

GROUP HOMOMORPHISMS OF SMALL COHOMOLOGICAL DIMENSION

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Communicated by U.U. Umirbaev

Key words: cohomological dimension, Lusternik-Schirelmann category, geometric dimension, group homomorphisms.

AMS Mathematics Subject Classification: 20J06, 55M30.

Abstract. We characterize group epimorphisms of cohomological dimension one under the additional assumption that their Lusternik–Schirelmann category is one. We further establish an Eilenberg–Ganea type theorem for epimorphisms between one-relator groups. Finally, we construct examples of epimorphisms between finite groups that have cohomological dimension two but infinite Lusternik–Schirelmann category.

DOI: <https://doi.org/10.32523/2077-9879-2026-17-3-44-53>

1 Introduction

The *cohomological dimension* of a discrete group is one of the fundamental numerical invariants in algebraic topology and geometric group theory. Recall that the cohomological dimension of a group Γ , denoted by $\text{cd}(\Gamma)$, is the largest integer n such that $H^n(\Gamma, M) \neq 0$ for some Γ -module M ; if no such integer exists, one sets $\text{cd}(\Gamma) = \infty$ (see Brown [4], [3]).

One of the most celebrated results in the subject is the *Stallings–Swan Theorem*, which states that a group has cohomological dimension one if and only if it is free. It was originally proved by Stallings [21] for finitely generated groups and later extended by Swan [22] to arbitrary groups.

Another fundamental invariant is the *geometric dimension* of a group, denoted by $\text{gd}(\Gamma)$. It is defined as the smallest dimension of a CW-complex that is a model for the classifying space $B\Gamma$. Equivalently, $B\Gamma$ is an Eilenberg–MacLane space of type $K(\Gamma, 1)$, and $\text{gd}(\Gamma)$ is the smallest dimension among all models of $K(\Gamma, 1)$. The *Eilenberg–Ganea Theorem* asserts that every group Γ with $\text{cd}(\Gamma) \geq 3$ satisfies $\text{gd}(\Gamma) = \text{cd}(\Gamma)$. The remaining case $\text{cd}(\Gamma) = 2$ is the subject of the classical *Eilenberg–Ganea Problem*, which asks whether every group of cohomological dimension two admits a two-dimensional classifying space.

In a 2013 MathOverflow discussion, Grant [14] proposed the notion of the *cohomological dimension of a group homomorphism*. For a homomorphism $\phi : \Gamma \rightarrow \Lambda$, this invariant, denoted by $\text{cd}(\phi)$, is defined in terms of the largest degree in which the induced homomorphism $\phi^* : H^k(\Lambda, M) \rightarrow H^k(\Gamma, M)$ is nontrivial for some Λ -module M . Although this definition is formally analogous to the classical cohomological dimension of a group, its properties are substantially different and have only recently begun to be understood.

The first systematic investigations of $\text{cd}(\phi)$ were carried out by Dranishnikov and De Saha [7], who developed homological and geometric techniques for studying cohomological dimensions of group homomorphisms. Earlier partial results appeared in [19, 10]. These works revealed that several classical theorems no longer hold in the homomorphism setting. In [10], the author and Dranishnikov constructed epimorphisms ϕ for which $\text{cd}(\phi) = 1$, $\text{gd}(\phi) = 2$, and $\text{cd}(\phi) = 2$, $\text{gd}(\phi) = 3$. More recently, Dranishnikov and De Saha [7] proved that for every integer $n \geq 2$ there exist homomorphisms satisfying $\text{cd}(\phi) = n$, $\text{gd}(\phi) = n + 1$. Thus, unlike the classical Eilenberg–Ganea Theorem, the cohomological and geometric dimensions of a homomorphism may differ in every positive dimension.

The principal objective of this paper is to investigate group homomorphisms of small cohomological dimension, with particular emphasis on epimorphisms. We first characterize epimorphisms of cohomological dimension zero (Theorem 3.2). We then establish a homomorphism analogue of the Stallings–Swan Theorem

by proving that an epimorphism satisfies $\text{cat}(\phi) = \text{cd}(\phi) = 1$ if and only if it factors through a free group via epimorphisms (Theorem 3.3). As an application, we show that an analogue of the Eilenberg–Ganea equality holds for epimorphisms between finitely generated one-relator groups, including surface groups and Baumslag–Solitar groups.

A striking difference from the classical theory arises in the presence of torsion. While every group with nontrivial torsion has infinite cohomological dimension, epimorphisms between finite groups may have finite cohomological dimension. In Theorem 4.1, we construct explicit examples of such epimorphisms, the Lusternik–Schnirelmann categories of which are nevertheless infinite. Consequently, the gap between $\text{cd}(\phi)$ and $\text{cat}(\phi)$ can be arbitrarily large, revealing a phenomenon with no analogue in the classical theory.

Our arguments combine several complementary techniques. They rely on the Bernstein–Schwarz universal cohomology class developed by Dranishnikov and Rudyak [11], the projective resolutions introduced by Dranishnikov and De Saha [7], the Ganea–Schwarz characterization of the Lusternik–Schnirelmann category, and classical methods from the cohomology of groups, including extension theory and dimension shifting.

The study of group homomorphisms and related algebraic structures is closely connected with questions concerning symmetries and group actions; see, for example, [12] for a study of symmetry groups arising from Pfaffians of symmetric matrices.

Throughout the paper, $H(\Gamma, M)$ denotes the cohomology of a group Γ with coefficients in a Γ -module M . If X is a connected CW-complex with fundamental group Γ , we write $H(X; M)$ for its cohomology with local coefficients. In particular, there is an isomorphism $H^*(\Gamma, M) \cong H^*(B\Gamma; M)$, where $B\Gamma$ denotes a classifying space for Γ , i.e., an Eilenberg–MacLane space $K(\Gamma, 1)$.

2 Preliminaries

In this section, we recall the algebraic and topological tools used throughout the paper. In particular, we review the cohomological dimension of group homomorphisms, the Bernstein–Schwarz cohomology class, the Lusternik–Schnirelmann category of maps, and the Ganea–Schwarz characterization of the LS-category.

2.1 Cohomological dimension of group homomorphisms

Let Γ be a discrete group. Recall that a *projective resolution* of the trivial Γ -module $\mathbb{Z}$ is an exact sequence of projective $\mathbb{Z}\Gamma$ -modules

$$\cdots \longrightarrow P_n(\Gamma) \longrightarrow P_{n-1}(\Gamma) \longrightarrow \cdots \longrightarrow P_1(\Gamma) \longrightarrow \mathbb{Z}\Gamma \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The application of the functor $\text{Hom}_\Gamma(-, M)$ to this resolution, where M is a Γ -module, yields a cochain complex whose homology is the group cohomology $H^*(\Gamma, M)$.

Following Grant [14], the *cohomological dimension* of a group homomorphism $\phi : \Gamma \longrightarrow \Lambda$ is defined by

$$\text{cd}(\phi) = \max \left\{ k \mid \phi^* : H^k(\Lambda, M) \longrightarrow H^k(\Gamma, M) \text{ is nontrivial for some } \Lambda\text{-module } M \right\},$$

where $\text{cd}(\phi) = \infty$ if this maximum does not exist.

The following characterization, established by Dranishnikov and De Saha [7], provides one of the principal algebraic tools used in this paper.

Theorem 2.1 ([7]). *Let $\phi : \Gamma \rightarrow \Lambda$ be a group homomorphism with $\text{cd}(\phi) = k$, and let $\phi_* : (P_*(\Gamma), \partial_*) \rightarrow (P_*(\Lambda), \partial'_*)$ be the chain map between projective resolutions induced by ϕ . Then ϕ_* is chain homotopic to a chain map $\psi_* : P_*(\Gamma) \rightarrow P_*(\Lambda)$ such that $\psi_n = 0$ for all $n > k$.*

2.2 Bernstein–Schwarz cohomology class

Let Γ be a discrete group and $I(\Gamma)$ denotes the augmentation ideal of the group ring $\mathbb{Z}\Gamma$. The *Bernstein–Schwarz class* $\beta_\Gamma \in H^1(\Gamma, I(\Gamma))$ is the primary obstruction to lifting the classifying map $u : X \rightarrow B\Gamma$ to the universal covering $ET \rightarrow B\Gamma$ (see [2, 11]).

The following universality property plays an important role in the proofs of our main results. For Γ -modules M and N , the cup product is the bilinear pairing

$$\smile : H^p(\Gamma, M) \times H^q(\Gamma, N) \longrightarrow H^{p+q}(\Gamma, M \otimes N).$$

In particular, if $\beta_\Gamma \in H^1(\Gamma, I(\Gamma))$, then $(\beta_\Gamma)^k = \underbrace{\beta_\Gamma \smile \cdots \smile \beta_\Gamma}_{k \text{ times}} \in H^k(\Gamma, I(\Gamma)^{\otimes k})$, where $I(\Gamma)^{\otimes k} = \underbrace{I(\Gamma) \otimes \cdots \otimes I(\Gamma)}_{k \text{ times}}.$

Theorem 2.2 (Universality [11, 20]). *For every cohomology class $\alpha \in H^k(\Gamma, L)$, there exists a homomorphism of Γ -modules $I(\Gamma)^{\otimes k} \longrightarrow L$, the induced homomorphism in cohomology of which sends $(\beta_\Gamma)^k \in H^k(\Gamma, I(\Gamma)^{\otimes k})$ to α .*

2.3 Lusternik–Schnirelmann category and sequential topological complexity

Let $f : X \rightarrow Y$ be a continuous map. For a positive integer r , let $X^r = X \times \cdots \times X$, $Y^r = Y \times \cdots \times Y$ denote the Cartesian products of r copies of X and Y , respectively. We write $f^r = f \times \cdots \times f : X^r \rightarrow Y^r$. Elements of X^r and Y^r will be denoted by $\bar{x} = (x_0, \dots, x_{r-1})$, $\bar{y} = (y_0, \dots, y_{r-1})$, respectively.

Definition 1. The *Lusternik–Schnirelmann category* of a map $f : X \rightarrow Y$, denoted by $\text{cat}(f)$, is the smallest integer k such that X admits an open cover $U_0, U_1, \dots, U_k$, for which the restriction $f|_{U_i} : U_i \rightarrow Y$ is null-homotopic for every $i = 0, \dots, k$.

Definition 2. Let $f : X \rightarrow Y$ be a continuous map.

1. A *sequential f -motion planner* on a subset $U \subset X^r$ is a map $f_U : U \rightarrow PY$ such that $f_U(\bar{x})\left(\frac{j}{r-1}\right) = f(x_j)$, $j = 0, \dots, r-1$.
2. The *sequential topological complexity* of f , denoted by $TC_r(f)$, is the smallest integer k such that X^r admits an open cover $U_0, \dots, U_k$, where each U_i admits a sequential f -motion planner. If no such integer exists, we set $TC_r(f) = \infty$.

Observe that when $r = 2$, we recover Scott's topological complexity of a map [19]. Moreover, if $f = \text{id}_X$, then $TC_r(f)$ coincides with Rudyak's sequential topological complexity of the space X [23].

By the homotopy invariance of classifying spaces, there is a one-to-one correspondence between group homomorphisms $\phi : \Gamma \rightarrow \Lambda$ and homotopy classes of maps $B\phi : B\Gamma \rightarrow B\Lambda$ that induce ϕ on the fundamental groups.

Definition 3. Let $\phi : \Gamma \rightarrow \Lambda$ be a group homomorphism.

1. The *Lusternik–Schnirelmann category* of ϕ is defined by $\text{cat}(\phi) = \text{cat}(B\phi)$, where $B\phi : B\Gamma \rightarrow B\Lambda$ is the induced map between the classifying spaces.
2. The *geometric dimension* of ϕ , denoted by $gd(\phi)$, is the smallest integer k such that the map $B\phi$ is homotopic to a map whose image lies in the k -skeleton of $B\Lambda$. Equivalently, $gd(\phi) = \min \{n \mid \text{there exists a map } f : B\Gamma \rightarrow B\Lambda^{(n)} \text{ with } f \simeq B\phi\}.$

The next lemma [8, Proposition 4.3] shows that the geometric dimension and the Lusternik–Schnirelmann category agree for group homomorphisms.

Lemma 2.1. ([8]) *Let $\phi : \Gamma \rightarrow \Lambda$ be a group homomorphism. Then $\text{cat}(\phi) = gd(\phi)$.*

2.4 Ganea–Schwarz approach to the Lusternik–Schnirelmann category

Recall that an element of the iterated join $X_0 * X_1 * \cdots * X_k$ of topological spaces $\{X_0, X_1, \dots, X_k\}$ is a formal convex combination $t_0x_0 + \cdots + t_kx_k$, where $x_i \in X_i$, $t_i \geq 0$, and $\sum_{i=0}^k t_i = 1$, with all terms of the form $0x_i$ omitted.

Let $f_i : X_i \rightarrow Y$, $i = 0, \dots, k$, be fibrations with connected normal space Y . Their *fiberwise join* is the space

$$X_0 *_Y X_1 *_Y \cdots *_Y X_k = \{t_0x_0 + \cdots + t_kx_k \in X_0 * \cdots * X_k \mid f_0(x_0) = \cdots = f_k(x_k)\}.$$

The corresponding fiberwise join fibration is

$$f_0 *_Y \cdots *_Y f_k : X_0 *_Y X_1 *_Y \cdots *_Y X_k \longrightarrow Y,$$

which sends $t_0x_0 + \cdots + t_kx_k$ to the common value $f_i(x_i)$, for any index i with $t_i \neq 0$.

When $X_i = X$ and $f_i = f : X \rightarrow Y$ for every i , the fiberwise join of spaces is denoted by $*_Y^{k+1}X$, and the corresponding fiberwise join fibration is denoted by $*_Y^{k+1}f$.

Let X be a path-connected space. Replacing the inclusion of a base point $i : \{x_0\} \rightarrow X$ by a fibration yields the path-space fibration $p_0^X : G_0(X) \rightarrow X$, whose fiber is the loop space ΩX . The k -th Ganea space is $G_k(X) = *_X^{k+1}G_0(X)$, and the corresponding Ganea fibration is $p_k^X : G_k(X) \rightarrow X$, obtained as the $(k+1)$ -fold fiberwise join of p_0^X . The fiber of p_k^X is the $(k+1)$ -fold join $*^{k+1}\Omega X$.

The following theorem gives the Ganea–Schwarz characterization of the Lusternik–Schnirelmann category of maps (see [19, 20, 10, 6]).

Theorem 2.3. *Let $f : X \rightarrow Y$ be a map between connected normal spaces. Then $\text{cat}(f) \leq k$ if and only if there exists a lift*

$$\begin{array}{ccc} & & G_k(Y) \\ & \nearrow & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with respect to the Ganea fibration $p_k^Y : G_k(Y) \rightarrow Y$.

3 Small cohomological dimensions

In this section we establish our main results on group homomorphisms of small cohomological dimension. We first recall a reduction theorem proved in the joint paper of the author and Dranishnikov [10], which shows that it is sufficient to consider epimorphisms. We then characterize epimorphisms of cohomological dimension zero.

3.1 Reduction to epimorphisms

Let $\phi : \Gamma \rightarrow \Lambda$ be a group homomorphism. We denote by $\phi' : \Gamma \rightarrow \text{im}(\phi)$ the induced epimorphism obtained by restricting the codomain of ϕ to its image.

The following result, proved in the joint paper of the author and Dranishnikov [10], shows that both the cohomological dimension and the Lusternik–Schnirelmann category depend only on the image of a homomorphism.

Theorem 3.1. ([10]) *For every group homomorphism $\phi : \Gamma \rightarrow \Lambda$, one has $\text{cat}(\phi) = \text{cat}(\phi')$ and $\text{cd}(\phi) = \text{cd}(\phi')$.*

By Theorem 3.1, it is sufficient to study epimorphisms. Therefore, throughout the remainder of this section we assume that $\phi : \Gamma \rightarrow \Lambda$ is an epimorphism.

3.2 Cohomological dimension zero

Our first result characterizes epimorphisms of cohomological dimension zero.

Theorem 3.2. *Let $\phi : \Gamma \twoheadrightarrow \Lambda$ be an epimorphism between finitely generated groups. Then $\text{cd}(\phi) = 0$ if and only if ϕ is the trivial homomorphism.*

Proof. Suppose first that ϕ is the trivial homomorphism. Since ϕ is surjective, the target group Λ is trivial. Consequently, its classifying space $B\Lambda$ is contractible, and hence $\tilde{H}^*(B\Lambda; M) = 0$ for every Λ -module M . Therefore, the induced homomorphism $\phi^* : H^*(B\Lambda; M) \rightarrow H^*(B\Gamma; M)$ is trivial in every positive degree. By the definition of cohomological dimension, it follows that $\text{cd}(\phi) = 0$.

Conversely, suppose that $\text{cd}(\phi) = 0$ and assume, for contradiction, that ϕ is a nontrivial epimorphism.

Via the homomorphism ϕ , every Λ -module may be regarded as a Γ -module. In particular, the augmentation ideal $I(\Lambda)$ is naturally a Γ -module.

Let $\beta_\Lambda \in H^1(\Lambda, I(\Lambda))$ be the Bernstein–Schwarz class of Λ . It is represented by the cocycle $f_\Lambda : P_1(\Lambda) \rightarrow I(\Lambda)$, $f_\Lambda(c') = \partial'_1(c')$.

Since ϕ is an epimorphism, the induced chain map $\phi_1 : P_1(\Gamma) \rightarrow P_1(\Lambda)$ is surjective. Hence, the composition $f_\Lambda \circ \phi_1 : P_1(\Gamma) \rightarrow I(\Lambda)$ is a nonzero cocycle representing the cohomology class $\phi^*(\beta_\Lambda) \in H^1(\Gamma, I(\Lambda))$. Therefore, $\phi^*(\beta_\Lambda) \neq 0$, which contradicts the assumption that $\text{cd}(\phi) = 0$. Hence, ϕ must be the trivial homomorphism. $\square$

As an immediate corollary, we obtain the following statement.

Corollary 3.1. *Let $\phi : F_n \rightarrow F_m$ be an epimorphism between free groups. Then $\text{cat}(\phi) = \text{cd}(\phi) = 1$.*

Proof. Since F_m is nontrivial, Theorem 3.2 implies that $\text{cd}(\phi) \geq 1$. On the other hand, $\text{cd}(\phi) \leq \text{cat}(\phi) \leq \text{cat}(F_n) = 1$, where the first inequality follows by [10]. Therefore, $\text{cat}(\phi) = \text{cd}(\phi) = 1$. $\square$

3.3 Cohomological dimension one

We now turn to epimorphisms of cohomological dimension one. The following theorem can be regarded as an analogue of the Stallings–Swan theorem for group homomorphisms. It shows that the condition $\text{cat}(\phi) = \text{cd}(\phi) = 1$ is equivalent to the existence of a factorization of ϕ through a free group.

Theorem 3.3. *Let $\phi : \Gamma \twoheadrightarrow \Lambda$ be an epimorphism. Then $\text{cat}(\phi) = \text{cd}(\phi) = 1$ if and only if ϕ factors through a free group via epimorphisms $\Gamma \twoheadrightarrow F_n \twoheadrightarrow \Lambda$.*

Proof. Suppose first that the epimorphism $\phi : \Gamma \twoheadrightarrow \Lambda$ factors through a free group F_n , that is, $\phi = g \circ h$, where $h : \Gamma \twoheadrightarrow F_n$, $g : F_n \twoheadrightarrow \Lambda$ are epimorphisms.

Since Λ is nontrivial, Theorem 3.2 implies that $\text{cd}(\phi) \geq 1$. Moreover, by [10], $\text{cd}(\phi) \leq \text{cat}(\phi)$. Therefore, it is enough to prove that $\text{cat}(\phi) \leq 1$.

Let $Bh : B\Gamma \rightarrow BF_n$, $Bg : BF_n \rightarrow B\Lambda$ be the induced maps between classifying spaces. Since $B\phi = Bg \circ Bh$, the monotonicity of the Lusternik–Schnirelmann category under composition yields the inequality

$$\text{cat}(\phi) = \text{cat}(B\phi) \leq \min\{\text{cat}(Bh), \text{cat}(Bg)\}.$$

Because $\text{cat}(Bh) \leq \text{cat}(BF_n)$ and $\text{cat}(Bg) \leq \text{cat}(BF_n)$, we obtain $\text{cat}(\phi) \leq \text{cat}(F_n) = 1$. Consequently, $1 \leq \text{cd}(\phi) \leq \text{cat}(\phi) \leq 1$, which proves that $\text{cat}(\phi) = \text{cd}(\phi) = 1$.

Conversely, suppose that $\text{cat}(\phi) = \text{cd}(\phi) = 1$. By the Ganea–Schwarz characterization of the Lusternik–Schnirelmann category (Theorem 2.3), the induced map $B\phi : B\Gamma \rightarrow B\Lambda$ admits a lift

It is well known that the first Ganea space $G_1(B\Lambda)$ has the homotopy type of a one-dimensional CW-complex (see [6, Example 1.61]). Therefore, $\pi_1(G_1(B\Lambda))$ is a free group, say F_n . Passing to the fundamental groups, the above lifting diagram induces a factorization $\Gamma \twoheadrightarrow F_n \twoheadrightarrow \Lambda$. Since the original homomorphism $\phi : \Gamma \twoheadrightarrow \Lambda$ is an epimorphism, the induced homomorphism $F_n \twoheadrightarrow \Lambda$ is also surjective. Hence, ϕ factors through the free group F_n via epimorphisms, completing the proof. $\square$

$$\begin{array}{ccc}
& & G_1(B\Lambda) \\
& \nearrow & \downarrow \\
B\Gamma & \longrightarrow & B\Lambda.
\end{array}$$

4 Epimorphisms between finite groups

Unlike the classical cohomological dimension of groups, which is infinite for groups with nontrivial torsion, the cohomological dimension of a group homomorphism may be finite even for epimorphisms between finite groups. In this section, we construct explicit examples illustrating this phenomenon and show that the gap between the cohomological dimension and the Lusternik–Schnirelmann category of a homomorphism can be arbitrarily large.

We recall the notation for the connected sum and wedge sum of two topological spaces. The notation $M \# N$ denotes the connected sum of M and N , while $M \vee N$ denotes their wedge sum.

Our first example is the following.

Theorem 4.1. *For every integer $k \geq 1$, the natural epimorphism $\phi : \mathbb{Z}_{p^{2k}} \twoheadrightarrow \mathbb{Z}_{p^k}$ is such that $\text{cd}(\phi) = 2$ and $\text{cat}(\phi) = \infty$.*

Proof. We first prove that $\text{cd}(\phi) \leq 2$. The proof is based on the following observation.

Claim 1. There exists a degree-zero map $f : L_{p^{2k}}^3 \longrightarrow L_{p^k}^3$ between lens spaces that induces the natural epimorphism $f_* : \mathbb{Z}_{p^{2k}} \twoheadrightarrow \mathbb{Z}_{p^k}$ on the fundamental groups.

We now construct the map f . Consider two copies of the circle S^1 equipped with free actions of $\mathbb{Z}_{p^{2k}}$ and $\mathbb{Z}_{p^k}$, respectively. A $\mathbb{Z}_{p^{2k}}$ -equivariant map $\psi : S^1 \rightarrow S^1$ has degree p^k . Consequently, the join $\psi * \psi : S^1 * S^1 \rightarrow S^1 * S^1$ has degree p^{2k} , and therefore, induces a map of the corresponding orbit spaces $\bar{\psi} : L_{p^{2k}}^3 \rightarrow L_{p^k}^3$ of degree p^{2k} . Let $q_t : S^3 \rightarrow L_{p^t}^3$, $t = k, 2k$, denote the quotient maps.

Define

$$f : L_{p^{2k}}^3 = L_{p^{2k}}^3 \# S^3 \longrightarrow L_{p^{2k}}^3 \vee S^3 \xrightarrow{1 \vee r} L_{p^{2k}}^3 \vee S^3 \xrightarrow{\bar{\psi} \vee q_2} L_{p^k}^3 \vee L_{p^k}^3 \xrightarrow{j} L_{p^k}^3,$$

where $r : S^3 \rightarrow S^3$ has degree -1 , and $j : L_{p^k}^3 \vee L_{p^k}^3 \rightarrow L_{p^k}^3$ identifies the two copies of $L_{p^k}^3$. Therefore, $\deg(f) = p^k - p^k = 0$.

Claim 2. The induced homomorphism $f^* : H^3(L_{p^k}^3; M) \rightarrow H^3(L_{p^{2k}}^3; M)$ is trivial for every $\mathbb{Z}_{p^k}$ -module M .

Suppose that $f^*(\alpha) \neq 0$ for some $\alpha \in H^3(L_{p^k}^3; M)$. By the Poincaré duality with local coefficients, $[L_{p^{2k}}^3] \cap f^*(\alpha) \neq 0$. Since f induces an epimorphism on the fundamental groups, the induced homomorphism on 0-dimensional homology, $f_* : H_0(L_{p^{2k}}^3; M) \rightarrow H_0(L_{p^k}^3; M)$, is an isomorphism. Hence,

$$0 \neq f_*([L_{p^{2k}}^3] \cap f^*(\alpha)) = f_*([L_{p^k}^3]) \cap \alpha = 0,$$

which is a contradiction. The map f extends to a map $\phi : L_{p^{2k}}^\infty \rightarrow L_{p^k}^\infty$.

Claim 3. The induced homomorphism $\phi^* : H^3(L_{p^k}^\infty; M) \rightarrow H^3(L_{p^{2k}}^\infty; M)$ is trivial for every $\mathbb{Z}_{p^k}$ -module M .

Indeed, the inclusion of the m -skeleton induces an injective homomorphism $H^m(X; M) \rightarrow H^m(X^{(m)}; M)$ for every CW-complex X and every $\pi_1(X)$ -module M .

The dimension-shifting argument in group cohomology [4] implies that $\phi^* : H^m(L_{p^k}^\infty; M) \rightarrow H^m(L_{p^{2k}}^\infty; M)$ is trivial for every $m \geq 3$. Therefore, $\text{cd}(\phi) \leq 2$.

The reverse inequality, $\text{cd}(\phi) \geq 2$, follows from computations with integral cohomology. Hence, $\text{cd}(\phi) = 2$. It remains to prove that $\text{cat}(\phi) = \infty$.

This follows from the fact that the reduced K -theory cup-length of ϕ is unbounded. By Atiyah's computation [1], $K(B\mathbb{Z}_{p^k}) = \mathbb{Z}[[\eta_k]]/(\eta_k^{p^k} - 1)$, where η_k is the pullback of the canonical line bundle over $\mathbb{C}P^\infty$ under the inclusion $\mathbb{Z}_{p^k} \hookrightarrow S^1$. The reduced K -theory of $B\mathbb{Z}_{p^k}$ is generated by $\eta_k - 1$. The induced homomorphism $\phi^* : K(B\mathbb{Z}_{p^k}) \rightarrow K(B\mathbb{Z}_{p^{2k}})$ sends η_k to $\eta_{2k}^{p^k}$, and therefore, $(\eta_k - 1)^m$ to $(\eta_{2k}^{p^k} - 1)^m$.

To prove that $(\eta_{2k}^{p^k} - 1)^m \neq 0$ in $\mathbb{Z}[[\eta_{2k}]]/(\eta_{2k}^{p^{2k}} - 1)$, it suffices to work modulo p . For $m = p^{2k} + 1$, we have $(\eta_{2k}^{p^k} - 1)^{p^{2k}+1} = \eta_{2k}^{p^k} - 1 \neq 0$ in $\mathbb{Z}_p[[\eta_{2k}]]/(\eta_{2k}^{p^{2k}} - 1)$. Therefore, $\text{cat}(\phi) \geq m$ for every positive integer m by Proposition 5.1 of [8]. Since m is arbitrary, it follows that $\text{cat}(\phi) = \infty$. $\square$

5 Applications

In this section we discuss several corollaries of the main results obtained above. Throughout this section, every homomorphism is assumed to be an epimorphism in view of Theorem 3.1.

Recall that every group homomorphism $\phi : \Gamma \rightarrow \Lambda$ induces, up to homotopy, a unique map $B\phi : B\Gamma \rightarrow B\Lambda$ between the corresponding classifying spaces. Consequently, homotopy invariants of maps naturally define invariants of group homomorphisms.

The *Lusternik–Schnirelmann category* of ϕ is defined by $\text{cat}(\phi) = \text{cat}(B\phi)$.

The *sequential topological complexity* of ϕ is defined by $TC_r(\phi) = TC_r(B\phi)$.

Theorem 4.1 demonstrates that the gap between the cohomological dimension and the Lusternik–Schnirelmann category of a group homomorphism can be arbitrarily large. This partially answers the question raised in [17] for homomorphisms between finite groups. Previously, in all known examples $\text{cat}(\phi) - \text{cd}(\phi) \leq 1$ (see [10, 7, 14]).

Moreover, Theorem 4.1 shows that no analogue of Rudyak's inequalities

$$r \text{cd}(\Gamma) \leq TC_r(\Gamma) \leq (r+1) \text{cd}(\Gamma)$$

holds for group homomorphisms between finite groups. Indeed, $\text{cat}(\phi) \leq TC_r(\phi)$, while Theorem 4.1 gives $\text{cat}(\phi) = \infty$ and $(r+1) \text{cd}(\phi) = 2(r+1)$. Hence, $TC_r(\phi) \not\leq (r+1) \text{cd}(\phi)$.

Another interesting application concerns the recently introduced distributional invariants. Dranishnikov and Jauhari [9] introduced the distributional Lusternik–Schnirelmann category and the distributional topological complexity of maps, denoted by dcat and dTC , respectively. They proved that these invariants are lower bounds for the classical Lusternik–Schnirelmann category and topological complexity.

Since the cohomological dimension is also a lower bound for the Lusternik–Schnirelmann category, it is natural to ask whether $\text{cd}(\phi)$ and $\text{dcat}(\phi)$ are comparable.

The examples below show that this is not the case.

Indeed, let $\phi : \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$ be natural projection. By Theorem 4.1 with $p = 2$ and $k = 1$, we conclude that $\text{cd}(\phi) = 2$, whereas $\text{dcat}(\phi) = 1$, since $\text{dcat}(\mathbb{Z}_2) = 1$ (see [9]).

On the other hand, let $p : S^n \rightarrow \mathbb{R}P^n$ be the double covering for $n \geq 2$. Since p is not null-homotopic, $\text{dcat}(p) = 1$, whereas $\text{cd}(p_*) = 0$, because S^n is simply connected.

We conclude with another application of Theorems 3.2 and 3.3.

The classical Eilenberg–Ganea equality $\text{cat}(\Gamma) = \text{cd}(\Gamma)$ does not admit a direct analogue for arbitrary group homomorphisms. Nevertheless, it is known to hold for several classes of finitely generated torsion-free groups, including free groups, finitely generated abelian groups, nilpotent groups, virtually nilpotent groups, and almost nilpotent groups (see [19, 15, 16, 17]). The following theorem provides another family for which the equality remains valid.

Theorem 5.1. *Let $\phi : \Gamma \twoheadrightarrow \Lambda$ be an epimorphism between finitely generated one-relator groups. Then $\text{cat}(\phi) = \text{cd}(\phi)$.*

Proof. If Γ is free, then the result follows immediately by Corollary 3.1.

Assume therefore that Γ is not free. Since finitely generated torsion-free non-free one-relator groups admit two-dimensional classifying spaces (see [4]), we have $\text{cd}(\Gamma) = \text{cat}(\Gamma) = 2$. By Theorem 3.2, $\text{cd}(\phi) \geq 1$.

There are two possibilities. If ϕ factors through a free group, then Theorem 3.3 implies that $\text{cat}(\phi) = \text{cd}(\phi) = 1$. Otherwise, Theorem 3.3 shows that $\text{cd}(\phi) \geq 2$. Since $\text{cd}(\phi) \leq \text{cat}(\phi) \leq \text{cat}(\Gamma) = 2$, it follows that $\text{cat}(\phi) = \text{cd}(\phi) = 2$. □

As an immediate corollary we obtain the following statement.

Corollary 5.1. *Let $\phi : \pi_1(M) \rightarrow \pi_1(N)$ be an epimorphism between the fundamental groups of closed orientable surfaces. Then $\text{cat}(\phi) = \text{cd}(\phi)$.*

Proof. Fundamental groups of closed orientable surfaces are one-relator groups, so the result follows directly from Theorem 5.1. □

Recall that the Baumslag–Solitar group is defined by $BS(m, n) = \langle a, b \mid ba^mb^{-1} = a^n \rangle$. Baumslag–Solitar groups provide classical examples of finitely presented groups that are not Hopfian (for example, $BS(2, 3)$), highlighting the subtle behavior of one-relator groups (see [5]).

Corollary 5.2. *Let $\phi : BS(m, n) \rightarrow BS(r, s)$ be an epimorphism of Baumslag–Solitar groups. Then $\text{cat}(\phi) = \text{cd}(\phi)$.*

Proof. Since every Baumslag–Solitar group is a one-relator group, the conclusion follows immediately from Theorem 5.1. □

Acknowledgments

The author thanks the referee for greatly improving the quality of the manuscript. The work was partially supported by the grant No. AP25796111 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

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Received: 15.09.2025