

ON SPECTRAL PROBLEMS FOR NONLOCAL ANALOGUES
OF THE POLYHARMONIC OPERATOR

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Abstract. In this paper, using mappings of the unit ball in $\mathbb{R}^n$, associated with orthogonal matrices, two types of nonlocal analogues of the polyharmonic operator are introduced. Spectral problems for the Dirichlet problem and Dirichlet-type problems for these operators are studied. The eigenfunctions and eigenvalues of the problems under consideration are found. Theorems on the completeness of systems of eigenfunctions in the space L_2 are proved.

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1 Introduction

This work is devoted to the study of spectral problems for some boundary value problems for nonlocal analogues of the polyharmonic operator. The considered equations enter the class of differential equations with shifts of spatial variables. There are various types of differential equations with shifts of arguments, and among them, a special place is occupied by equations with involutive shifts of arguments. Recall that a mapping S is called an involution if $S^2(x) = x$.

Differential equations with involution have numerous applications in various fields of science. For example, in [7, 20] the use of such equations in modeling of a specific physical process of thermal diffusion is described, and in [9, 16, 17] equations with involutively transformed arguments, which have applications in modeling optical systems, are considered. A detailed description of applications, as well as methods for solving such equations, are given in monographs [3, 15, 23].

Spectral issues arising in connection with ordinary differential operators with involution were considered the first-order operators in [1, 2], the second-order in [10, 11, 12, 18], and the higher-order operators in [8, 14].

Note that spectral issues for differential operators with involution in the case of multiple variables are insufficiently studied. In this area, we can only note works [21, 22].

Let us consider formulation of such problems. Let Ω be the unit ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $\partial\Omega$ be the unit sphere, and Q be a real $n \times n$ orthogonal matrix such that $Q^l = E$ for some $l \in \mathbb{N}$ and $Q^k \neq E$ for $0 < k < l$. Here, E is the unit matrix.

Note that if $x \in \Omega$, or $x \in \partial\Omega$, then for any $k \in \mathbb{N}$ inclusions $Q^k x \in \Omega$ or $Q^k x \in \partial\Omega$ take place. This is the case because the transformation of the space $\mathbb{R}^n$ with the orthogonal matrix Q preserves the norm: $|x|^2 = (x, x) = (Q^T Q x, x) = (Q x, Q x) = |Q x|^2$.

Let also $P_1, P_2, \dots, P_n$ be real symmetric commutative matrices such that $P_j^2 = E$. As an example, we can consider matrices of the mappings $P_j x = (x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n)$, $j = 1, 2, \dots, n$.

If we consider all possible products of mappings P_j , then the total number of such mappings, including the identity mapping $P_0 x = x$, is equal to 2^n .

To number such mappings, we use the binary number system. Namely, if $0 \leq i < 2^n$, then in the binary number system the following representation holds:

$$i \equiv (i_n \dots i_1)_2 = i_1 + 2i_2 + \dots + 2^{n-1}i_n,$$

where $i_k = 0, 1$. Then, for any index $i \in \{0, 1, \dots, 2^n - 1\}$ we can consider mappings of the type $P_n^{i_n} \dots P_2^{i_2} P_1^{i_1}$.

Using the mappings Q and $P_n^{i_n} \dots P_2^{i_2} P_1^{i_1}$ we introduce the following operators:

$$L_Q u(x) = \sum_{i=0}^{l-1} a_i (-\Delta)^m u(Q^i x),$$

$$L_P u(x) = \sum_{i=0}^{2^n-1} a_i (-\Delta)^m u(P_n^{i_n} \dots P_2^{i_2} P_1^{i_1} x),$$

where Δ is the Laplace operator, $m \in \mathbb{N}$, and $a_0, \dots, a_r$ are given real numbers and $r = l - 1$ for L_Q and $r = 2^n - 1$ for L_P .

Let us consider the following problems in the unit ball Ω .

Problem 1. Find a function $u \not\equiv 0$ on Ω belonging to the class $C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ and a number $\lambda \in \mathbb{C}$ that satisfy the conditions

$$L_Q u(x) = \lambda u(x), \quad x \in \Omega, \quad (1.1)$$

$$\frac{\partial^j u}{\partial \nu^j}(x) = 0, \quad j = 0, 1, \dots, m-1, \quad x \in \partial\Omega, \quad (1.2)$$

where ν is the unit outer normal to the sphere $\partial\Omega$.

Problem 2. Find a function $u \not\equiv 0$ on Ω belonging to the class $C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ and a number $\lambda \in \mathbb{C}$ that satisfy the conditions

$$L_P u(x) = \lambda u(x), \quad x \in \Omega, \quad (1.3)$$

and boundary conditions (1.2).

Problem 3. Find a function $u \not\equiv 0$ on Ω belonging to the class $C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ and a number $\lambda \in \mathbb{C}$ that satisfy equation (1.1) and the conditions

$$\Delta^j u(x) = 0, \quad j = 0, 1, \dots, m-1, \quad x \in \partial\Omega. \quad (1.4)$$

Problem 4. Find a function $u \not\equiv 0$ on Ω belonging to the class $C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ and a number $\lambda \in \mathbb{C}$ that satisfy equation (1.2) and boundary conditions (1.4).

Note that Problems 1 and 2 for the case of a nonlocal Laplace operator are studied in [21, 22].

2 Eigenfunctions and eigenvalues of Problem 1

Let us assume that a function w and a number μ are eigenfunctions and eigenvalues of the Dirichlet problem

$$(-\Delta)^m w(x) = \lambda w(x), \quad x \in \Omega, \quad (2.1)$$

$$\frac{\partial^j w}{\partial \nu^j}(x) = 0, \quad j = 0, 1, \dots, m-1, \quad x \in \partial\Omega. \quad (2.2)$$

Let $\varepsilon_j = e^{i2\pi j/l}$, $j = 0, 1, \dots, l-1$ be all roots of the equation $\varepsilon^l = 1$, $\varepsilon \in \mathbb{C}$, w and μ be an eigenfunction and an eigenvalue, respectively, of Dirichlet problem (2.1), (2.2).

Let us consider the functions

$$u_j(x) = \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^k w(Q^k x), \quad x \in \Omega, \quad j = 0, 1, \dots, l-1. \quad (2.3)$$

Lemma 2.1. *If w and μ are an eigenfunction and an eigenvalue of Dirichlet problem (2.1), (2.2), then the functions u_j satisfy the conditions of problem (1.1), (1.2) for all $j = 0, 1, \dots, l-1$. Moreover, the representation*

$$w(x) = \sum_{j=0}^{l-1} u_j(x), x \in \Omega.$$

is valid.

Proof. Let

$$\Lambda = r \frac{\partial}{\partial r}, \Lambda^{[k]} = \Lambda(\Lambda - 1) \dots (\Lambda - (k - 1)), k \in \mathbb{N},$$

and, for any $n \times n$ orthogonal matrix $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$I_S u(x) = u(Sx), x \in \Omega.$$

It was proved in [6] that if $x \in \partial\Omega$, then the equality

$$\frac{\partial^j u}{\partial \nu^j}(x) = \Lambda^{[k]} u(x)$$

holds. Moreover, it was also proved in [6] that the operators I_S and Λ , as well as I_S and Δ , commute. That is, the following equalities hold:

$$I_S(\Lambda u(x)) = \Lambda I_S u(x), \quad I_S(\Delta u(x)) = \Delta I_S u(x), x \in \Omega.$$

Then,

$$\begin{aligned} (-\Delta)^m u_j(x) &= \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^k (-\Delta)^m I_{Q^k} w(x) = \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^k I_{Q^k} (-\Delta)^m w(x) = \frac{\mu}{l} \sum_{k=0}^{l-1} \varepsilon_j^k I_{Q^k} w(x) \\ &= \frac{\mu}{l} \sum_{k=0}^{l-1} \varepsilon_j^k w(Q^k x) = \mu u_j(x), x \in \Omega. \end{aligned}$$

Moreover, if $x \in \partial\Omega$, then $Q^k x \in \partial\Omega$ for $k = 0, 1, \dots, l-1$ and, thus, (2.2) implies the following equalities:

$$\begin{aligned} \left. \frac{\partial^j w}{\partial \nu^j}(Q^k x) \right|_{\partial\Omega} &= \left. \Lambda^{[j]} I_{Q^k} w(x) \right|_{\partial\Omega} = \left. I_{Q^k} \Lambda^{[j]} w(x) \right|_{\partial\Omega} \\ &= \left. I_{Q^k} \frac{\partial^j w}{\partial \nu^j}(x) \right|_{\partial\Omega} = 0, 0 \leq j \leq m-1, 0 \leq k \leq l-1. \end{aligned} \quad (2.4)$$

Hence, the functions u_j satisfy all conditions of Problem 1.

We note that the following equality holds (see, for example, [5]):

$$\sum_{k=0}^{l-1} \varepsilon_j^k = \begin{cases} l, & k = 0 \\ 0, & k = 1, 2, \dots, l-1, \end{cases} \quad (2.5)$$

Then, using equality (2.3), we obtain:

$$\sum_{j=0}^{l-1} u_j(x) = \sum_{j=0}^{l-1} \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^k w(Q^k x), x \in \Omega.$$

By changing the order of summation in the last expression, we have:

$$\sum_{j=0}^{l-1} \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^k w(Q^k x) = \left(\sum_{k=0}^{l-1} \frac{1}{l} w(Q^k x) \right) \cdot \left(\sum_{j=0}^{l-1} \varepsilon_j^k \right) = \frac{1}{l} w(x) \cdot l = w(x).$$

□

Further, it is easy to verify that all functions u_j satisfy for all $x \in \Omega$ the following equalities:

$$\begin{aligned} u_j(Qx) &= \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^k w(Q^{k+1}x) = \frac{1}{l} \bar{\varepsilon}_j \varepsilon_j \sum_{k=0}^{l-1} \varepsilon_j^k w(Q^{k+1}x) = \frac{1}{l} \bar{\varepsilon}_j \sum_{k=0}^{l-1} \varepsilon_j^{k+1} w(Q^{k+1}x) \\ &= \bar{\varepsilon}_j \frac{1}{l} \sum_{k=0}^{l-1} \varepsilon_j^{k+1} w(Q^{k+1}x) = \bar{\varepsilon}_j \frac{1}{l} \left[\varepsilon_j^1 w(Qx) + \dots + \varepsilon_j^{l-1} w(Q^{l-1}x) + \varepsilon_j^0 w(x) \right] = \bar{\varepsilon}_j u_j(x). \end{aligned}$$

Hence, for any $1 \leq q \leq l-1$ the equality

$$u_j(Q^q x) = \bar{\varepsilon}_j^q u_j(x) \quad (2.6)$$

is valid. As in eigenfunction problem (2.1), (2.2), using equality (2.6), we obtain

$$\begin{aligned} (-\Delta)^m \left(\sum_{q=0}^{l-1} a_q u_j(Q^q x) \right) &= (-\Delta)^m \left(\sum_{q=0}^{l-1} a_q \bar{\varepsilon}_j^q u_j(x) \right) \\ &= (-\Delta)^m u_j(x) \left(\sum_{q=0}^{l-1} a_q \bar{\varepsilon}_j^q \right) = \mu u_j(x) \sigma_j = \lambda_j u_j(x), \end{aligned}$$

where the notations

$$\lambda_j = \mu \sigma_j, \sigma_j = \sum_{q=0}^{l-1} a_q \bar{\varepsilon}_j^q$$

are used.

Thus, the following assertion is proved.

Lemma 2.2. *Let a function w and a number μ be an eigenfunction and eigenvalue, respectively, of Dirichlet problem (2.1), (2.2). Moreover, let $\varepsilon_j, j = 0, 1, \dots, l-1$ be all roots of the equation $\varepsilon^l = 1$. Then each function $u_j(x)$ defined by (2.3) and the number μ is an eigenfunction and an eigenvalue, respectively, of Problem 1.*

Let $\{w_k, \mu_k\}_{k=1}^\infty$ be a complete system of eigenfunctions and eigenvalues of Dirichlet problem (2.1), (2.2) and let $\varepsilon_j, j = 0, 1, \dots, l-1$ be all roots of the equation $\varepsilon^l = 1$.

For each function w_k , consider l functions

$$u_{k,j}(x) = \frac{1}{l} \sum_{s=0}^{l-1} \varepsilon_j^s w_k(Q^s x), \quad (2.7)$$

and define the numbers

$$\lambda_{k,j} = \mu_k \sigma_j, \sigma_j = \sum_{q=0}^{l-1} a_q \bar{\varepsilon}_j^q. \quad (2.8)$$

Theorem 2.1. *Let the coefficients $a_q, q = 0, 1, \dots, l-1$ in Problem 1 be such that the conditions $\sigma_j = \sum_{q=0}^{l-1} a_q \bar{\varepsilon}_j^q \neq 0$ are satisfied for $j = 0, 1, \dots, l-1$. Then the functions $u_{k,j}$ defined by (2.7) and the numbers $\lambda_{k,j}$ defined by (2.8) are eigenfunctions and eigenvalues, respectively, of Problem 1. The system of functions $\{u_{k,j}\}_{k=1; j=0}^{\infty; l-1}$ is complete in the space $L_2(\Omega)$.*

Proof. The assertion that $u_{k,j}$ and $\lambda_{k,j}$ are eigenfunctions and eigenvalues, respectively, of Problem 1 follows by Lemma 2.2. Let us prove the completeness of the system $\{u_{k,j}\}_{k=1; j=0}^{\infty; l-1}$. Let a function $f \in L_2(\Omega)$ be orthogonal to all elements of the system $\{u_{k,j}\}_{k=1; j=0}^{\infty; l-1}$. Then, using [6, Lemma 5.1] we can write

$$\begin{aligned} 0 &= (u_{k,j}, f) = \int_{\Omega} u_{k,j}(x) \overline{f(x)} dx = \frac{1}{l} \int_{\Omega} \left[\sum_{s=0}^{l-1} \varepsilon_j^s w_k(Q^s x) \right] \overline{f(x)} dx \\ &= \frac{1}{l} \sum_{s=0}^{l-1} \varepsilon_j^s \int_{\Omega} w_k(Q^s x) \overline{f(x)} dx = \frac{1}{l} \int_{\Omega} w_k(x) \left(\sum_{s=0}^{l-1} \varepsilon_j^s \overline{f(Q^s x)} \right) dx, \end{aligned}$$

where $k \in \mathbb{N}, j = 0, 1, \dots, l-1$. As the system $\{w_k\}_{k=1}^\infty$ is complete in $L_2(\Omega)$ (see, for example, [13]), the equalities

$$\sum_{s=0}^{l-1} \varepsilon_j^s \overline{f(Q^s x)} = 0$$

must hold. Then, summing the last equalities over j and using (2.5), we obtain

$$0 = \sum_{j=0}^{l-1} \sum_{s=0}^{l-1} \varepsilon_j^s \overline{f(Q^s x)} = \sum_{s=0}^{l-1} \overline{f(Q^s x)} \sum_{j=0}^{l-1} \varepsilon_j^s = \overline{f(x)}$$

which means that $f(x) = 0$ for all $x \in \Omega$. This implies the completeness of the system $\{u_{k,j}\}_{k=1; j=0}^{\infty; l-1}$. $\square$

3 Eigenfunctions and eigenvalues of Problem 2

Suppose w_k and μ_k , $k \in \mathbb{N}$ are eigenfunctions and eigenvalues, respectively, of Dirichlet problem (2.1), (2.2).

Let $p, q \in \{0, 1, \dots, 2^n - 1\}$ and $\langle p, q \rangle = p_1 q_1 + p_2 q_2 + \dots + p_n q_n$. Using w_k introduce the following functions

$$u_k^p(x) = \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} w_k(P_n^{q_n} \dots P_1^{q_1} x). \quad (3.1)$$

Lemma 3.1. *If w_k and μ_k , $k \in \mathbb{N}$, are eigenfunctions and eigenvalues, respectively, of problem (2.1), (2.2), then the function u_k^p is also an eigenfunction of Dirichlet problem (2.1), (2.2), corresponding to the eigenvalue μ_k .*

Proof. As

$$(-\Delta)^m w(x) = \mu w(x)$$

and

$$\Delta(w(P_k x)) = \Delta w(P_k x),$$

for any $q_j = 0, 1, \dots, 2^n - 1, j = 0, 1, \dots, n$ the following equality holds:

$$(-\Delta)^m w_k(P_n^{q_n} \dots P_1^{q_1} x) = \mu_k w_k(P_n^{q_n} \dots P_1^{q_1} x), x \in \Omega.$$

Hence,

$$\begin{aligned} (-\Delta)^m u_k^p(x) &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} (-\Delta)^m w_k(P_n^{q_n} \dots P_1^{q_1} x) \\ &= \frac{\mu_k}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} w_k(P_n^{q_n} \dots P_1^{q_1} x) = \mu_k u_k^p(x). \end{aligned} \quad (3.2)$$

In addition, if $x \in \partial\Omega$, then for any $0 \leq q \leq 2^n - 1$ points $P_n^{q_n} \dots P_1^{q_1} x$ also belong to $\partial\Omega$ as $(P_k)^t = P_k$ [5]. Using equalities (2.4) with Q replaced by P_k we verify that the conditions

$$\left. \frac{\partial^j u_k^p}{\partial \nu^j}(x) \right|_{\partial\Omega} = 0, j = 0, 1, \dots, m-1, \quad (3.3)$$

are satisfied. Hence, all functions u_k^p defined by (3.1) are also eigenfunctions of Dirichlet problem (2.1), (2.2). $\square$

Lemma 3.2. *If for any $p = (p_n \dots, p_1)_2$ and any $x \in \Omega$ equalities*

$$\sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} f(P_n^{q_n} \dots P_1^{q_1} x) = 0 \quad (3.4)$$

are satisfied, then $f \equiv 0$ on Ω .

Proof. Let us sum up equalities (3.4) over all $p = (p_n \dots, p_1)_2$

$$0 = \sum_{p=0}^{2^n-1} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} f(P_n^{q_n} \dots P_1^{q_1} x) = \sum_{q=0}^{2^n-1} f(P_n^{q_n} \dots P_1^{q_1} x) \sum_{p=0}^{2^n-1} (-1)^{p_1 q_1 + \dots + p_n q_n} \quad (3.5)$$

It is easy to see that

$$\sum_{p=0}^{2^n-1} (-1)^{p_1 q_1 + \dots + p_n q_n} = \sum_{p_1=0}^1 (-1)^{p_1 q_1} \dots \sum_{p_n=0}^1 (-1)^{p_n q_n}.$$

As

$$\sum_{p_1=0}^1 (-1)^{p_1 q_1} = \begin{cases} 2, & q_1 = 0 \\ 0, & q_1 = 1 \end{cases},$$

we have

$$\sum_{p=0}^{2^n-1} (-1)^{p_1 q_1 + \dots + p_n q_n} = \begin{cases} 2^n, & q = (q_n \dots q_1)_2 = 0 \\ 0, & q \neq 0. \end{cases}.$$

Hence,

$$\sum_{q=0}^{2^n-1} f(P_n^{q_n} \dots P_1^{q_1} x) \sum_{p=0}^{2^n-1} (-1)^{p_1 q_1 + \dots + p_n q_n} = 2^n f(P_n^0 \dots P_1^0 x) = 2^n f(x),$$

and, therefore,

$$\sum_{q=0}^{2^n-1} f(P_n^{q_n} \dots P_1^{q_1} x) \sum_{k=0}^{2^l-1} (-1)^{p_1 q_1 + \dots + p_n q_n} = 2^n f(P_n^0 \dots P_1^0 x) = 2^n f(x),$$

which means that substituting the found value into (3.4) we obtain $f(x) = 0$. □

Now, let us denote

$$\theta_p = \sum_{i=0}^{2^n-1} (-1)^{\langle p, i \rangle} a_i, \quad (3.6)$$

where $p \in \{0, 1, \dots, 2^n - 1\}$.

Theorem 3.1. *Let the coefficients a_j , $j = 0, 1, \dots, 2^n - 1$ in Problem 2 be such that the conditions*

$$\theta_p = \sum_{i=0}^{2^n-1} (-1)^{\langle p, i \rangle} a_i \neq 0$$

are satisfied for $p \in \{0, 1, \dots, 2^n - 1\}$. Then the elements of the system $\{u_k^p\}_{k=0, p=0}^{\infty, 2^n-1}$ are eigenfunctions of Problem 2, and the corresponding eigenvalues are determined by the equality

$$\lambda_k^p = \mu_k \sum_{i=0}^{2^n-1} (-1)^{\langle p, i \rangle} a_i \equiv \mu_k \theta_p.$$

Moreover, the system of functions $\{u_k^p\}_{k=0, p=0}^{\infty, 2^n-1}$ is complete and orthonormal in the space $L_2(\Omega)$.

Proof. Let us first consider the function $u_k^p(P_1x)$. Since equality (3.2) holds for the function $u_k^p(x)$ and the matrices $P_1, P_2, \dots, P_n$ are commutative, we can write

$$\begin{aligned} (-\Delta)^m u_k^p(P_1^{i_1}x) &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} (-\Delta)^m w_k(P_n^{q_n} \dots P_1^{q_1} P_1^{i_1}x) \\ &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} (-\Delta)^m w_k(P_n^{q_n} \dots P_1^{q_1+i_1}x) \\ &= \frac{\mu_k}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle - p_1 i_1} w_k(P_n^{q_n} \dots P_1^{q_1}x) = (-1)^{-p_1 i_1} \mu_k u_k^p(x) = (-1)^{p_1 i_1} \mu_k u_k^p(x). \end{aligned}$$

In the equalities above, the replacement of the index was made $q_1 \rightarrow q_1 - i_1$, which implies that

$$\langle p, q \rangle = \sum_{k=1}^n p_k q_k \rightarrow \sum_{k=1}^n p_k q_k - p_1 i_1 = \langle p, q \rangle - p_1 i_1.$$

Similarly, in the general case, using the commutativity of the matrices $P_1, P_2, \dots, P_n$, we obtain

$$\begin{aligned} (-\Delta)^m u_k^p(P_1^{i_1} P_2^{i_2} \dots P_n^{i_n}x) &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} (-\Delta)^m w_k(P_n^{q_n} \dots P_1^{q_1} P_1^{i_1} \dots P_n^{i_n}x) \\ &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} (-\Delta)^m w_k(P_n^{q_n+i_n} \dots P_1^{q_1+i_1}x). \end{aligned}$$

Changing the index in the last sum $q \rightarrow q - i$, we get

$$\langle p, q \rangle = \sum_{k=1}^n p_k q_k \rightarrow \sum_{k=1}^n p_k (q_k - i_k) = \langle p, q \rangle - \langle p, i \rangle,$$

and we obtain

$$\begin{aligned} (-\Delta)^m u_k^p(P_1^{i_1} P_2^{i_2} \dots P_n^{i_n}x) &= \frac{\mu_k}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle - \langle p, i \rangle} w_k(P_n^{q_n} \dots P_1^{q_1}x) \\ &= (-1)^{-\langle p, i \rangle} \mu_k u_k^p(x) = (-1)^{\langle p, i \rangle} \mu_k u_k^p(x). \end{aligned}$$

Hence,

$$\begin{aligned} L_P u_k^p(x) &= \sum_{i=0}^{2^n-1} a_i (-\Delta)^m u_k^p(P_n^{i_n} \dots P_2^{i_2} P_1^{i_1}x) \\ &= \left(\sum_{i=0}^{2^n-1} (-1)^{p_1 i_1 + p_2 i_2 + \dots + p_n i_n} a_i \right) \mu_k u_k^p(x) = \theta_p \mu_k u_k^p(x) = \lambda_k^p u_k^p(x), \end{aligned}$$

where θ_p is defined by (3.6). Therefore, the function u_k^p satisfies the following equation

$$L_P u_k^p(x) = \lambda_k^p u_k^p(x).$$

On the other hand, as shown above, for the system of functions u_k^p , equalities (3.3) hold.

Thus, the elements of the system $\{u_k^p\}_{k=1, p=0}^{\infty, 2^n-1}$ are eigenfunctions, and the numbers $\lambda_k^p = \theta_p \mu_k$ for $k \in \mathbb{N}$ and $p = 0, 1, \dots, 2^n - 1$ corresponding to them are the eigenvalues of Problem 2.

Let us prove the completeness of the system $\{u_k^p\}_{k=1, p=0}^{\infty, 2^n-1}$. Let a function $f \in L_2(\Omega)$ be orthogonal to all elements of the system $\{u_k^p\}_{k=1, p=0}^{\infty, 2^n-1}$. Then we have:

$$\begin{aligned} 0 &= (u_k^p, f) = \int_{\Omega} u_k^p(x) \overline{f(x)} dx = \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} \int_{\Omega} w_k(P_n^{q_n} \dots P_1^{q_1} x) \overline{f(x)} dx \\ &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} \int_{\Omega} w_k(x) \overline{f(P_n^{q_n} \dots P_1^{q_1} x)} dx \\ &= \int_{\Omega} w_k(x) \left[\frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} \overline{f(P_n^{q_n} \dots P_1^{q_1} x)} \right] dx. \end{aligned}$$

As the system w_k is complete in $L_2(\Omega)$, we have

$$\int_{\Omega} w_k(x) \left[\frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} \overline{f(P_n^{q_n} \dots P_1^{q_1} x)} \right] dx = 0.$$

Hence, in the domain Ω the equality

$$\sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} \overline{f(P_n^{q_n} \dots P_1^{q_1} x)} = 0$$

is valid almost everywhere. Then, using the assertion of Lemma 3.2, we obtain that $f = 0$ almost everywhere on Ω . $\square$

4 Eigenfunctions and eigenvalues of Problem 3

In the case of Problem 3, to construct the eigenfunctions and eigenvalues of this problem, we use the Dirichlet spectral problem for the Laplace operator. Namely, we assume that w_k and μ_k for $k \in \mathbb{N}$ are respectively the eigenfunctions and eigenvalues of the following Dirichlet problem

$$-\Delta w(x) = \mu w(x), x \in \Omega; \quad w(x) = 0, x \in \partial\Omega. \quad (4.1)$$

Note that the system $\{w_k\}_{k=1}^{\infty}$ has the following properties (see, for example, [13, 4]):

- 1) the system is complete and orthonormal in the space $L_2(\Omega)$;
- 2) the system belongs to the class $C^{\infty}(\Omega)$.

Lemma 4.1. *The functions w_k and numbers μ_k , $k \in \mathbb{N}$ satisfy the following conditions:*

$$(-\Delta)^m w(x) = \mu^m w(x), x \in \Omega; \quad \Delta^j w(x) = 0, j = 0, 1, \dots, m-1, x \in \partial\Omega. \quad (4.2)$$

Proof. As $\Delta w_k(x) = -\mu_k w_k(x)$, then successively applying the Laplace operator Δ to this equality we obtain

$$\Delta^m w_k(x) = (-1)^m \mu_k^m w_k(x), x \in \Omega$$

and

$$w_k(x) = 0, \Delta w_k(x) = 0, \dots, \Delta^{m-1} w_k(x) = 0, x \in \partial\Omega.$$

Thus, the functions w_k satisfy all conditions of problem (4.2). $\square$

Now, as in the case of Problem 1, assuming that $\varepsilon_j, j = 0, 1, \dots, l-1$ are all roots of the equation $\varepsilon^l = 1$, for each function w_k , we consider l functions $u_{k,j}, j = 0, 1, \dots, l-1$ of type (2.7) and define the numbers $\lambda_{k,j}$ as in (2.8).

Note that the functions $u_{k,j}(x), j = 0, 1, \dots, l-1$ satisfy the equalities

$$-\Delta u_{k,j}(x) = \frac{1}{l} \sum_{s=0}^{l-1} \varepsilon_j^s (-\Delta) w_k(Q^s x) = \frac{\mu_k}{l} \sum_{s=0}^{l-1} \varepsilon_j^s w_k(Q^s x) = \mu_k u_{k,j}(x), x \in \Omega,$$

moreover, $u_{k,j}(x)|_{\partial\Omega} = 0$. This means that the functions $u_{k,j}, j = 0, 1, \dots, l-1$ satisfy all conditions of problem (4.1). Then, by Lemma 4.1, they also satisfy all conditions of problem (4.2). Since the equalities $u_{k,j}(Q^q x) = \varepsilon_j^q u_{k,j}(x)$ hold for the functions $u_{k,j}, j = 0, 1, \dots, l-1$, we obtain

$$\begin{aligned} L_Q u_{k,j}(x) &= (-\Delta)^m \left[\sum_{s=0}^{l-1} a_s u_{k,j}(Q^s x) \right] = (-\Delta)^m \left[\sum_{s=0}^{l-1} a_s \varepsilon_j^s u_{k,j}(x) \right] \\ &= \mu_k^m \sum_{s=0}^{l-1} a_s \varepsilon_j^s u_{k,j}(x) = \mu_k^m u_{k,j}(x) \left(\sum_{s=0}^{l-1} a_s \varepsilon_j^s \right) = \mu_k^m \sigma_j u_{k,j}(x) = \lambda_{k,j} u_{k,j}(x), \end{aligned}$$

where $\sigma_j = \sum_{i=0}^{l-1} a_i \varepsilon_j^i$, $\lambda_{k,j} = \mu_k^m \sigma_j$.

Thus, the following assertion is proved.

Theorem 4.1. *Let the coefficients $a_q, q = 0, 1, \dots, l-1$ in Problem 3 be such that for all $j = 0, 1, \dots, l-1$ the conditions $\sigma_j = \sum_{q=0}^{l-1} a_q \varepsilon_j^q \neq 0$ are satisfied, and let the functions w_k and numbers μ_k be eigenfunctions and eigenvalues, respectively, of Problem (4.1). Then the functions*

$$u_{k,j}(x) = \frac{1}{l} \sum_{s=0}^{l-1} \varepsilon_j^s w_k(Q^s x)$$

and the numbers $\lambda_{k,j} = \mu_k^m \sigma_j$ are eigenfunctions and eigenvalues, respectively, of Problem 3. The system of functions $\{u_{k,j}\}_{k=1; j=0}^{\infty; l-1}$ is complete in the space $L_2(\Omega)$.

5 Eigenfunctions and eigenvalues of Problem 4

As in the case of Problem 3, to construct the eigenfunctions and eigenvalues of Problem 4, we use Dirichlet spectral problem (4.1).

Let w_k and $\mu_k, k \in \mathbb{N}$ be eigenfunctions and eigenvalues, respectively, of problem (4.1). For an arbitrary $p \in \{0, 1, \dots, 2^n - 1\}$, we construct functions of form (3.1), i.e. functions

$$u_k^p(x) = \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} w_k(P_n^{q_n} \dots P_1^{q_1} x).$$

Theorem 5.1. *Let the functions w_k and numbers $\mu_k, k \in \mathbb{N}$ be eigenfunctions and eigenvalues, respectively, of problem (4.1). If in Problem 4 the coefficients $a_j, j = 0, 1, \dots, 2^n - 1$ are such that for any $p \in \{0, 1, \dots, 2^n - 1\}$ the conditions*

$$\theta_p = \sum_{i=0}^{2^n-1} (-1)^{\langle p, i \rangle} a_i \neq 0$$

are satisfied, then the system of functions $\{u_k^p(x) : k \in \mathbb{N}, p \in \{0, 1, \dots, 2^n - 1\}\}$ is a system of eigenfunctions of Problem 4, and the corresponding eigenvalues are defined by the equalities $\lambda_k^p = \theta_p \mu_k^m$. Moreover, the system $\{u_k^p(x) : k \in \mathbb{N}, p \in \{0, 1, \dots, 2^n - 1\}\}$ is a complete and orthonormal system of functions in the space $L_2(\Omega)$.

Proof. According to Lemma 3.1 (for $m = 1$) the functions u_k^p are eigenfunctions of Dirichlet problem (2.1), (2.2), corresponding to the eigenvalues μ_k . Then they must also satisfy the conditions of problem (4.2). Indeed, as

$$-\Delta u_k^p(x) = \mu_k u_k^p(x), x \in \Omega,$$

then according to (3.3), $u_k^p(x)|_{\partial\Omega} = 0$. Hence, for any $j = 1, 2, \dots, m-1$, $\Delta^j u_k^p(x)|_{\partial\Omega} = 0$ and

$$(-\Delta)^m u_k^p(x) = \mu_k^m u_k^p(x), \quad x \in \Omega.$$

Moreover, the function u_k^p satisfies the equality

$$u_k^p(P_1^{i_1} P_2^{i_2} \dots P_n^{i_n} x) = (-1)^{p_1 i_1 + p_2 i_2 + \dots + p_n i_n} u_k^p(x).$$

Indeed, similarly to the proof of Theorem 3.1, using the commutativity of the matrices $P_1, P_2, \dots, P_n$, we can write

$$\begin{aligned} u_k^p(P_1^{i_1} P_2^{i_2} \dots P_n^{i_n} x) &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} w_k(P_n^{q_n} \dots P_1^{q_1} P_1^{i_1} \dots P_n^{i_n} x) \\ &= \frac{1}{2^n} \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle} (-\Delta)^m w_k(P_n^{q_n+i_n} \dots P_1^{q_1+i_1} x). \end{aligned}$$

Replacing the index in the last sum as $q \rightarrow q - i$, and taking into account that

$$\langle p, q \rangle = \sum_{k=1}^n p_k q_k \rightarrow \sum_{i=1}^n p_k (q_k - i_k) = \langle p, q \rangle - \langle p, i \rangle,$$

we obtain:

$$\begin{aligned} u_k^p(P_1^{i_1} P_2^{i_2} \dots P_n^{i_n} x) &= \sum_{q=0}^{2^n-1} (-1)^{\langle p, q \rangle - \langle p, i \rangle} w_k(P_n^{q_n} \dots P_1^{q_1} x) \\ &= (-1)^{-\langle p, i \rangle} u_k^p(x) = (-1)^{\langle p, i \rangle} u_k^p(x). \end{aligned}$$

Then, we obtain:

$$\begin{aligned} L_P u_k^p(x) &= \sum_{i=0}^{2^n-1} a_i (-\Delta)^m u_k^p(P_n^{i_n} \dots P_1^{i_1} x) = \mu_k^m \sum_{i=0}^{2^n-1} a_i u_k^p(P_n^{i_n} \dots P_1^{i_1} x) \\ &= \mu_k^m \sum_{i=0}^{2^n-1} a_i (-1)^{p_1 i_1 + \dots + p_n i_n} u_k^p(x) = \mu_k^m u_k^p(x) \sum_{i=0}^{2^n-1} a_i (-1)^{p_1 i_1 + \dots + p_n i_n} \\ &= \theta_p \mu_k^m u_k^p(x) = \lambda_k^p v_k^p(x), \end{aligned}$$

for all $x \in \Omega$.

Thus, the functions u_k^p satisfy all conditions of Problem 4. The completeness of the system $\{u_k^p(x) : k \in \mathbb{N}, p \in \{0, 1, \dots, 2^n - 1\}\}$ is proved by the same method as in the case of Problem 2. $\square$

6 Conclusion

In this paper, spectral problems for nonlocal analogues of the polyharmonic operator are studied. The Dirichlet and Dirichlet-type boundary value problems for two types of nonlocal analogues of the polyharmonic operator are considered. The eigenfunctions and eigenvalues of these problems are determined. Theorems on the completeness of systems of eigenfunctions in the space $L_2(\Omega)$ are proved. The aim of further research is to study initial boundary value problems for nonlocal analogues of parabolic and hyperbolic equations with nonlocal polyharmonic operators.

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