

APPLICATION OF A PARTIAL-ORDER RELATION TO THE PROBLEM OF ABSENCE OF NON-TRIVIAL SOLUTIONS TO INEQUALITIES IN THE COMPLEX SPACE

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Abstract. This article examines the possibility of constructing a definition of an order relation of complex numbers and its applications to solving mathematical problems. As a result, we derive a condition for the absence of weak solutions to second-order semilinear equations and inequalities, and expand the concept of Hölder's inequality in the complex space based on the definition of a partial order.

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1 Introduction

It is known that comparing two complex numbers is not always possible, but in some problems we need to compare two complex values. This is where the relevance of our research comes from.

The article is divided into two main sections. In the first section, we define ordered complex sets and thus define an ordered complex subspace of a complex space and look at the properties of this space.

In the second section we apply the definitions contained in the first section to three mathematical problems. We first expand the concepts of Young's inequality and Hölder's inequality, which are based on the concept of a partial order. Then, we study the absence of weak solutions to second-order semilinear equations and inequalities and obtain a condition for the absence of a non-trivial solution. Finally, we study holomorphic functions based on the concept of a partial order.

In our research we will need the following basic information.

Definition 1 ([4]). In order for A to be a subspace of $\mathbb{C}$, it must satisfy the conditions:

- (1) for any $z_1, z_2 \in A$ it follows that $z_1 + z_2 \in A$,
- (2) for any $z \in A$, then $\lambda z \in A$, where λ is any real number.

Definition 2 ([4]). A sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of a metric space X is called fundamental if for any $\epsilon > 0$ there exists such a number $N_\epsilon \in \mathbb{N}$, that $\rho(x_n, x_m) < \epsilon$ for all $n \geq m > N_\epsilon$.

Definition 3 ([4]). If any fundamental sequence converges in the space X , then this space is called *complete*.

Definition 4 ([1]). Let $E \subset \mathbb{R}^n$ be an open set, $\alpha \in \mathbb{N}_0^n$, $\alpha \neq 0$ and $f, g \in L_{loc}^q(E)$. A function g is a weak derivative of a function f of order α on E (for short $g = D_w^\alpha f$) if

$$\forall \varphi \in C_0^\infty(E) \quad \int_E f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_E g \varphi dx. \quad (1.1)$$

Lemma 1.1. (*Young's inequality*). If $a \geq 0$ and $b \geq 0$ are non-negative real numbers and if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (1.2)$$

Equality holds if and only if $a^p = b^q$.

2 Definition of a partial-order relation of complex numbers

We define the following partial-order relation in the complex space $\mathbb{C}$.

Definition 5. If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, $(x_1, x_2, y_1, y_2 \in \mathbb{R})$ are some complex numbers, then we write $z_1 \geq z_2$ or, equivalently, $z_2 \leq z_1$ if the following inequalities hold:

$$x_1 \geq x_2, \quad y_1 \geq y_2. \quad (2.1)$$

If at least one inequality in (2.1) is strict, then we write $z_1 > z_2$ or, equivalently, $z_2 < z_1$.

Definition 6. Let us define an *ordered set* $C_A \subset \mathbb{C}$ as a set for which any two elements $z_1, z_2 \in C_A$ satisfy either $z_1 \geq z_2$ or $z_1 \leq z_2$.

This means that if $C_A = \{z_j\}_{j=1}^n$ and the real parts have the following order: $x_1 \geq x_2 \geq \dots \geq x_n$, then the imaginary parts have the same order: $y_1 \geq y_2 \geq \dots \geq y_n$. For example:

- (1) $C_{A_1} = \{-2 - 3i, -1 + i, 1 + 2i\}$. It is obvious that $-2 \leq -1 \leq 1$ have the same order as $-3 \leq 1 \leq 2$,
- (2) $C_{A_2} = \{-6 - 3i, -3 - i, 1 + 5i, 7 + 8i\}$, here also $-6 \leq -3 \leq 1 \leq 7$ have the same order as $-3 \leq -1 \leq 5 \leq 8$.

We note that if $z_1 \geq z_2$, then $0 \geq x_1 \geq x_2 \geq x_1 + x_2$. However, this does not imply that $y_1 \geq y_2 \geq y_1 + y_2$. For example, for $z_1 = -2 - 3i, z_2 = -1 + i$ one has $x_1 + x_2 < x_1 < x_2$, but $y_1 + y_2 \not\leq y_1 < y_2$.

Definition 7. We say that ordered set $C_A \subset \mathbb{C}$ is a *strictly ordered set* if its elements satisfy the following conditions:

- (1) numbers $x_1, x_2, x_3 = x_1 + x_2$ have the same order as $y_1, y_2, y_3 = y_1 + y_2$,
- (2) for any $\alpha \in \mathbb{R}$ and $\alpha x_j \geq x_j$, then $\alpha y_j \geq y_j$, or if $\alpha x_j \leq x_j$, then $\alpha y_j \leq y_j$.

Notation: $z_1 \geq z_2$.

For example, let $z_1 = -3 - i, z_2 = 1 + 4i$. According to the above $z_1 \leq z_2$ because $x_1 = -3 \leq x_1 + x_2 = -2 \leq x_2 = 1$ and $y_1 = -1 \leq y_1 + y_2 = 3 \leq y_2 = 4$.

However, $z_1 = -3 - i \not\leq z_2 = 1 + 4i$, because $x_1 = -3 \leq x_1 + x_2 = -2 \leq x_2 = 1$ and $y_1 = -1 \leq y_1 + y_2 = 3 \leq y_2 = 4$.

We note that in the example after Definition 6 the set C_{A_1} is ordered, but not strictly ordered, and the set C_{A_2} is strictly ordered.

Remark 1. Let $C_A \subset \mathbb{C}$ be a strictly ordered set. If for any $\alpha \in \mathbb{R}$, any $n \in \mathbb{N}$ and any $z_j \in C_A, j = 1, \dots, n$,

$$\alpha \sum_{j=1}^n z_j \in C_A, \quad (2.2)$$

then C_A is a subspace of $\mathbb{C}$. We call it an *ordered complex subspace* of the space $\mathbb{C}$.

Proof. Let $\alpha = 1, n = 2$, then for any $z_1, z_2 \in C_A$ it follows that $\sum_{j=1}^2 z_j = z_1 + z_2 \in C_A$.

Let $\alpha \in \mathbb{R}, n = 1$, then for any $z \in C_A$ it follows that $\alpha z \in C_A$. By Definition 1 C_A is a subspace of $\mathbb{C}$. $\square$

Let us give some examples of such spaces.

- (1) $C_{A_1} = \{z \in \mathbb{C} | \text{Im}(z) = 0\} = \mathbb{R}$ is an ordered complex subspace of the space $\mathbb{C}$.
- (2) $C_{A_2} = \{z \in \mathbb{C} | \text{Re}(z) = 0\} = \{z = yi \mid y \in \mathbb{R}\}$ is an ordered complex subspace of the space $\mathbb{C}$.
- (3) Let $z_0 = a + ib$ where a, b are real numbers, simultaneously positive or negative. Then it is easy to prove that $C_{A_3} = \{\alpha z_0\}_{\alpha \in \mathbb{R}}$ is an ordered complex subspace of the space $\mathbb{C}$.

- (4) Let $z_0 = a - ib$, where a, b are positive numbers. Then $C_{A_3} = \{\alpha z_0\}_{\alpha \in \mathbb{R}}$ is an unordered complex subspace of the space $\mathbb{C}$, because if $z_0 = 1 - i$ and let $\alpha_1 = 1, \alpha_2 = -2$ then we get $z_1 = 1 - i, z_2 = -2 + 2i$. In this case $x_1 > x_2, y_1 < y_2$.

We can notice that this subspace is complete. To prove that we need the following remark.

Remark 2. A necessary and sufficient condition for a sequence $\{z_n = x_n + iy_n\}_{n \in \mathbb{N}}$ to be fundamental in $\mathbb{C}$ is that both sequences $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$ should be fundamental in $\mathbb{R}$.

Theorem 2.1. *Let C_A be ordered complex subspace of the space $\mathbb{C}$. Then C_A is complete.*

Proof. Let the sequence $\{z_n\}_{n \in \mathbb{N}} \subset C_A$ be fundamental and

$$\lim_{n \rightarrow \infty} z_n = z$$

. Then the theorem is true if we prove that $z \in C_A$.

Let $z_0 \in C_A$ be such that $z_0 \leq z_j$ for each $z_j \in \{z_n\}_{n \in \mathbb{N}}$. Then we have the following:

- (1) $z_0 \leq z_n$ with $n \rightarrow \infty$ we get $z_0 \leq z$;
- (2) Let $x_0 \leq x_n \leq x_0 + x_n \Leftrightarrow y_0 \leq y_n \leq y_0 + y_n$ with $n \rightarrow \infty$ we get by Remark 2

$$x_0 \leq x \leq x_0 + x \Leftrightarrow y_0 \leq y \leq y_0 + y;$$

- (3) Let $\lambda x_n \leq x_n \Leftrightarrow \lambda y_n \leq y_n$ with $n \rightarrow \infty$ we get $\lambda x \leq x \Leftrightarrow \lambda y \leq y$ and vice versa. Let $\lambda x_n \geq x_n \Leftrightarrow \lambda y_n \geq y_n$ with $n \rightarrow \infty$ we get $\lambda x \geq x \Leftrightarrow \lambda y \geq y$.

From the above we find that $z \in C_A$ and C_A is complete. □

Example. Obviously, C_{A_1} and C_{A_2} are complete spaces, and let us prove that C_{A_3} is also a complete space. Let $\{z_n = \alpha_n z_0\}_{n \in \mathbb{N}}$ be a sequence in C_A such that α_n is a fundamental sequence in $\mathbb{R}$ and $\lim_{n \rightarrow \infty} \alpha_n = \alpha \in \mathbb{R}$. Then $\{z_n\}$ is fundamental in C_A because for any $\epsilon > 0$ there is $N_\epsilon = |z_0| \epsilon_1 > 0$ where

$$|z_n - z_m| = |\alpha_n z_0 - \alpha_m z_0| \leq |z_0| |\alpha_n - \alpha_m| \leq |z_0| \epsilon_1 \leq \epsilon$$

for any $n, m > N_\epsilon$ and $\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} \alpha_n z_0 = \alpha z_0 \in C_A$.

Definition 8. Let $C_F = \{f_j\}_{j \in \mathbb{N}}$ be a family of functions $f_j : G_j \rightarrow \mathbb{C}$, where $G_j \subseteq \mathbb{C}$ is the domain of f_j , and assume that $\bigcap_{j \in \mathbb{N}} G_j \neq \emptyset$. We say that C_F is a *strictly ordered function set* if, for any $z \in \bigcap_{j \in \mathbb{N}} G_j$ either $f_j(z) \leq f_k(z)$ or $f_j(z) \geq f_k(z)$.

This means that:

- (1) if $u_j \geq u_l$ then $v_j \geq v_k$, where $f_j(z) = u_j(x, y) + iv_j(x, y)$ and $f_k(z) = u_k(x, y) + iv_k(x, y)$,
- (2) $u_j, u_k, u_j + u_k$ have the same number order as $v_j, v_k, v_j + v_k$,
- (3) for any $\alpha \in \mathbb{R}, j \in \mathbb{N}$, if $\alpha u_j \geq u_j$ one has $\alpha v_j \geq v_j$ or if $\alpha u_j \leq u_j$ then $\alpha v_j \leq v_j$.

Definition 9. Let $C_F = \{f_j\}_{j=1}^\infty$ be a strictly ordered set of functions. For any $\alpha \in \mathbb{R}$ and any $n \in \mathbb{N}$, if the element $\alpha \sum_{j=1}^n f_j \in C_F$, then C_F is an ordered complex subspace.

3 Application of previous concepts to solving mathematical problems

In this section, we will use the previous definitions to solve some mathematical problems. For example, we will study the solutions to inequalities that involve complex variables.

Consider the differential inequality

$$\frac{\partial u}{\partial x}(x, y) + y + 2i \leq x - i \frac{\partial v}{\partial y}(x, y), \quad u(0, y) = W(y), \quad v(x, 0) = M(x). \quad (3.1)$$

for functions $u(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$, $v(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that u , $\frac{\partial u}{\partial x}$, v , and $\frac{\partial v}{\partial y}$ are continuous on $\mathbb{R}^2$ and the functions $W, M : \mathbb{R} \rightarrow \mathbb{R}$ are continuous on $\mathbb{R}$. By Definition 5 this means for all $x, y \in \mathbb{R}$

$$\frac{\partial u}{\partial x}(x, y) \leq x - y; \quad \frac{\partial v}{\partial y}(x, y) \leq -2.$$

Then, for all $x, y \in \mathbb{R}$

$$\int_0^x \frac{\partial u}{\partial x}(\xi, y) d\xi \leq \int_0^x (\xi - y) d\xi, \quad \int_0^y \frac{\partial v}{\partial y}(x, \eta) d\eta \leq \int_0^y (-2) d\eta.$$

Hence,

$$u(x, y) \leq \frac{x^2}{2} - xy + W(y); \quad v(x, y) \leq -2y + M(x). \quad (3.2)$$

Thus, $f(z) = u + iv \leq g(z) = \frac{x^2}{2} - xy + W(y) + i(-2y + M(x))$. The set of all solutions to differential inequality (3.1) is the ordered set of all functions C_f .

3.1 Young's and Hölder's inequalities

In this subsection, we will extend the concept of Young's and Hölder's inequalities to complex space on the base of the definition of a partial order.

Theorem 3.1. *If $a_s, b_s \in \mathbb{C}$, $s = 1, 2, \dots, m$, where $\text{Arg}(a_s) + \text{Arg}(b_s) \in [0, \frac{\pi}{2}]$, and if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sum_{s=1}^m a_s b_s \leq (1 + i) \left(\sum_{s=1}^m |a_s|^p \right)^{1/p} \left(\sum_{s=1}^m |b_s|^q \right)^{1/q}; \quad m \in \mathbb{N}. \quad (3.3)$$

Proof. Let $a_s = |a_s|e^{i\theta_s}$, $b_s = |b_s|e^{i\vartheta_s} \in \mathbb{C}$, then

$$a_s b_s = |a_s| |b_s| e^{i(\theta_s + \vartheta_s)} = |a_s b_s| e^{i(\theta_s + \vartheta_s)}; \quad \theta_s + \vartheta_s \in [0, \frac{\pi}{2}]$$

$$|a_s b_s| = |a_s| |b_s| |e^{i(\theta_s + \vartheta_s)}| = |a_s| |b_s|$$

and we have

$$0 \leq \cos(\theta_s + \vartheta_s) \leq 1, \quad 0 \leq \sin(\theta_s + \vartheta_s) \leq 1.$$

Then, by Definition 5

$$0 \leq |a_s| |b_s| \cos(\theta_s + \vartheta_s) + |a_s| |b_s| i \sin(\theta_s + \vartheta_s) \leq |a_s| |b_s| (1 + i).$$

$$\begin{aligned} \sum_{s=1}^m |a_s b_s| e^{i(\theta_s + \vartheta_s)} &= \sum_{s=1}^m \left(|a_s b_s| \cos(\theta_s + \vartheta_s) \right) + i \sum_{s=1}^m \left(|a_s b_s| \sin(\theta_s + \vartheta_s) \right) \\ &\leq \sum_{s=1}^m |a_s b_s| + i \sum_{s=1}^m |a_s b_s| = \sum_{s=1}^m |a_s b_s| (1 + i). \end{aligned}$$

By Hölder's inequality we get

$$\sum_{s=1}^m |a_s b_s| e^{i(\theta_s + \vartheta_s)} \leq \left(\sum_{s=1}^m |a_s|^p \right)^{1/p} \left(\sum_{s=1}^m |b_s|^q \right)^{1/q} (1+i).$$

□

Lemma 3.1. *Let $a, b \in \mathbb{C}$ and let $p > 1$ and $q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $\text{Arg}(a) + \text{Arg}(b) \in [\pi, \frac{3}{2}\pi]$, then*

$$ab \geq \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right) (-1 - i). \quad (3.4)$$

(ii) *If $\text{Arg}(a) + \text{Arg}(b) \in (\frac{\pi}{2}, \pi)$, then*

$$-\left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right) < ab < i \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right). \quad (3.5)$$

(iii) *If $\text{Arg}(a) + \text{Arg}(b) \in (\frac{3}{2}\pi, 2\pi)$, then*

$$-i \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right) < ab < \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right). \quad (3.6)$$

Proof. Let $a = |a|e^{i\theta}$, $b = |b|e^{i\vartheta} \in \mathbb{C}$, then $ab = |a||b| e^{i(\theta+\vartheta)}$. By Lemma 1.1 we have $|a| |b| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$,

(i) If $\theta + \vartheta \in [\pi, \frac{3}{2}\pi]$, then

$$0 \geq \text{Re}(ab) = |a| |b| \cos(\theta + \vartheta) \geq -|a| |b| \geq -\left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right),$$

$$0 \geq \text{Im}(ab) = |a| |b| \sin(\theta + \vartheta) \geq -|a| |b| \geq -\left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right).$$

According to Definition 5 we get

$$ab = |a| |b| \cos(\theta + \vartheta) + i|a| |b| \sin(\theta + \vartheta) \geq \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right) (-1 - i).$$

(ii) If $\theta + \vartheta \in (\frac{\pi}{2}, \pi)$, by Definition 5 we get

$$-1 + 0i < \cos(\theta + \vartheta) + i\sin(\theta + \vartheta) = e^{i(\theta+\vartheta)} < 0 + i$$

and

$$-\left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right) < -|a||b| < |a||b|e^{i(\theta+\vartheta)} < i|a||b| < i \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right).$$

(iii) If $\theta + \vartheta \in (\frac{3}{2}\pi, 2\pi)$, we have by Definition 5

$$0 - i < \cos(\theta + \vartheta) + i\sin(\theta + \vartheta) = e^{i(\theta+\vartheta)} < 1 + 0i$$

and

$$-i \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right) < -i|a||b| < |a||b|e^{i(\theta+\vartheta)} < |a||b| < \left(\frac{|a|^p}{p} + \frac{|b|^q}{q} \right).$$

□

Remark 3. (Hölder's inequality for functions of complex variable) If $f(z), g(z)$ are functions defined and continuous in the domain D , γ is piecewise smooth curve lying in D , and if $p > 1, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $\text{Arg}(f) + \text{Arg}(g) \in [0, \frac{\pi}{2}]$ one has

$$\int_{\gamma} f g dz \leq \left(\int_{\gamma} |f|^p dx \right)^{1/p} \left(\int_{\gamma} |g|^q dx \right)^{1/q} + i \left(\int_{\gamma} |f|^p dy \right)^{1/p} \left(\int_{\gamma} |g|^q dy \right)^{1/q}, \quad (3.7)$$

where $z = x + iy$.

Proof. The assumption $\text{Arg}(f) + \text{Arg}(g) \in [0, \frac{\pi}{2}]$ gives

$$0 \leq e^{(\text{Arg}(f) + \text{Arg}(g))i} = \cos(\text{Arg}(f) + \text{Arg}(g)) + i \sin(\text{Arg}(f) + \text{Arg}(g)) \leq 1 + i,$$

$$\int_{\gamma} f g dz = \int_{\gamma} |f g| e^{(\text{Arg}(f) + \text{Arg}(g))i} dz \leq (1 + i) \int_{\gamma} |f g| dz, \quad (3.8)$$

$$\int_{\gamma} |f g| dz = \int_{\gamma} |f g| dx + i \int_{\gamma} |f g| dy. \quad (3.9)$$

Now, by using Hölder's inequality for the real and imaginary parts of (3.9) and by Definition 5 we obtain

$$\int_{\gamma} f g dz \leq \left(\int_{\gamma} |f|^p dx \right)^{1/p} \left(\int_{\gamma} |g|^q dx \right)^{1/q} + i \left(\int_{\gamma} |f|^p dy \right)^{1/p} \left(\int_{\gamma} |g|^q dy \right)^{1/q}, \quad (3.10)$$

and the statement of the remark is proved. $\square$

Note that if we use Young's inequality for the real and imaginary parts of the complex integral (3.9), we obtain

$$\int_{\gamma} |f g| dx \leq \frac{1}{p} \int_{\gamma} |f|^p dx + \frac{1}{q} \int_{\gamma} |g|^q dx, \quad (3.11)$$

$$\int_{\gamma} |f g| dy \leq \frac{1}{p} \int_{\gamma} |f|^p dy + \frac{1}{q} \int_{\gamma} |g|^q dy. \quad (3.12)$$

By Definition 5 we can write

$$\int_{\gamma} f g dz \leq \left(\frac{1}{p} \int_{\gamma} |f|^p dz + \frac{1}{q} \int_{\gamma} |g|^q dz \right) (1 + i). \quad (3.13)$$

3.2 Absence of non-trivial solutions to inequalities

In this subsection we consider obtaining a condition for the absence of a weak and nontrivial solution to first-order evolution problems with complex variables.

Let the functions $H(f), L(f), G(f) \in C_F$, for any $f(z, t) \in L_{loc}^q(\mathbb{C}^{n+1}), q \geq 1$, where C_F is an ordered complex subspace, be defined in the following way:

$$\left. \begin{aligned} L(f(z, t)) &= \sum_{|\alpha|=m} (D)^{\alpha} (a_{\alpha}(z, t) f^p(z, t)); p \geq 1, \\ H(f(z, t)) &= \frac{\partial f(z, t)}{\partial t}, \quad G(f(z, t)) = f^q(z, t). \end{aligned} \right\} \quad (3.14)$$

Here, $a_{\alpha}(z, t)$ are bounded measurable functions defined for $t \geq 0, z \in \mathbb{C}^n$. Let us now study the existence and absence of solutions to the following first-order evolution problem:

$$\left\{ \begin{aligned} H(f(z, t)) &\geq L(f(z, t)) + G(f(z, t)); \quad (z, t) \in B^n \times (0, \infty), \\ f(z, 0) &= f_0(z) \geq 0; \quad z \in B^n. \end{aligned} \right. \quad (3.15)$$

The absence of a solution to this problem in real space has already been studied by V.A. Galaktionov, Yu.V. Egorov, V.A. Kondratiev, and S.I. Pokhozhaev [2], [3], and they got the following result.

Definition 10 ([5]). A weak solution f of (3.15) where $z = x \in \mathbb{R}^n$ is defined as a non-negative function $f(x, t)$ satisfying the inequality

$$-\int_0^T \int_{\mathbb{R}^n} f \frac{\partial \varphi}{\partial t} dx dt - \int_{\mathbb{R}^n} f_0 \varphi(x, 0) dx \geq \int_0^T \int_{\mathbb{R}^n} f^p L^*(\varphi) dx dt + \int_0^T \int_{\mathbb{R}^n} f^q \varphi dx dt \quad (3.16)$$

for any function $\varphi \in C_0^m(\mathbb{R}^{n+1})$, $\varphi(x, t) \geq 0$, where

$$L^*(\varphi) = \sum_{|\alpha|=m} a_\alpha(z, t) (-D)^\alpha \varphi^p(z, t).$$

Theorem 3.2 ([5]). If a non-negative function $f(x, t)$, belonging to the class $L_{loc}^1(\mathbb{R}_+^{n+1}) \cap L_{loc}^q(\mathbb{R}_+^{n+1})$ and such that $f_0(x) \in L_{loc}^1(\mathbb{R}_+^{n+1})$, satisfies inequality (3.16) and if $1 \leq p < q \leq p + m/n$, then $f(x, t) \equiv 0$.

By Definition 8 and inequality (3.15) we get

$$\left. \begin{aligned} \operatorname{Re}(H(f(z, t))) &\geq \operatorname{Re}(L(f(z, t))) + \operatorname{Re}(G(f(z, t))), \\ \operatorname{Im}(H(f(z, t))) &\geq \operatorname{Im}(L(f(z, t))) + \operatorname{Im}(G(f(z, t))), \\ \operatorname{Re}(f_0(z)) &\geq 0; \operatorname{Im}(f_0(z)) \geq 0. \end{aligned} \right\} \quad (3.17)$$

To make it easier to calculate f^p, f^q , we will use polar coordinates.

Let $z = \rho e^{i\theta} \in \mathbb{C}^n$, $\rho \in ([0, \infty))^n$, $\theta \in ([0, 2\pi])^n$, $f(z, t) = R(\rho, \theta, t) e^{i\vartheta(\rho, \theta, t)}$ and $a_\alpha(z, t) = A_\alpha(\rho, \theta, t) e^{i\phi_\alpha(\rho, \theta, t)}$, where $R \in L_{loc}^q(V)$, $V = ([0, \infty))^n \times ([0, 2\pi])^n \times (0, \infty)$ and $\vartheta : V \rightarrow [0, 2\pi]$, (3.17) takes the form:

$$\left. \begin{aligned} \frac{\partial(R \cos \vartheta)}{\partial t} &\geq L_{\operatorname{Re}}(f) + R^q \cos(q\vartheta), \\ \frac{\partial(R \sin \vartheta)}{\partial t} &\geq L_{\operatorname{Im}}(f) + R^q \sin(q\vartheta), \\ R_0 \cos(\vartheta_0) &= R(\rho, \theta, 0) \cos(\vartheta(\rho, \theta, 0)) \geq 0, \\ R_0 \sin(\vartheta_0) &= R(\rho, \theta, 0) \sin(\vartheta(\rho, \theta, 0)) \geq 0, \end{aligned} \right\} \quad (3.18)$$

where

$$\left. \begin{aligned} L_{\operatorname{Re}}(f) &= \sum_{|\alpha|=m} D^\alpha (A_\alpha R^p \cos(\phi_\alpha + p\vartheta)), \\ L_{\operatorname{Im}}(f) &= \sum_{|\alpha|=m} D^\alpha (A_\alpha R^p \sin(\phi_\alpha + p\vartheta)). \end{aligned} \right\} \quad (3.19)$$

According to Definition 10, the non-negative functions $R(\rho, \theta, t) \cos(\vartheta(\rho, \theta, t))$, $R(\rho, \theta, t) \sin(\vartheta(\rho, \theta, t))$ satisfy the inequalities

$$\left. \begin{aligned} -\int_V R \cos \vartheta \frac{\partial \varphi}{\partial t} dV - \int_S R_0 \cos(\vartheta_0) \varphi(\rho, \theta, 0) dS &\geq \int_V R^p L_c(\varphi) dV + \int_V R^q \cos(q\vartheta) \varphi dV, \\ -\int_V R \sin \vartheta \frac{\partial \varphi}{\partial t} dV - \int_S R_0 \sin(\vartheta_0) \varphi(\rho, \theta, 0) dS &\geq \int_V R^p L_s(\varphi) dV + \int_V R^q \sin(q\vartheta) \varphi dV, \end{aligned} \right\} \quad (3.20)$$

where $\varphi \geq 0$, $\varphi \in C_0^m(\mathbb{R}^{2n+1})$,

$$\left. \begin{aligned} \int_S g(\rho, \theta) dS &= \int_0^{+\infty} \dots \int_0^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} g(\rho, \theta) d\rho_1 \dots d\rho_n d\theta_1 \dots d\theta_n, \\ \int_V g(\rho, \theta) dV &= \int_0^T \int_0^{+\infty} \dots \int_0^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} g(\rho, \theta) d\rho_1 \dots d\rho_n d\theta_1 \dots d\theta_n dt, \end{aligned} \right\} \quad (3.21)$$

and

$$\left. \begin{aligned} L_c(\varphi) &= \sum_{|\alpha|=m} (A_\alpha \cos(\phi_\alpha + p\vartheta) (-D)^\alpha(\varphi)), \\ L_s(\varphi) &= \sum_{|\alpha|=m} (A_\alpha \sin(\phi_\alpha + p\vartheta) (-D)^\alpha(\varphi)). \end{aligned} \right\} \quad (3.22)$$

Then, taking into account inequalities (3.20) and Definition 5, we come to the following definition.

Definition 11. A function $f(z, t) \in L_{loc}^1(\mathbb{C}^{n+1}) \cap L_{loc}^q(\mathbb{C}^{n+1})$ such that $f_0(z) \in L_{loc}^1(\mathbb{C}^n)$ is called a weak solution to problem (3.15) if $\text{Arg}(f) \in [0, \frac{\pi}{2q}]$ and the following inequality holds:

$$-\int_V f \frac{\partial \varphi}{\partial t} dV - \int_S f_0 \varphi_0 dS \geq \int_V f^p L^*(\varphi) dV + \int_V f^q \varphi dV \quad (3.23)$$

for any function $\varphi \geq 0$, $\varphi \in C_0^m(\mathbb{R}_+^{2n+1})$, where $L^*(\varphi) = L_c(\varphi) + iL_s(\varphi)$.

In its traditional formulation, it is usually assumed that the function f is real-valued and non-negative. Moreover, f may be interpreted as a physical quantity, such as density, concentration, or temperature. This model is based on the interaction of two fundamental mechanisms: diffusion and growth.

In the present work, we begin by addressing the following fundamental question: can the traditional formulation of inequality (3.23) be extended so that the function f becomes complex-valued, thereby allowing it to be represented simultaneously in terms of amplitude and phase, while preserving the physical interpretation of the model?

Such an extension involves more than merely changing the mathematical nature of the unknown function. It introduces a new mechanism into the dynamics of the amplitude: its evolution is determined not only by the interplay between diffusion and growth, but also depends directly on the phase and its spatial gradient. Thus, the proposed extension provides a foundation for further development of the model and its subsequent application to problems involving wave propagation.

Theorem 3.3. If a function $f(z, t) \in L_{loc}^1(\mathbb{C}^{n+1}) \cap L_{loc}^q(\mathbb{C}^{n+1})$ is such that $f_0(z) \in L_{loc}^1(\mathbb{C}^n)$, $\text{Arg}(f) \in [0, \frac{\pi}{2q}]$, satisfies inequality (3.15) and

$$1 \leq \frac{p+1}{2} < q \leq \frac{n+\nu}{n+\nu-m} p, \quad \nu \geq 2, \quad (3.24)$$

then $f(x, t) \equiv 0$.

Proof. We have $\text{Arg}(f) \in [0, \frac{\pi}{2q}] \subset [0, \pi/2]$, then the real and imaginary parts of the function $f(z, t)$ are non-negative and $f(z, t) \geq 0 + 0i$. Let

$$\varphi_0(s) = \begin{cases} 1 & s \leq 1, \\ 0 & s \geq 2, \end{cases} \quad (3.25)$$

then $-\int_S f_0 \varphi_0 dS \leq 0$. By Definition 11 we get the following:

$$0 \leq \int_V f^q \varphi dV \leq -\int_V f \frac{\partial \varphi}{\partial t} dV - \int_V f^p L^*(\varphi) dV, \quad (3.26)$$

where $\varphi \in C_0^m(\mathbb{R}_+^{2n+1})$ and $f \geq 0$, then $-\int_V f \frac{\partial \varphi}{\partial t} dV \leq 0$, and we get

$$0 \leq \int_V f^q \varphi dV \leq -\int_V f^p L^*(\varphi) dV. \quad (3.27)$$

Using inequality (3.13), we obtain

$$\int_V R^q E \varphi dV \leq c_1 \int_V \frac{|L^*(\varphi)|^{q/(q-p)}}{\varphi^{p/(q-p)}} dV, \quad (3.28)$$

where $E(\vartheta, q, p) = e^{iq\vartheta} + \frac{p}{q} e^{\frac{5\pi}{4}i}$ and $c_1 = \frac{(q-p)e^{i\pi/4}}{q} \in \mathbb{C}$. Now, we choose the function

$$\varphi(\rho, \theta, t) = \varphi_0 \left(\frac{\rho^2 + t^{2/\nu}}{B^2} \right), \quad B > 0, \quad \nu \geq 2, \quad (3.29)$$

where φ_0 is defined in (3.25).

Next, we change the variables $t = B^\nu \tau$, $\rho = B\zeta$ and we get:

$$\int_V \frac{|L^*(\varphi)|^{q/(q-p)}}{\varphi^{p/(q-p)}} dV = c_2 B^{n+\nu-\frac{qm}{q-p}},$$

where

$$c_2 = \int_{\theta} \int_{\zeta} \int_{1 \leq (\tau^{1/\nu} + |\zeta| \leq \sqrt{2})} \frac{|L^*(\varphi_0)|^{q/(q-p)}}{\varphi_0^{p/(q-p)}} d\zeta_1 \dots d\zeta_n d\theta_1 \dots d\theta_n d\tau < \infty.$$

Substituting the last estimate into inequality (3.28), we get

$$\int_V R^q E \varphi dV \leq c_1 c_2 B^{n+\nu-\frac{mq}{q-p}}. \quad (3.30)$$

If $B \rightarrow \infty$ and $q < \frac{n+\nu}{n+\nu-m}p$, then $\int_V R^q E \varphi dV = 0$ and $f \equiv 0$.

□

3.3 Holomorphic functions

In this subsection we consider the possibility of proving that a function is holomorphic, based on the notions greater or less for a holomorphic function. The main contribution of this work is the development of a comparison criterion for establishing the holomorphy of complex-valued functions. The criterion relates the function under consideration to a holomorphic comparison of a function and a system of differential inequalities. Consequently, the holomorphy can be established indirectly, without requiring the direct verification of the classical Cauchy–Riemann conditions.

Theorem 3.4. *Let C_F be an ordered complex subspace. If $f(z) = u + iv \in C_F$ is a holomorphic function on $\mathbb{C}$ and there is a function $g(z) = w + ih$ satisfying the following system of inequalities:*

$$\left. \begin{aligned} \frac{du}{d\bar{z}} &\geq \frac{dw}{d\bar{z}} \\ \frac{dv}{dz} &\geq \frac{dh}{dz} \end{aligned} \right\} \forall z \in D, \quad (3.31)$$

where $D \subseteq \mathbb{C}$ is the non-empty intersection of the domains of f, g . Then $g(z)$ will be a holomorphic function on D if and only if $g(z) = f(z) + c$.

Proof. The function $f(z)$ is holomorphic, which means

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (3.32)$$

We can rewrite inequalities (3.31) in more detailed form:

$$\left. \begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} &\geq \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y}; \\ \frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} &\geq \frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y}. \end{aligned} \right\} \quad (3.33)$$

By Definition 5 and inequalities (3.33) we have

$$\left\{ \begin{aligned} \frac{\partial u}{\partial x} &\geq \frac{\partial w}{\partial x}; & \frac{\partial u}{\partial y} &\geq \frac{\partial w}{\partial y}; \\ \frac{\partial v}{\partial x} &\geq \frac{\partial h}{\partial x}; & \frac{\partial v}{\partial y} &\leq \frac{\partial h}{\partial y}. \end{aligned} \right\} \quad (3.34)$$

Now, let us prove that $\frac{\partial w}{\partial x} - \frac{\partial h}{\partial y} = 0$, $\frac{\partial w}{\partial y} + \frac{\partial h}{\partial x} = 0$,

$$\left. \begin{aligned} \frac{\partial w}{\partial x} - \frac{\partial h}{\partial y} &\leq \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0; \\ \frac{\partial w}{\partial y} + \frac{\partial h}{\partial x} &\leq \frac{\partial u}{\partial y} + \frac{\partial h}{\partial x} = 0. \end{aligned} \right\} \quad (3.35)$$

From this we obtain that $g(z)$ will be a holomorphic function on D if

$$\left. \begin{aligned} 0 &\leq \frac{\partial w}{\partial x} - \frac{\partial h}{\partial y} \\ 0 &\leq \frac{\partial w}{\partial y} + \frac{\partial h}{\partial x} \end{aligned} \right\} \Leftrightarrow \frac{dw}{d\bar{z}} \geq (-i) \frac{dh}{d\bar{z}}. \quad (3.36)$$

By (3.34) there holds

$$\begin{aligned} \frac{\partial h}{\partial y} &\leq \frac{\partial w}{\partial x} \leq \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \leq \frac{\partial h}{\partial y}, \\ -\frac{\partial h}{\partial x} &\leq \frac{\partial w}{\partial y} \leq \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \leq -\frac{\partial h}{\partial x}. \end{aligned}$$

From this we conclude that

$$\left. \begin{aligned} \frac{\partial h}{\partial y} &= \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x}, \\ -\frac{\partial h}{\partial x} &= -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} \end{aligned} \right\} \Leftrightarrow g(z) = f(z) + c. \quad (3.37)$$

□

Remark 4. Let C_F be an ordered complex subspace. If $f(z) = u + iv \in C_F$ is a holomorphic function on $\mathbb{C}$ and there is a function $g(z) = w + ih$ satisfying

$$\left. \begin{aligned} \frac{du}{d\bar{z}} &\leq \frac{dw}{d\bar{z}} \\ \frac{dv}{dz} &\leq \frac{dh}{dz} \end{aligned} \right\} \forall z \in D \subset \mathbb{C}, \quad (3.38)$$

then $g(z)$ will be a holomorphic function on D if and only if $g(z) = f(z) + c$.

Proof. We can rewrite inequalities (3.38)

$$\left. \begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} &\geq \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} &\geq \frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} \frac{\partial u}{\partial x} &\geq \frac{\partial w}{\partial x}, & \frac{\partial u}{\partial y} &\geq \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} &\geq \frac{\partial h}{\partial x}, & \frac{\partial v}{\partial y} &\leq \frac{\partial h}{\partial y} \end{aligned} \right\}. \quad (3.39)$$

Now, let us prove that $\frac{\partial w}{\partial x} - \frac{\partial h}{\partial y} = 0$, $\frac{\partial w}{\partial y} + \frac{\partial h}{\partial x} = 0$ and

$$\left. \begin{aligned} \frac{\partial w}{\partial x} - \frac{\partial h}{\partial y} &\geq \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0; \\ \frac{\partial w}{\partial y} + \frac{\partial h}{\partial x} &\geq \frac{\partial u}{\partial y} + \frac{\partial h}{\partial x} = 0. \end{aligned} \right\} \quad (3.40)$$

From this we obtain that $g(z)$ will be a holomorphic function on D if

$$\frac{\partial w}{\partial x} - \frac{\partial h}{\partial y} \leq 0; \quad \frac{\partial w}{\partial y} + \frac{\partial h}{\partial x} \leq 0. \quad (3.41)$$

By (3.39) and (3.32) there hold

$$\left. \begin{aligned} \frac{\partial h}{\partial y} &\geq \frac{\partial w}{\partial x} \geq \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \geq \frac{\partial h}{\partial y}, \\ -\frac{\partial h}{\partial x} &\geq \frac{\partial w}{\partial y} \geq \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \geq -\frac{\partial h}{\partial x} \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} \frac{\partial h}{\partial y} &= \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x}, \\ -\frac{\partial h}{\partial x} &= -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} \end{aligned} \right\} \quad (3.42)$$

and $g(z) = f(z) + c$ on D . □

Both statements lead us to the following conclusion: if $f(z) = u + iv$ is a holomorphic function on $\mathbb{C}$ and there are functions $w(x, y), h(x, y) \in \mathbb{R}^2$ satisfying $\frac{du}{d\bar{z}}, \frac{dw}{d\bar{z}}, \frac{dv}{dz}, \frac{dh}{dz} \in C_F$, which is an ordered set of functions, then the function $w(x, y) + ih(x, y) = g(z)$ will be a holomorphic function in C_F if and only if $g(z) = f(z) + c$.

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