by $B_{n_*, \tilde{a}}$, and $\tilde{\Psi}^{(\tilde{a})}$ (see (5.13)) — via $\Psi_{n_*, \tilde{a}, 0}$, with some $\tilde{a} \in \{0, \pm 1/2\}$. Thus, representation (3.22) takes place for the right-hand side of (3.11).

Similarly, also basing on Theorem 3.1, we get

$$\|f\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})} \approx \|\lambda_f^{(a)}\|_{b_{pq}^{s-\kappa_*-\alpha}(w)},$$

where the sequence $\lambda_f^{(a)}$ with $a \in \{0, \pm 1/2\}$ is defined in (3.23) with $\Phi^{(a)}$ and $\Psi^{(a)}$ (see (5.14)), specified by $B_{m_*, a}$ and $\Psi_{m_*, a, 0}$. That is, for the left-hand side of (3.11) the following relation holds:

$$\|f\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})} \approx \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{0\tau}^{(a)}| \chi_{Q_{0\tau}} \right\|_{L^p(\mathbb{R}, w)} + \left(\sum_{d \in \mathbb{N}} 2^{qd(s-\kappa_*-\alpha)} \left\| \sum_{\tau \in \mathbb{Z}} |\lambda_{d\tau}^{(a)}| \chi_{Q_{(d-1)\tau}} \right\|_{L^p(\mathbb{R}, w)}^q \right)^{\frac{1}{q}}. \quad (3.42)$$

Next, for arbitrary d_0 and τ_0 , the function $h_{*, \tau_0} = h_{\tau_0}^*$ is introduced into consideration (see (3.34)). In this case (see (3.35)),

$$\begin{aligned} \gamma_{n^*} 2^{\alpha d_0} I_+^\alpha h_{\tau_0}^*(x) &= \Psi_{n^*, \bar{a}, \bar{s}; m^*(1), \alpha(0)}(2^{d_0-1}y - \tau_0) \\ &= \sum_{|l| \leq m^*} \lambda_{|l|}(m^*) (-1)^{|l|} \Psi_{n^*, \bar{a}, \bar{s}}(2^{d_0-1}y - \tau_0 - l/2) =: \sum_{|l| \leq m^*} \lambda_{|l|}(m^*) (-1)^{|l|} h_{*, t_0, l}(y). \end{aligned}$$

We will use representation (3.42) for $f = h_{*, \tau_0}$ and (3.22) for $h_{*, t_0, l}$ with $|l| \leq m^*$ instead of $I_+^\alpha f$. First, we estimate the left-hand side of (3.11). To do this, we write (see (3.42))

$$\begin{aligned} \|h_{*, \tau_0}\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})}^p &\gtrsim 2^{d_0(s-\kappa_*-\alpha)p} |\lambda_{d_0\tau_0}^{(a)}|^p [w(Q_{(d_0-1)\tau_0})] \\ &= 2^{d_0(s-\kappa_*-\alpha)p} 2^{d_0 p/2} |\langle h_{*, \tau_0}, \Psi_{(d_0-1)\tau_0}^{(a)} \rangle|^p [w(Q_{(d_0-1)\tau_0})] \\ &= 2^{d_0(s-\kappa_*-\alpha)p} 2^{d_0 p} |\langle \Psi_{m^*, \bar{a}, \bar{s}; n^*(1), \alpha(1)}(2^{d_0-1} \cdot - \tau_0), \Psi_{m^*, a, 0}(2^{d_0-1} \cdot - \tau_0) \rangle|^p [w(Q_{(d_0-1)\tau_0})]. \end{aligned}$$

Just as in (3.37), it suffices to fix here $\bar{s}$ in a way that allows to reduce the intersection of the supports of $\Psi_{m^*, \bar{a}, \bar{s}; n^*(1), \alpha(1)}(2^{d_0-1} \cdot - \tau_0)$ and $\Psi_{m^*, a, 0}$ with $\bar{a} = a = 0$ to $\mathbf{B}_{m^*}$ -intersection of the supports of the rightmost B_{m^*} in the formula for $\Psi_{m^*, \bar{a}, \bar{s}; n^*(1), \alpha(1)}(2^{d_0-1} \cdot - \tau_0)$ and the leftmost B_{m^*} in representation (3.42) using $\Psi_{m^*, a, 0}$. This will lead us to the estimate

$$\|h_{*, \tau_0}\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})} \gtrsim 2^{d_0(s-\kappa_*-\alpha)} [w(Q_{(d_0-1)\tau_0})]^{\frac{1}{p}} \Theta(m^*). \quad (3.43)$$

On the other hand, we will estimate $\|I_+^\alpha h_{*, \tau_0}\|_{B_{pq}^{s, u}(\mathbb{R})}^p$ from above via $2^{(s-\alpha)d_0} \Lambda_{n^*}^2$ by analogy with (3.39) and (3.40). Together with (3.43) this will lead us to the inequality

$$C \gtrsim 2^{-d_0 \kappa_*} [u(Q_{(d_0-1)\tau_0})]^{-\frac{1}{p}} [w(Q_{(d_0-1)\tau_0})]^{\frac{1}{p}}, \quad (3.44)$$

and, hence, to the required lower bound for the best constant C in inequality (3.11). $\square$

For the operators $I_{c\pm}^\alpha$ with $c \in \mathbb{R}$ the following analogue of Theorem 3.4 is true.

Theorem 3.5. *Let $p > 1$, $0 < q \leq \infty$, $s \in \mathbb{R}$, weights $u, v, w \in \mathcal{A}_\infty^{\text{loc}}$, $\alpha \in \mathbb{N}$ and $f \in L_{\text{loc}}(\mathbb{R})$. Suppose that $f \equiv 0$ on $(-\infty, c)$ in the case of the operator I_{c+}^α and $f \equiv 0$ on (c, ∞) for I_{c-}^α .*

(i) $I_{c\pm}^\alpha$ is bounded from $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ to $B_{pq}^{s, u}(\mathbb{R})$ with some $\kappa^* \in \mathbb{R}$ if and only if

$$\widetilde{\mathfrak{C}}_\pm^\alpha(\kappa^*, c) = \widetilde{\mathfrak{M}}_\pm^\alpha(1, c) + \widetilde{\mathfrak{M}}_\pm^\alpha(0, c) + \sup_{d \in \mathbb{N}} \widetilde{\mathfrak{M}}_\pm(d, \kappa^*, c; v, u) < \infty,$$

where

$$\widetilde{\mathfrak{M}}_\pm(d, \kappa, c; v, u) := \sup_{\tau \in \pm\mathbb{N}_0} 2^{-d\kappa} [v(Q_{(d-1)\tau}^{(c)})]^{-\frac{1}{p}} [u(Q_{(d-1)\tau}^{(c)})]^\frac{1}{p}.$$

Moreover, the norm $\|I_{c\pm}^\alpha\|_{B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R}) \rightarrow B_{pq}^{s, u}(\mathbb{R})}$ of the operator $I_{c\pm}^\alpha$ is equivalent to $\widetilde{\mathfrak{C}}_\pm^\alpha(\kappa^*, c)$ with constants depending only on the numerical parameters.

(ii) Let $\kappa_* \in \mathbb{R}$ and assume that $0 < \widetilde{\mathfrak{C}}$ is the best and independent of f constant in the inequality

$$\|f\|_{B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})} \leq \widetilde{\mathfrak{C}} \|I_{c\pm}^\alpha f\|_{B_{pq}^{s, u}(\mathbb{R})}, \quad (3.45)$$

where $I_{c\pm}^\alpha f \in B_{pq}^{s, u}(\mathbb{R})$. Inequality (3.45) holds if and only if

$$\widetilde{\mathfrak{C}}(\kappa_*) := \sup_{d \in \mathbb{N}_0} \widetilde{\mathfrak{M}}_\pm(d, \kappa_*, c; u, w) < \infty,$$

where

$$\widetilde{\mathfrak{M}}_\pm(d, \kappa, c; u, w) := \sup_{\tau \in \pm\mathbb{N}_0} \begin{cases} [u(Q_{0\tau}^{(c)})]^{-\frac{1}{p}} [w(Q_{0\tau}^{(c)})]^\frac{1}{p}, & d = 0, \\ 2^{-d\kappa} [u(Q_{(d-1)\tau}^{(c)})]^{-\frac{1}{p}} [w(Q_{(d-1)\tau}^{(c)})]^\frac{1}{p}, & d \in \mathbb{N}. \end{cases}$$

Moreover, $\widetilde{\mathfrak{C}} \approx \widetilde{\mathfrak{C}}(\kappa_*)$.

Proof. Specifics of the proof can be found in [32, § 6]. □

Remark 1. The characteristics found in Theorem 1.1 (i) differ from those presented in Theorem 3.4 (i) by the supreme components over $d \in \mathbb{N}$, namely,

$$\sup_{d \in \mathbb{N}} [\mathcal{M}_\pm^\alpha(d, \kappa^*, 1) + \mathcal{M}_\pm^\alpha(d, \kappa^*, 0)] \geq \sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u).$$

The characteristics obtained in part (ii) of Theorem 1.1 are also redundant compared with the new sharp ones established in Theorem 3.4(ii) since $\mathbb{C}_\pm^\alpha(\kappa_*) \geq \mathfrak{C}(\kappa_*)$.

The same is true for the results of Theorems 1.2 and 3.5.

At the end of this Section, let us give an example.

Example 1. Let $u(x) = w(x) = (1 + |x|)^{-t}$ and $v(x) = (1 + |x|)^{-t+sp}$ with $t > \max\{1, p\alpha - (p-1)\}$ and $\min\{1, \alpha\} < \alpha \leq s$. For the operator I_{0+}^α acting from the space $B_{pq}^{s-\alpha, v}(\mathbb{R})$ to the space $B_{pq}^{s, u}(\mathbb{R})$ we have the following estimates:

$$\widetilde{\mathfrak{M}}_+(0, 0) \lesssim \begin{cases} \sup_{\tau \in \mathbb{N}_0} (1 + \tau)^{-\frac{t}{p} + \alpha}, & \alpha p \leq t < sp, \\ \sup_{\tau \in \mathbb{N}_0} (1 + \tau)^{-s+1}, & t \geq sp \end{cases}$$

and

$$\widetilde{\mathfrak{M}}_+(1, 0) \lesssim \begin{cases} \sup_{\tau \in \mathbb{N}_0} (1 + \tau)^{-\frac{t}{p} + \alpha - \frac{1}{p'}}, & t < sp, \\ \sup_{\tau \in \mathbb{N}_0} (1 + \tau)^{-s+\alpha}, & t \geq sp, \end{cases}$$

in particular, the quantities $\widetilde{\mathfrak{M}}_+^\alpha(i, 0)$, $i = 0, 1$, are finite. The condition

$$\widetilde{\mathfrak{M}}_+(d, 0, 0; v, u) < \infty$$

is also satisfied and $\widetilde{\mathfrak{M}}_+(d, 0, 0; u, w) < \infty$ (see also [14]). Therefore, under the assumed conditions and by virtue of Theorem 3.5, the operator I_{0+}^α is bounded from $B_{pq}^{s-\alpha, v}(\mathbb{R})$ to $B_{pq}^{s, u}(\mathbb{R})$. Moreover, the norm inequality of form (3.45) corresponding to the embedding $B_{pq}^{s, u}(\mathbb{R}) \rightarrow B_{pq}^{s-\alpha, w}(\mathbb{R})$ holds (since $\widetilde{\mathfrak{M}}_+(d, 0, 0; u, w) < \infty$).

For I_+^α , assume that $t + (p - 1) + p(\alpha - 1) < sp$ and $\alpha p < t$. Then,

$$\mathbf{M}_+^\alpha(1) \lesssim \sup_{\tau \in \mathbb{N}} (1 + \tau)^{-\frac{t}{p} + \alpha - \frac{1}{p'}} + \sup_{\tau \in -\mathbb{N}_0} (1 - \tau)^{\frac{t}{p} - s + \alpha - \frac{1}{p}} < \infty,$$

$$\mathbf{M}_+^\alpha(0) \lesssim \sup_{\tau \in \mathbb{N}} (1 + \tau)^{-\frac{t}{p} + \alpha} + \sup_{\tau \in -\mathbb{N}_0} (1 - \tau)^0 < \infty.$$

Moreover, $\mathfrak{M}(d, 0; v, u) < \infty$ and $\mathfrak{M}(d, 0; u, w) < \infty$ for all $d \in \mathbb{N}_0$. Hence, by Theorem 3.4, the operator I_+^α is bounded from $B_{pq}^{s-\alpha, v}(\mathbb{R})$ to $B_{pq}^{s, u}(\mathbb{R})$ and inequality (3.11) holds with $\kappa_* = 0$.

For the estimates in Example 1, the «power rule» for sums was applied (see [4, Lemma 1]).

4 Estimates under additional conditions on weights

To simplify the quantitative characteristics obtained in Theorems 3.4 and 3.5, we can consider them on certain subclasses of weight functions.

4.1 Weights with homogeneous anti-derivatives

The parameters κ_* and κ^* in the definitions of the function spaces $B_{pq}^{s+\kappa^*-\alpha, v}(\mathbb{R})$ and $B_{pq}^{s-\kappa_*-\alpha, w}(\mathbb{R})$ (see Theorems 3.4 and 3.5) depend on the weight functions v and w , as well as on the weight u of $B_{pq}^{s, u}(\mathbb{R})$. They can be refined by narrowing the functionals $\sup_{d \in \mathbb{N}} \mathfrak{M}(d, \kappa^*; v, u)$ and $\sup_{d \in \mathbb{N}_0} \mathfrak{M}(d, \kappa_*; u, w)$ in Theorem 3.4, as well as $\sup_{d \in \mathbb{N}} \widetilde{\mathfrak{M}}_\pm(d, \kappa^*; c; v, u)$ and $\sup_{d \in \mathbb{N}_0} \widetilde{\mathfrak{M}}_\pm(d, \kappa_*; c; u, w)$ to certain subclasses of weights.

Consider the collection of the so-called positively homogeneous weights. Recall that a function $h: \mathbb{R} \rightarrow \mathbb{R}$ is positively homogeneous of degree $s(h)$ if for any $x \in \mathbb{R}$ in the function's domain and any $\lambda > 0$

$$h(\lambda x) = \lambda^{s_h} h(x).$$

The number s_h is called the *order of homogeneity*.

In our situation, the appropriate values of the parameter λ are 2^{-d} , $d \in \mathbb{N}$. Given that the anti-derivative of each weight σ_i , $i = 1, 2$, is a homogeneous function Σ_i of degree s_{Σ_i} , one can transform the functional $\widetilde{\mathfrak{M}}_\pm(d, \kappa_*, c; \sigma_1, \sigma_2)$ to a more transparent form in the case of $d \in \mathbb{N}$. Namely,

$$\widetilde{\mathfrak{M}}_\pm(d, \kappa, c; \sigma_1, \sigma_2) = 2^{-d\kappa} 2^{\frac{d-1}{p}(s_{\Sigma_1} - s_{\Sigma_2})} \widetilde{\mathfrak{M}}_\pm(0, \kappa, c; \sigma_1, \sigma_2).$$

Assuming $\kappa = (s_{\Sigma_1} - s_{\Sigma_2})/p$, we obtain that

$$\sup_{d \in \mathbb{N}_0} \widetilde{\mathfrak{M}}_\pm(d, \kappa, c; \sigma_1, \sigma_2) \simeq \widetilde{\mathfrak{M}}_\pm(0, \kappa, c; \sigma_1, \sigma_2).$$

Examples are $\Sigma_i(x) \simeq |x|^\zeta$ on $|x| \geq |c|$ for I_{c+}^α with $c > 0$ or I_{c-}^α with $c < 0$.

To simplify the functionals of the type $\mathfrak{M}_\pm$, one can also use direct (2.2) and/or inverse (2.3) doubling conditions for the Muckenhoupt-type functions in combination with Lemma 2.1.

4.2 Usage of an average type condition

The functionals $\mathbf{M}_\pm^\alpha(\varepsilon)$, as well as $\widetilde{\mathbf{M}}_\pm^\alpha(\varepsilon, c)$, $\varepsilon = 0, 1$, in assertions (i) of Theorems 3.4 and 3.5 can be reduced to an integral form if weights σ satisfy the condition

$$\int_r^{r+1} \sigma \approx \sigma(\tau) \quad \text{for some } \tau \in [r, r+1] \quad (4.1)$$

(see [30, 34]). The result, if $\sigma = u$ and $\sigma = v$ are of type (4.1), is as follows:

$$\begin{aligned} \mathbf{M}_+^\theta(\varepsilon) &\approx \sup_{\tau \in \mathbb{Z}} \left(\int_{\tau}^{\infty} \frac{u(x) dx}{(x - \tau + 1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\tau} \frac{[v(y)]^{1-p'} dy}{(\tau - y + 1)^{-p'(\theta-1)(1-\varepsilon)}} \right)^{\frac{1}{p'}}, \\ \mathbf{M}_-^\theta(\varepsilon) &\approx \sup_{\tau \in \mathbb{Z}} \left(\int_{-\infty}^{\tau} \frac{u(x) dx}{(\tau - x + 1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\int_{\tau}^{\infty} \frac{[v(y)]^{1-p'} dy}{(y - \tau + 1)^{-p'(\theta-1)(1-\varepsilon)}} \right)^{\frac{1}{p'}}, \\ \widetilde{\mathbf{M}}_+^\theta(\varepsilon) &\approx \sup_{\tau \in \mathbb{N}_0} \left(\int_{\tau+c}^{\infty} \frac{u(x) dx}{(x - \tau - c + 1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\int_c^{\tau+c} \frac{[v(y)]^{1-p'} dy}{(\tau + c - y + 1)^{-p'(\theta-1)(1-\varepsilon)}} \right)^{\frac{1}{p'}}, \\ \mathbf{M}_-^\theta(\varepsilon) &\approx \sup_{\tau \in -\mathbb{N}_0} \left(\int_{-\infty}^{\tau+c} \frac{u(x) dx}{(\tau + c - x + 1)^{-p(\theta-1)\varepsilon}} \right)^{\frac{1}{p}} \left(\int_{\tau+c}^c \frac{[v(y)]^{1-p'} dy}{(y - \tau - c + 1)^{-p'(\theta-1)(1-\varepsilon)}} \right)^{\frac{1}{p'}}. \end{aligned}$$

5 Appendix

Here we give precise definitions of elements of the Battle–Lemarié type wavelet systems, in terms of which decomposition theorems in the Besov spaces $B_{pq}^{s,w}(\mathbb{R})$ are given in §3.1.

We start with defining a basic spline of order $n \in \mathbb{N}_0$: namely, $B_0 = \chi_{[0,1]}$ and

$$B_n(x) := (B_{n-1} * B_0)(x) = \int_0^1 B_{n-1}(x-t) dt.$$

The key property of B_n in our case is the property of its derivative $B_n^{(k)}$ of order k :

$$B_n^{(k)}(\cdot) = \sum_{l=0}^k \binom{k}{l} B_{n-k}(\cdot - l) \quad (n \geq k \in \mathbb{N}). \quad (5.1)$$

Other properties of B_n can be seen, for example, in [1] and [32].

The B_n is a scaling function, i.e. it generates in $L^2(\mathbb{R})$ the so-called multiresolution analysis KMA_{B_n} in the form of expanding subspaces $V_d \subset V_{d+1}$, $d \in \mathbb{N}_0$, such that $\text{clos}_{L^2(\mathbb{R})}(\bigcup_{d \in \mathbb{Z}} V_d) = L^2(\mathbb{R})$ and $\bigcap_{d \in \mathbb{Z}} V_d = \{0\}$. Moreover, for each $d \in \mathbb{N}_0$ the system

$$\{B_n(2^d \cdot -\tau)\}_{\tau \in \mathbb{Z}}$$

of integer shifts of dyadic dilations of B-spline B_n , which has compact support $[0, n+1]$, forms a non-orthogonal but Riesz basis in $V_d \subset L^2(\mathbb{R})$.

Let W_d denote the orthogonal complement of V_d to V_{d+1} , i.e. $V_{d+1} = V_d \oplus W_d$ for all $d \in \mathbb{N}$. Then, $W_i \perp W_j$ for $i \neq j$. From this, in particular, we obtain that

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{l \geq 0} W_l \right).$$

The spaces W_d , $d \in \mathbb{Z}$, are called *wavelet spaces*. Just as the spaces V_d are generated by integer shifts of dyadic dilations of the scaling function B_n , the spaces W_d are generated by basis functions–wavelets, constructed from B_n . The multiresolution analysis KMA_ϕ with $\phi = B_n$ allows one to construct a wavelet Riesz basis in $L^2(\mathbb{R})$ using integer shifts of B_n and the same type of shifts of dyadic dilations of the corresponding B_n wavelets. Within the framework of KMA_{B_n} , one can move from the scaling function B_n to another, orthonormal one, satisfying the relation

$$\hat{\phi}_n(\omega) = \frac{\beta_n \hat{B}_n(\omega)}{(e^{i\omega} r_1 + 1) \dots (e^{i\omega} r_n + 1)} = \beta_n \sum_{l_1=0}^{\infty} (-r_1 e^{i\omega})^{l_1} \dots \sum_{l_n=0}^{\infty} (-r_n e^{i\omega})^{l_n} \hat{B}_n(\omega). \quad (5.2)$$

Here $\beta_n := (r_1 + 1) \dots (r_n + 1)$, where $\{r_j\}_{j=1}^n \subset (0, 1)$ are the roots of the Euler trigonometric polynomial (see, for example, [33]). The function ϕ_n is named after G. Battle [2, 3] and P.G. Lemarié–Rieusset [11]. One can write (see e.g. [15, 33, 32])

$$\phi_n(x) = \beta_n \sum_{l_1=0}^{\infty} (-r_1)^{l_1} \dots \sum_{l_n=0}^{\infty} (-r_n)^{l_n} B_n(x + l_1 \dots + l_n).$$

The Battle–Lemarié wavelets ψ_n corresponding to ϕ_n satisfy the following condition (see [33, 32]):

$$\begin{aligned} \hat{\psi}_{n,\mathbf{s}}(\omega) &= \frac{\gamma_n}{2e^{i\omega\mathbf{s}}} (1/r_1 - e^{-i\omega/2}) \dots (1/r_n - e^{-i\omega/2}) \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} e^{(n-k)i\omega/2} \\ &\times \prod_{j=1}^n \sum_{m_j=0}^{\infty} (-r_j e^{-i\omega})^{m_j} \sum_{l_j=0}^{\infty} (-r_j e^{i\omega/2})^{l_j} \hat{B}_n(\omega/2) \quad (\mathbf{s} \in \mathbb{Z}), \end{aligned} \quad (5.3)$$

where $\gamma_n = 2^{-n} \beta_n (r_1 \dots r_n)$ and $\mathbf{s}$ is fixed. Their integer shifts form an orthonormal basis in $W_0 \subset L^2(\mathbb{R})$. By analogy with ϕ_n , one can write:

$$\begin{aligned} \psi_{n,\mathbf{s}}(\cdot + \mathbf{s}) &= \gamma_n \sum_{m_1=0}^{\infty} (-r_1)^{m_1} \sum_{l_1=0}^{\infty} (-r_1)^{l_1} \dots \sum_{m_n=0}^{\infty} (-r_n)^{m_n} \sum_{l_n=0}^{\infty} (-r_n)^{l_n} \\ &\quad \left[\sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} B_n(2 \cdot -k - 2m_1 + l_1 - \dots - 2m_n + l_n) \right. \\ &\quad - \left\{ \sum_{j_1 \in \{1, \dots, n\}} \frac{1}{r_{j_1}} \right\} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} B_n(2 \cdot -k + 1 - 2m_1 + l_1 - \dots - 2m_n + l_n) + \\ &\quad \left\{ \sum_{\substack{j_1, j_2 \in \{1, \dots, n\} \\ j_1 \neq j_2}} \frac{1}{r_{j_1} r_{j_2}} \right\} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} B_n(2 \cdot -k + 2 - 2m_1 + l_1 - \dots - 2m_n + l_n) + \dots \\ &\quad \dots + \left\{ \sum_{\substack{j_1, \dots, j_c \in \{1, \dots, n\} \\ j_1 \neq \dots \neq j_c}} \frac{(-1)^c}{r_{j_1} \dots r_{j_c}} \right\} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} B_n(2 \cdot -k + c - 2m_1 + l_1 - \dots - 2m_n + l_n) + \dots \\ &\quad \left. + \dots + \frac{(-1)^n}{r_1 \dots r_n} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} B_n(2 \cdot + (n-k) - 2m_1 + l_1 - \dots - 2m_n + l_n) \right]. \end{aligned}$$

The Fourier transform of $\psi_{n,\mathbf{s}}$ in (5.3) can be represented as (see details in [32], (2.31))

$$\begin{aligned} \hat{\psi}_{n,\mathbf{s}}(\omega) &= \frac{\gamma_n}{2e^{i\omega\mathbf{s}}} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} e^{(n-k)i\omega/2} \\ &\times \prod_{j=1}^n \frac{|1 - e^{i\omega/2} r_j|^2}{r_j} \prod_{j=1}^n \sum_{m_j=0}^{\infty} (-r_j e^{-i\omega})^{m_j} \sum_{l_j=0}^{\infty} (r_j^2 e^{i\omega})^{l_j} \hat{B}_n(\omega/2). \end{aligned}$$

The orthonormal wavelet basis in $L^2(\mathbb{R})$ generated by the pair $\{\phi_n, \psi_{n,\mathbf{s}}\}$ is formed as follows: it is $\sqrt{2} \phi_n(x - \tau)$ and $2^{d-1} \psi_{n,\mathbf{s}}(2^{d-1} x - \tau)$ for all $\tau \in \mathbb{Z}$ and $d \in \mathbb{N}_0$.

Instead of B_n one can take the function $B_{n,\mathbf{a}}(\cdot) := B_n(\cdot - \mathbf{a})$ with $\mathbf{a} \in \mathbb{R}$. This would form another $KMA_{B_n,\mathbf{a}}$ and, generally speaking, some other system $\{\phi_{n,\mathbf{a}}, \psi_{n,\mathbf{a},\mathbf{s}}\}$, generating an orthonormal basis of Battle–Lemarié wavelets in $L^2(\mathbb{R})$.

Since $\phi_{n,\mathbf{a}}$ and $\psi_{n,\mathbf{a},\mathbf{s}}$ have unbounded supports, in [29], with the aim of effective using spline systems of this type in decomposition theorems, the so-called localisation algorithm was proposed. It consists in forming, within the framework of $KMA_{B_n,\mathbf{a}}$, new basis elements which are finite linear combinations of integer shifts of the scaling or wavelet Battle–Lemarié functions.

As a localised analogue of $\phi_{n,\mathbf{a}}$ we use the function $\Phi_{n,\mathbf{a}} = B_{n,\mathbf{a}}$, which turns out to be a linear combination of $n + 1$ integer shifts of $\phi_{n,\mathbf{a}}$. Indeed, by (5.2),

$$\hat{\phi}_{n,\mathbf{a}}(\omega) \prod_{j=1}^n \frac{1}{r_j + 1} (e^{i\omega} r_j + 1) = \hat{B}_{n,\mathbf{a}}(\omega) =: \hat{\Phi}_{n,\mathbf{a}}(\omega). \quad (5.4)$$

Therefore, we have

$$\text{supp } \Phi_{n,\mathbf{a}} = [\mathbf{a}, \mathbf{a} + n + 1] \quad \text{and} \quad \Phi_{n,\mathbf{a}}(\cdot) = \beta_n^{-1} \sum_{\varkappa=-n}^0 \alpha'_{\varkappa} \cdot \phi_{n,\mathbf{a}}(\cdot - \varkappa), \quad (5.5)$$

where $\alpha'_{\varkappa} > 0$ for all $\varkappa = -n, \dots, 0$ and $\sum_{\varkappa=-n}^0 \alpha'_{\varkappa} = \beta_n$. Recall that integer shifts of $\phi_{n,\mathbf{a}}$ in (5.5) are mutually orthogonal, moreover, $0 < r_j < 1$ for all $j = 1, \dots, n$ in (5.4).

As a localised analogue of $\psi_{n,\mathbf{a},\mathbf{s}}$, we consider the function $\Psi_{n,\mathbf{a},\mathbf{s}}$ satisfying

$$\begin{aligned} \hat{\Psi}_{n,\mathbf{a},\mathbf{s}}(\omega) &= \gamma_n^{-1} \hat{\psi}_{n,\mathbf{a},\mathbf{s}}(\omega) \prod_{j=1}^n (1 + e^{-i\omega} r_j) (1 - e^{i\omega} r_j^2) \\ &= \frac{\hat{B}_{n,\mathbf{a}}(\omega/2)}{2e^{i\omega\mathbf{s}}} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} e^{(n-k)i\omega/2} \prod_{j=1}^n \frac{|1 - e^{i\omega/2} r_j|^2}{r_j} \\ &= \frac{\hat{B}_{n,\mathbf{a}}(\omega/2)}{2e^{i\omega\mathbf{s}}} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{k!(n+1-k)!} e^{(n-k)i\omega/2} \prod_{j=1}^n [\rho_j - 2 \cos(j\omega/2)], \end{aligned} \quad (5.6)$$

where $\rho_j := r_j + 1/r_j = 2(2\alpha_j - 1)$ with $\alpha_j > 1$, $j = 1, \dots, n$ (see details in [32]). From this, on one hand, it is clear that $\Psi_{n,\mathbf{a},\mathbf{s}}$ is a linear combination of $2n + 1$ mutually orthogonal integer shifts of $\psi_{n,\mathbf{s},\mathbf{a}}$. On the other hand, $\Psi_{n,\mathbf{a},\mathbf{s}}$ is a finite linear combination of integer and a half shifts of $B_{n,\mathbf{a}}(2\cdot)$. Since all r_j lie inside $(0, 1)$, the system $\{\Psi_{n,\mathbf{a},\mathbf{s}}(\cdot - \tau)\}_{\tau \in \mathbb{Z}}$ forms a Riesz basis in $W_0(B_{n,\mathbf{a}})$. Also observe that

$$\text{supp } \Psi_{n,\mathbf{a},\mathbf{s}} = [\mathbf{s} + \mathbf{a} - n, \mathbf{s} + \mathbf{a} + n + 1] \quad \text{and} \quad \Psi_{n,\mathbf{a},\mathbf{s}}(\cdot) = \gamma_n^{-1} \sum_{|\varkappa| \leq n} \alpha''_{\varkappa} \cdot \psi_{n,\mathbf{a},\mathbf{s}}(\cdot - \varkappa), \quad (5.7)$$

where

$$\gamma_n^{-1} \sum_{|\varkappa| \leq n} \alpha''_{\varkappa} = 2^n \prod_{j=1}^n \frac{(1 + r_j)(1 - r_j^2)}{r_j(1 + r_j)} = 2^n \prod_{j=1}^n (1/r_j - r_j) := \Lambda_n > 0.$$

To formulate decomposition theorems, we also need the functions of the form

$$\hat{\Psi}_{n,\mathbf{a},\mathbf{s};m(\mathbb{k}),\alpha(\zeta)}(\omega) = \hat{\Psi}_{n,\mathbf{a},\mathbf{s}}(\omega) \left[\prod_{j=1}^m [\rho_j(m) - 2 \cos(\omega/2)] \right]^{\mathbb{k}} \left[\sum_{\iota=0}^{2\alpha} \frac{(-1)^{\iota} (2\alpha)!}{\iota! (2\alpha - \iota)!} e^{(\alpha - \iota)i\omega/2} \right]^{\zeta},$$

where $\mathbb{k}, \zeta \in \{0, 1\}$. Due to the fact that

$$r_j(m) [\rho_j(m) - 2 \cos(\omega/2)] = |1 - e^{i\omega/2} r_j(m)|^2$$

and

$$\sum_{\iota=0}^{2\alpha} \frac{(-1)^{\iota} (2\alpha)!}{\iota! (2\alpha - \iota)!} e^{(\alpha - \iota)i\omega/2} = e^{\alpha i\omega/2} (1 - e^{-i\omega/2})^{2\alpha},$$

the element $\Psi_{n,\mathbf{a},\mathbf{s};m(1),\alpha(\zeta)}$, $\zeta = 0, 1$, is a linear combination of integer and a half shifts of $\Psi_{n,\mathbf{a},\mathbf{s}}$. Observe that

$$\text{supp } \Psi_{n,\mathbf{a},\mathbf{s};m(\mathbb{k}),\alpha(\zeta)} = [\mathbf{s} + \mathbf{a} - n - m\mathbb{k}/2 - \alpha\zeta/2, \mathbf{s} + \mathbf{a} + n + 1 + m\mathbb{k}/2 + \alpha\zeta/2] \quad (5.8)$$

and

$$\Psi_{n,\mathbf{a},\mathbf{s};m(1),\alpha(0)}(\cdot) = \sum_{|\varkappa| \leq m} \lambda_{|\varkappa|}(m) (-1)^{|\varkappa|} \Psi_{n,\mathbf{a},\mathbf{s}}(\cdot - \varkappa/2), \quad (5.9)$$

$$\Psi_{n,a,s;m(1),\alpha(1)}(\cdot) = \sum_{|\varkappa| \leq m} \lambda_{|\varkappa|}(m) (-1)^{|\varkappa|} \sum_{\zeta=0}^{2\alpha} \binom{2\alpha}{\zeta} (-1)^\zeta \Psi_{n,a,s}(\cdot - \zeta/2 + \alpha/2 - \varkappa/2)$$

with $\sum_{|\varkappa| \leq m} \lambda_{|\varkappa|}(m) (-1)^{|\varkappa|} = \prod_{j=1}^m \frac{1 - r_j^2(m)}{r_j(m)} = 2^{-m} \Lambda_m,$ (5.10)

(see [32, p. 18]). Moreover,

$$\prod_{j=1}^m [(r_j + 1/r_j) - 2 \cos(\omega/2)] =: \sum_{j=0}^m (-1)^j \lambda_j \cos(j\omega/2), \quad (5.11)$$

where $\lambda_m = 2$ and $0 < \lambda_j = \lambda(\rho_1, \dots, \rho_m)$ in the case $j \neq m$, and λ_j are even for $j \neq 0$. For completeness, let us also write down that (see (5.6))

$$\Psi_{n,a,s}(\cdot) = \gamma_n \sum_{|s| \leq n} \lambda_{|s|}(n) (-1)^{|s|} \sum_{k=0}^{n+1} (-1)^k \binom{1+n}{k} B_n(2 \cdot + (n-k) - s). \quad (5.12)$$

In conclusion, for $a \in \mathbb{R}$ we define pairs of functions of the form

$$\tilde{\Phi} = \Phi_{n,a} \quad \text{and} \quad \tilde{\Psi}^{(\mathbb{k}, \zeta)} = \Psi_{n,a,s;m(\mathbb{k}),\alpha(\zeta)} / \Lambda_n, \quad \text{where } \mathbb{k}, \zeta \in \{0, 1\}; \quad (5.13)$$

$$\Phi^{(a)} = \Phi_{n,a} \quad \text{and} \quad \Psi^{(a)} = \Psi_{n,a,s} / \Lambda_n. \quad (5.14)$$

In (5.13) and (5.14) the first components $\tilde{\Phi}$ and $\Phi^{(a)}$ (the scaling functions or, in other words, the father wavelets) are identical to each other. We denote them differently only to distinguish from each other the wavelet systems $\{\tilde{\Phi}, \tilde{\Psi}^{(\mathbb{k}, \zeta)}\}$ and $\{\Phi^{(a)}, \Psi^{(a)}\}$. The difference in the systems is in the second components $\tilde{\Psi}^{(\mathbb{k}, \zeta)}$ and $\Psi^{(a)}$ or the wavelet functions (the mother wavelets). Here $\tilde{\Psi}^{(\mathbb{k}, \zeta)}$ is a more general function, which, however, coincides with $\Psi^{(a)}$ if $\mathbb{k} = \zeta = 0$.

The order of smoothness of the functions in (5.13) and (5.14) is n . For $x \in \mathbb{R}$ we denote:

$$\tilde{\Phi}_\tau(x) := \tilde{\Phi}(x - \tau) \quad \text{and} \quad \tilde{\Psi}_{d\tau}^{(\mathbb{k}, \zeta)}(x) := 2^{d/2} \tilde{\Psi}^{(\mathbb{k}, \zeta)}(2^d x - \tau) \quad (\tau \in \mathbb{Z}, d \in \mathbb{N}_0), \quad (5.15)$$

and also

$$\Phi_\tau^{(a)}(x) := \Phi^{(a)}(x - \tau) \quad \text{and} \quad \Psi_{d\tau}^{(a)}(x) := 2^{d/2} \Psi^{(a)}(2^d x - \tau) \quad (\tau \in \mathbb{Z}, d \in \mathbb{N}_0). \quad (5.16)$$

As noted in [32, Remark 2.4], the construction of $\Psi_{n,a,s;m(\mathbb{k}),\alpha(\zeta)}(\cdot - \tau)$, $\mathbb{k}, \zeta \in \{0, 1\}$, starts with $B_{n,2(s+a+\tau)}(\cdot)$. As for observations (5.7) and (5.8) concerning the supports of $\Psi_{n,a,s;m(\mathbb{k}),\alpha(\zeta)}$, they are slightly tweaked in this paper in comparison with the information presented in [32, § 2].

Systems (5.16), as well as (5.15), if they are generated by elements (5.13) with $\mathbb{k} = \zeta = 0$, form semi-orthogonal Riesz bases in $L^2(\mathbb{R})$ (i.e. elements from V_0 are orthogonal to W_l with $l \geq 0$, and functions of the space W_{l_0} with fixed $l_0 \in \mathbb{N}_0$ are orthogonal to W_l if $l \neq l_0$). The special structure of the elements of such systems allows them to be used for obtaining lower bounds (or proofs of necessity) in decomposition theorems (see §3.1). Upper bounds in such theorems (or proofs of sufficiency) are less demanding and allow representation with respect to combined elements of type (5.15), when they are generated by functions (5.13) with $\mathbb{k} = 1$ and/or $\zeta = 1$.

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Elena Pavlovna Ushakova

V.A. Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences
65 Profsoyuznaya St, Moscow 117997, Russian Federation,

Steklov Mathematical Institute of the Russian Academy of Sciences
8 Gubkin St, Moscow 119991, Russian Federation

and

Computing Center of the Far Eastern Branch of the Russian Academy of Sciences
65 Kim Yu Chen St, Khabarovsk 680000, Russian Federation.
E-mail: elenau@inbox.ru

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