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GENERALIZED HAMEL BASIS AND BASIS EXTENSION IN CONVEX CONES AND UNIQUELY DIVISIBLE SEMIGROUPS

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Key words: Hamel basis, convex cone, sublinear independence, divisible semigroup, uniquely divisible semigroup, cancellation law.

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Abstract. In the work, a concept of sublinear independence in an arbitrary convex cone is introduced and the corresponding generalization of Hamel basis is studied. Applying these results to the cones generated by uniquely divisible semigroups ((UD)–semigroups) allows us to extend obtained results for the class of (UD)–semigroups. Some applications are considered.

Introduction

Hamel basis (or algebraic basis) is a fundamental concept in the theory of linear spaces that is closely connected to the concept of linear independence (see, e.g. [8]). This concept was introduced originally by Hamel [15] where an example of a real discountinuous additive function was constructed. The base of such a construction is the remarkable fact of infinite linear dimension of $\mathbb{R}$ over $\mathbb{Q}$. It was explained soon an essential difference between algebraical and topological dimensions in case of the infinite-dimensional linear spaces over $\mathbb{R}$. Remind, e.g., that in any such a space Hamel basis is uncountable.

Remind also that a non-algorithmicity of constructing Hamel basis in infinite dimensional case overcomes often by the following significant result (the so-called Basic Lemma): each linearly independent subset of a linear space can be added to some Hamel basis in this space. In particular, each topological basis (Riecz basis) in a topological vector space can be added to such a basis. The numerous applications of Hamel bases are well known and turned our attention to them.

Let us pass to the problem of constructing an analogue of a Hamel basis in an abstract convex cone. It is clear that this problem is sufficiently important in modern non-smooth analysis because the convex and other types of cones play there the role that is analogous to the role of linear spaces in smooth analysis. But a series of essential obstacles takes place on this way. We list some of them.

1. First, the transition from linear spaces over $\mathbb{R}$ to the cones over $\mathbb{R}_+$ requires finding an appropriate replacement to the basic concept of linear independence itself.

2. Secondly, using of the positive coefficients leads us to the necessity of replacing of expansions of the form $x = \sum \lambda_k h_k$ by "bi-expansions" of the form $x + \sum \lambda_k h_k = \sum \mu_k h_k$.

3. Finally, "bi-expansions" are non-unique.

Thus, the present work is devoted to overcoming of the above mentioned obstacles and to researching the below cone-analogue of Hamel basis. In the section 1.1 we study the concepts of sublinear independence and bi-expansion in an abstract convex cone, in Section 1.2 we study the concept of sublinear Hamel basis therein, in Section 1.3 – some applications. Next, in Section 2 we generalize the means above over the frame of convex cones by using the concept of uniquely divisible semigroup that is recently got involved by us in [19].

1 Hamel basis in convex cone

1.1 Sublinear independence and bi-expansions

As is known, the concept of linear independence in the frame of linear spaces is based on the representation of an arbitrary vector in the form of a linear combination of the basis vectors. But in the frame of the convex cones we need to pass to a more general representation of the given vector by means of a couple of linear combinations of the basis vectors with non-negative coefficients, i.e. by means of the so-called bi-expansion. This construction leads us to the concept of sublinear independence.

Definition 1. Let X be a convex cone, $M \subset X$. Let us introduce two types of the envelopes of M:

(*i*) $L_+(M) = \{\sum_{k=1}^n \lambda_k h_k \mid h_k \in M; \lambda_k \in \mathbb{R}_+; n \in \mathbb{N}\}$ – the *plus-linear span* of M;

(*ii*) $L_{sub}(M) = \{h \in X \mid h + \sum_{k=1}^n \lambda_k h_k = \sum_{k=1}^n \mu_k h_k; h_k \in M, \lambda_k \geq 0, \mu_k \geq 0, n \in \mathbb{N}\}$ (1.1)

– the *sublinear span* of M.

We call the representation (1.1) *bi-expansion* (or *subinear expansion*) of h. If (1.1) takes place, we say that h *subinearly depends* on M.

Remark 1. 1. Using the concept of formal difference $\ominus$ (see [22]), the expansion (1.1) can be written in the form

$$L_{sub}(M) = L_+(M) \ominus L_+(M).$$

2. In the plus-linear combinations of the form $\sum_{k=1}^n \lambda_k h_k$ the vectors h_k can be repeated.

3. Since the vector $h = \sum_{k=1}^n \lambda_k h_k$ allows the trivial bi-expansion $h + 0 = \sum_{k=1}^n \lambda_k h_k$, then the following inclusion

$$L_+(M) \subset L_{sub}(M)$$

holds for any $M \subset X$. Here it is possible $L_{sub}(M) \neq L_+(M)$. Let, e.g., $X = \mathbb{R}, h_1 = 1, h_2 = -1$. Then $h_2 + (\lambda + 1)h_1 = \lambda h_1$, whence $h_2 \in L_{sub}(h_1)$ follows. However, $h_2 \notin L_+(h_1)$.

As for any convex cone the following implication

$$(x + y_1 = z_1,\ x + y_2 = z_2) \Rightarrow (y_1 + z_2 = y_2 + z_1), \tag{1.2}$$

is valid, then the corresponding property of the *sub-uniqueness* takes place.

Proposition 1.1. *The following implication (in short form)*

$$\begin{aligned} &\Big(h + \sum\nolimits^1 \lambda_k h_k = \sum\nolimits^1 \mu_k h_k \ ;\ h + \sum\nolimits^2 \lambda_k h_k = \sum\nolimits^2 \mu_k h_k \Big) \Rightarrow \\ &\Rightarrow \Big(\sum\nolimits^1 \lambda_k h_k + \sum\nolimits^2 \mu_k h_k = \sum\nolimits^2 \lambda_k h_k + \sum\nolimits^1 \mu_k h_k \Big). \end{aligned} \tag{1.3}$$

holds. But the inverse statement:

$$\left(h+\sum^{1}\lambda_k h_k=\sum^{1}\mu_k h_k\ ;\ \sum^{1}\lambda_k h_k+\sum^{2}\mu_k h_k=\sum^{2}\lambda_k h_k+\sum^{1}\mu_k h_k\right)\Rightarrow$$
$$\Rightarrow\left(h+\sum^{2}\lambda_k h_k=\sum^{2}\mu_k h_k\right)\ . \tag{1.4}$$

holds only for cones that satisfy the cancellation law (see [5]):

$$(x+y=x+z)\Rightarrow(y=z). \tag{1.5}$$

Proof. So, (1.3) follows from (1.2) immediately. In case of (1.4) we obtain (briefly):

$$\left(h+\sum\nolimits_{\lambda}^{1}=\sum\nolimits_{\mu}^{1}\right)\Rightarrow\left(h+\sum\nolimits_{\lambda}^{1}+\sum\nolimits_{\mu}^{2}=\sum\nolimits_{\mu}^{1}+\sum\nolimits_{\mu}^{2}\right)\Rightarrow$$
$$\Rightarrow\left(\left(h+\sum\nolimits_{\lambda}^{2}\right)+\sum\nolimits_{\mu}^{1}=\sum\nolimits_{\mu}^{2}+\sum\nolimits_{\mu}^{1}\right)\Rightarrow\left(h+\sum\nolimits_{\lambda}^{2}=\sum\nolimits_{\mu}^{2}\right).$$

□

Definition 2. A set $M\subset X$ is called *sublinearly closed*, if $L_{sub}(M)=M$.

Proposition 1.2. *For any $M\subset X$ the set $L_{sub}(M)$ is sublinearly closed, i.e.*

$$L_{sub}(L_{sub}(M))=L_{sub}(M). \tag{1.6}$$

Proof. Let $N=L_{sub}(M)$, then

$$L_{sub}(N)=\left\{h\in X\,\middle|\,h+\sum_{k=1}^{n}\lambda_k h_k=\sum_{k=1}^{n}\mu_k h_k\,\middle|\,\lambda_k,\mu_k\in\mathbb{R}_+;\ h_k\in N\right\}. \tag{1.7}$$

Here, in view of $h_k\in L_{sub}(M)$, we obtain

$$h_k+\sum_{l=1}^{m}\lambda_{kl}h_{kl}=\sum_{l=1}^{m}\mu_{kl}h_{kl},\ \text{for some } h_{kl}\in M \text{ and } \lambda_{kl},\mu_{kl}\mathbb{R}_+. \tag{1.8}$$

Substitution of (1.8) into (1.7), leads to $h+\sum\limits_{k=1}^{n}\sum\limits_{l=1}^{m}(\lambda_k\lambda_{kl})h_{kl}=\sum\limits_{k=1}^{n}\sum\limits_{l=1}^{m}(\mu_k\mu_{kl})h_{kl}$, whence (1.6) follows. □

1.2 Sublinear Hamel basis in convex cone

Here we describe constructing a sublinear analogue of the algebraic basis with the difference, consisting in the replacement of the standard expansion for a linear space by the bi-expansion of type (1.7)

1. Let $card(X)=J$ be the minimal order type of the cardinality $card(X)$ (see, e.g.,[11]). Choose some representative from each cone ray $\mathbb{R}_+\cdot x\in X$ and let X_R be a set of such representatives. Obviously, $card(X_R)=J$. Let us index by $j\in J$ all elements from $X_R: X_R=\{x_j\}_{j\in J}$.

2. Now, let us begin constructing a sublinear basis. Set $h_1 = x_1 = x_{j_1}$. Consider the set

$$L^R_{sub}(h_1) = \{x_j \in X_R \mid x_j + \lambda_1 h_1 = \mu_1 h_1, (\lambda_1, \mu_1 \in \mathbb{R}_+)\}. \tag{1.9}$$

Since $L_{sub}(h_1) \supset L_+(h_1)$, then $L^R_{sub}(h_1) = L_{sub}(h_1) \cap X_R$. Thus, two cases are possible.

   a) $L_{sub}(h_1) = X$. In this case any element $x \in X$ allows a bi-expansion of the type (1.9), and we can introduce Hamel basis $H = \{h_1\}$.

   b) $L_{sub}(h_1) \subsetneqq X$. In this case let us choose the minimal index $j_2 \in J, j_2 \succ j_1$, such that $x_{j_2} \notin L^R_{sub}(h_1)$, and denote by $h_2 = x_{j_2}$ the second element of a sublinear Hamel basis H.

3. Let us describe, for clearness, the second step of the constructing, too. Consider the set

$$L^R_{sub}(h_1, h_2) = \{x_j \in X_R \mid x_j + (\lambda_1 h_1 + \lambda_2 h_2) = (\mu_1 h_1 + \mu_2 h_2) \qquad (\lambda_i, \mu_i \in \mathbb{R}_+)\} \tag{1.10}$$

   The following two cases are possible.

   a) $L_{sub}(h_1, h_2) = X$. In this case, any element $x \in X$ allows a bi-expansion of form (1.10) and we can introduce sublinear Hamel basis $H = \{h_1, h_2\}$.

   b) $L_{sub}(h_1, h_2) \neq X$. In this case let us choose the minimal index $j_3 \succ j_2$ from J, such that $x_{j_3} \notin L^R_{sub}(h_1, h_2)$ and denote $h_3 = x_{j_3}$ the third element of sublinear Hamel basis H.

4. The general step of the induction. Suppose that the elements h_i of sublinear Hamel basis are chosen for $i \prec i_0$ and, in addition,

$$j_i = min\left\{j \succ j_{i'}, i' \prec i : x_{j_i} \notin L^R_{sub}(\{h_{i'}\}_{i' \prec i})\right\}.$$

   The two cases are possible:

   a) $L_{sub}(\{h_i\}_{i \preceq i_0}) = X$. In this case any element $x \in X$ allows a bi-expansion of the form

$$x + \sum_{k=1}^{n} \lambda_k h_{i_k} = \sum_{k=1}^{n} \mu_k h_{i_k} \qquad (i_k \preceq i_0), \tag{1.11}$$

   and it makes it possible to introduce the sublinear Hamel basis $H = L_{sub}(\{h_i\}_{i \preceq i_0})$.

   b) $L_{sub}(\{h_i\}_{i \preceq i_0}) \neq X$. In this case let us choose such minimal index $j_{i_1} > j_{i_0}$ from J that $x_{j_{i_1}} \notin L^R_{sub}(\{h_i\}_{i \preceq i_0})$, and set $h_{i_1} = x_{j_{i_1}}$ as the following element of sublinear Hamel basis H. Then the inductive construction can be continued.

5. Finally, according to the transfinite induction principle, we obtain a sublinear Hamel basis $H = \{h_i\}_{i \in I}$, where I belongs to J and is cofinal with J. In addition, each element $x \in X$ allows a bi-expansion of form (1.11).

6. The sub-uniqueness of bi-expansion follows from Proposition 1.2 (briefly):

$$\left(x + \sum\nolimits_{\lambda}^{1} = \sum\nolimits_{\mu}^{1}; \; x + \sum\nolimits_{\lambda}^{2} = \sum\nolimits_{\mu}^{2}\right) \Rightarrow \left(\sum\nolimits_{\lambda}^{1} + \sum\nolimits_{\mu}^{2} = \sum\nolimits_{\lambda}^{2} + \sum\nolimits_{\mu}^{1}\right). \tag{1.12}$$

Note that in the case of linear spaces condition (1.12) is equivalent to the uniqueness of the usual expansion respective to the algebraic basis:

$$x = \sum_{k=1}^{n} (\mu_k - \lambda_k) h_{i_k}.$$

□

Let us formulate the obtained result.

Theorem 1.1. *For any convex cone X there exists a sublinear Hamel basis $H = \{h_i\}_{i \in I}$ with the following properties:*

1) the elements of the basis H are sublinearly independent;

2) the basis H is minimal;

3) any vector $x \in X$ allows bi-expansion by the basis H of the form

$$x + \sum_{k=1}^{n} \lambda_k h_{i_k} = \sum_{k=1}^{n} \mu_k h_{i_k} \quad (\lambda_k \geq 0, \mu_k \geq 0), \tag{1.13}$$

and the expansion is sub-unique.

Let us consider a consequence of Theorem 1.1. First, introduce a concept of the sublinear kernel of an arbitrary convex cone.

Definition 3. Let X be a convex cone, $x \in X$. Say that x is an *invertible element* of X if there exists such element $x^- \in X$ that $x + x^- = 0$. The set of the all invertible elements of X denote by $Ker^-(X)$ (or, simply, X^-) and call it a *sublinear kernel* of X.

Note some obvious properties of the sublinear kernels.

Proposition 1.3. *Let X^- be a sublinear kernel of X. Then:*

(i) $\forall x \in X^- \, \exists! \, x^- \in X^- : \; x + x^- = 0$;

(ii) $\forall x \in X^- : \; (x^-)^- = x$;

(iii) setting $(-\lambda) \cdot x = \lambda \cdot x^-$ for $\lambda \geq 0$ we can represent X as a linear space.

Consider, as a concrete example, a dihedral angle, formed by two semihyperplanes in a linear space.

Example 1. Let E be a real linear space, H_1 and H_2 be some homogeneous hyperplanes in E, $L = H_1 \cap H_2$ ($codim H_1 = codim H_2 = 1, \; codim L = 2.$)

Then L divides each hyperplane H_i into two semihyperplanes, e.g., H_i^- and H_i^+. Set $X = co(H_1^- \cup H_2^-)$. It is easy to see that X is a convex cone and $L = ker^-(X)$. In the present case the decomposition into direct sum

$$X = X^- \oplus X^+ \; , \; X^+ = co(G_1^- \cup G_2^-),$$

where G_i^- are the corresponding semihyperplanes in H_i, is evident.

Note that in this example the cone X^+ plays a role of "planar angle" with respect to "dihedral angle" X.

Let us find out now a form of bi-extension (1.13) for the elements of the sublinear kernel X^- of an arbitrary convex cone X.

1) Construct a sublinear Hamel basis in X in a certain way, by choosing at first all possible invertible basis vectors $h_i^-(i \in I^-)$ and denoting by $H^- = \{h_i^-\}_{i\in I^-}$. All consequent basis vectors denote by $h_i^+(i \in I^+), H^+ = \{h_i^+\}_{i\in I^+}, I = I^-\dot{\cup}I^+$. Selecting in the bi-extension (1.13) the summands from H^- and H^+ leads to the equality

$$h + \left(\sum_k {}^-\lambda_k h_{i_k}^- + \sum_k {}^+\lambda_k h_{i_k}^+\right) = \left(\sum_k {}^-\mu_k h_{i_k}^- + \sum_k {}^+\mu_k h_{i_k}^+\right).$$

From here it follows

$$\left(h + \sum_k {}^-\lambda_k h_{i_k}^- - \sum_k {}^-\mu_k h_{i_k}^-\right) + \sum_k {}^+\lambda_k h_{i_k}^+ = \sum_k {}^+\mu_k h_{i_k}^+. \tag{1.14}$$

2) By denoting by h^+ the expression in parentheses in (1.14) and by setting

$$h^- = \sum_k {}^-\mu_k h_{i_k}^- - \sum_k {}^-\lambda_k h_{i_k}^-,$$

we rewrite (1.14) in a form of the system

$$\begin{cases} h^- = \sum_k {}^-\mu_k h_{i_k}^- - \sum_k {}^-\lambda_k h_{i_k}^-; \ (h_{i_k}^- \in H^-); \\ h^+ + \sum_k {}^+\lambda_k h_{i_k}^+ = \sum_k {}^+\mu_k h_{i_k}^+; \ (h_{i_k}^+ \in H^+); \\ h = h^- + h^+. \end{cases} \tag{1.15}$$

Let us mention that h^- does not depend on the choice of Hamel basis, but h^+ does depend on it.

Definition 4. Call the set $H^+ = \{h^+ | h \in X\}$ the *sublinear cokernel* of the cone X and denote it by $Coker^+(X)$.

So, taking into account the equalities (1.15) let us formulate the obtained result.

Theorem 1.2. *An arbitrary convex cone X allows decomposition into the direct sum*

$$X = X^- \oplus X^+ = Ker^-(X) \oplus Coker^+(X). \tag{1.16}$$

Here the elements from X^- allow usual "uni-expansion":

$$h^- = \sum_{k=1}^n \nu_k \cdot h_{i_k}^- \quad (\nu_k \in \mathbb{R});$$

but the elements from cokernel X^+ assume bi-expansion only:

$$h^+ + \sum_{k=1}^n \lambda_k h_{i_k}^+ = \sum_{k=1}^n \mu_k h_{i_k}^+.$$

Note also that uniqueness of h^+ is not assumed; so the term "direct sum" is meant here in generalized sense. Apparently, Theorem 1.2 can be interpreted as a "general theorem on dihedral angle".

1.3 Functional separability and embedding of a convex cone into a linear space

First of all, let us show that for each convex cone X that satisfies (CL), a linear space X'_+ of the all $\mathbb{R}_+$-linear functionals on X separates points in X. Note that it is necessary to check precisely separability of any couple $x \neq x'$ in X. In contrast to the case of linear space, we cannot restrict ourselves to checking separability of any non-zero point from zero only.

1) So, let $x \neq x'$ and some sublinear Hamel basis $H = \{h_i\}_{i \in I}$ in X be chosen. By virtue of Proposition 1.1 and Theorem 1.1, corresponding bi-expansions

$$x + \sum_k \lambda_k h_{i_k} = \sum_k \mu_k h_{i_k} \quad \text{and} \quad x' + \sum_k \lambda'_k h_{i_k} = \sum_k \mu'_k h_{i_k}$$

are not equivalent, i.e.

$$\sum_k \lambda_k h_{i_k} + \sum_k \mu'_k h_{i_k} \neq \sum_k \lambda'_k h_{i_k} + \sum_k \mu_k h_{i_k}.$$

Hence, such a number k_0 exists that

$$\lambda_{k_0} \cdot h_{i_{k_0}} + \mu'_{k_0} \cdot h_{i_{k_0}} \neq \lambda'_{k_0} \cdot h_{i_{k_0}} + \mu_k \cdot h_{i_{k_0}}. \tag{1.17}$$

2) Now, let us define a functional f_0 on H as follows:

$$f_0(h_{i_{k_0}}) = 1, \ f_0(h_i) = 0 \ (\text{при } i \neq i_{k_0}).$$

Next, continue f_0 on any vector $x \in X$ by (+)–linearity with the help of bi-expansion:

$$f_0(x + \sum_k \lambda_k h_{i_k}) = f(\sum_k \mu_k h_{i_k}).$$

From here it follows

$$f_0(x) = \sum_k (\mu_k - \lambda_k) f(h_{i_k}) = (\mu_{k_0} - \lambda_{k_0}) f(h_{i_{k_0}}) = \mu_{k_0} - \lambda_{k_0}.$$

Here, in view of (1.17), $x' \neq x$ implies

$$f_0(x') = \mu'_{k_0} - \lambda'_{k_0} \neq \mu_{k_0} - \lambda_{k_0} = f_0(x).$$

Thus, f_0 separates the point x and x' in X. Let us formulate the obtained result.

Theorem 1.3. *If X is a convex cone that satisfies (CL) then its dual linear space X'_+ separates points in X.*

Remark 2. Note that in case of absence (CL) in X the preceding scheme of proof is not applicable, in view of possible presence of repeating elements in sublinear Hamel basis. In fact, the inverse to Theorem 1.3 is also valid. Let, e.g. $x + y = x + z$, but $y \neq z$. Then, for any $f \in X'_+$ we obtain $(f(x) + f(y) = f(x) + f(z)) \Rightarrow (f(y) = f(z))$, i.e. y and z are not functionally separable with respect to X'_+. Analogous statement holds in case of absence of the distributive law $(\lambda + \mu)x \neq (\lambda x + \mu x)$ for a non-convex cone.

Now, let us pass to the bi-dual space $X''_+ = (X'_+)'$ and consider a so called "evaluation map" $\Lambda_x(f) = f(x)$ where $x \in X, f \in X'_+$. It is easy to see that

$$\Lambda_{x+y}(f) = f(x) + f(y) = \Lambda_x(f) + \Lambda_y(f); \quad \Lambda_{\alpha x}(f) = \alpha \cdot f(x) = \alpha \cdot \Lambda_x(f) \ (\alpha \in \mathbb{R}_+);$$

$$\Lambda_x(f+g) = f(x) + g(x) = \Lambda_x(f) + \Lambda_x(g); \quad \Lambda_x(\beta f) = \beta f(x) = \beta \cdot \Lambda_x(f) \ (\beta \in \mathbb{R}).$$

Thus, $\Lambda_x \in X''_+$. Next, define an embedding $X \mapsto X''_+$ in a standard way: $x \in X \mapsto \Lambda_x \in X''_+$. Then the injectivity of the given embedding follows from Theorem 1.3:

$$(x \neq x') \Rightarrow (\exists f \in X'_+ : f(x) \neq f(x')) \Leftrightarrow (\exists f \in X'_+ : \Lambda_x(f) \neq \Lambda_{x'}(f)) \Leftrightarrow (\Lambda_x \neq \Lambda_{x'}).$$

So, by using Hamel basis we come to one more proof of the classical principle. Let us formulate the obtained result.

Theorem 1.4. *If a convex cone X satisfies (CL) then X assumes a linear injective embedding into some linear space E : $X \overset{(lin)}{\hookrightarrow} E$.*

Corollary 1.1. *Under the embedding $x \mapsto \Lambda_x$ the linear kernel $Ker^-(X)$ passes to some linear subspace of E. This is a maximal subspace of E that is contained in the image of X with respect to the given embedding.*

2 Hamel basis in a cone generated by a uniquely divisible semigroup

The theory of abstract convex cones has ancient origins (see G. Birkhoff [6], H. Rädström [23]). Its active development was started with the works of G. Godini [13], R. E. Worth [27] and R. Urbanski [26]. In the last decades development of the theory is connected with the works of K. Keimel and W. Roth [17], B. Fuchssteiner, W. Lusky, T. A. Abreu, E. Corbacho and other mathematicians (see [12]–[25]). Special types of convex cones in function spaces are actively researched in the works of Russian and Soviet mathematicians M. L. Goldman, P. P. Zabreiko, V. D. Stepanov and E. Bahtigareeva (see [14]–[3]). The different classes of abstract convex cones were researched also by and F. S. Stonyakin and me (see [20],[25]).

Last time a steady interest in creation of an efficient theory of abstract non-convex cones arises. Such theory should make it possible to extend essentially the frame of application of convex and nonsmooth analysis. Actually, the problem is reduced to a choice among the two following approaches:

a) either total refusal of second distributive law $(DL)_2$: $(\lambda + \mu)x = \lambda x + \mu x$, (see, e.g. [18]–[16]);

b) or such generalization of $(DL)_2$ that allows us to keep valid most of the means of the convex cones theory (see our recent work [19]).

By following the second way, we are based in [19] on the concept of a *divisible semigroup* that is well known in algebra and analysis (see, e.g. [7]–[10]). To this condition we add a requirement of the *"unique divisibility"* that makes it possible to introduce in the given semigroup a so called *"additive product"* of scalars by vectors. Thus, we can construct an additive embedding of the given semigroup into some convex cone. So, let us explain briefly the mentioned results. Let us note that the scheme of exposition in this paper differs significantly from those in [19].

2.1 Uniquely divisible semigroups and their additive completions

In what follows, $(X,+)$ is an Abelian semigroup (with additive terminology). Make the notation:

$$\sum_n x = x + \ldots + x \ (\forall x \in X, n \in \mathbb{N}); \ \sum_0 x = 0.$$

Introduce a unique divisibility condition by combining the classical divisibility condition (2.1) with the additional exact divisibility conditions (2.2)–(2.3).

Definition 5. Say that an Abelian semigroup $(X,+)$ is *uniquely divisible* (or, (UD)-semigroup), if the following conditions:

$$(i) \quad \forall x \in X \ \forall n \in \mathbb{N} \ \exists y \in X: \ \sum_n y = x; \tag{2.1}$$

$$(ii) \quad \left(\sum_n y_1 = \sum_n y_2, \ n \neq 0\right) \Rightarrow (y_1 = y_2); \tag{2.2}$$

$$(iii) \quad \left(\sum_{n_1} y = \sum_{n_2} y, \ y \neq 0\right) \Rightarrow (n_1 = n_2) \tag{2.3}$$

are hold.

Let us note some properties of (UD)-semigroups that follow from (2.1)–(2.3) and allows us later to introduce correctly "additive multiplication".

Theorem 2.1. *If X is a (UD)-semigroup then:*

$$(i) \qquad \forall x \in X \ \forall n \in \mathbb{N} \ \exists! y \in X: \ \sum_n y = \sum_m x;$$

$$(ii) \quad (m_1/n_1 = m_2/n_2) \Rightarrow \left[\left(\sum_{n_1} y = \sum_{m_1} x\right) \Leftrightarrow \left(\sum_{n_2} y = \sum_{m_2} x\right)\right].$$

Let us consider a way to construct an extensive enough class of semigroups.

Example 2. Say that Abelian semigroup X is an "exact semigroup" if the condition (iii) from Definition 5

$$\left(\sum_{n_1} y = \sum_{n_2} y, \ y \neq 0\right) \Rightarrow (n_1 = n_2)$$

holds. Next if X is an exact semigroup then Let us carry out a factorization of X as follows:

$$(y_1 \mathcal{R} y_2) \Leftrightarrow (\exists n \in \mathbb{N}: \sum_n y_1 = \sum_n y_2).$$

It is easy to see that the factor-semigroup $X/\mathcal{R}$ forms a (UD)-semigroup with respect to the corresponding factor-addition.

Now, introduce in X "additive multiplication", first by non-negative rational scalars and mention correctness of the definition.

Definition 6. Let $x \in X, \ r = m/n \in \mathbb{Q}_+$. Set

$$(y = r * x) :\Leftrightarrow \left(\sum_n y = \sum_m x\right).$$

Theorem 2.2. *The Definition 6 is well-defined; moreover, $(X,+,*)$ forms a convex cone over $\mathbb{Q}_+$.*

Next, to extend the cone above up to a convex cone over $\mathbb{R}_+$, let us carry out some "additive completion" of X by means of "additive Dedekind sections".

Definition 7. For any $x \in X$ call an *additive Dedekind section* in $\mathbb{Q}_+ * x = \{r * x |\ r \in \mathbb{Q}_+\}$ such a partition $\mathbb{Q}_+ * x = A_1 \dot{\cup} A_2$ that A_1 and A_2 are non-empty sets and

$$(r_1 * x \in A_1,\ r_2 * x \in A_2) \Rightarrow (r_1 < r_2).$$

Denote such section by $A_1|A_2$. Like a scalar case, let us extract three possible kinds of Dedekind sections:

(i) $A_1|A_2$ is of *the first kind* if either A_1 contains the maximal element or A_2 contains the minimal element (but not simultaneously);

(ii) $A_1|A_2$ is of *the second kind* if both A_1 and A_2 contain the extremal elements;

(iii) $A_1|A_2$ is of *the third kind* if neither A_1 nor A_2 contain the extremal elements.

Like scalar case, from the uniqueness conditions (2.2)–(2.3) it easy follows.

Theorem 2.3. *If X is a (UD)-semigroup and $x \in X$ then each Dedekind section in $\mathbb{Q}_+ * x$ is either of the first or of the second kind, only.*

Now, denote by $[x]$ the first kind section generated by $x \in X$ with x belonging to the lower class of section and let us construct an additive completion of X.

Definition 8. If X is a (UD)-semigroup and $x \in X$ call a set of all Dedekind sections in $\mathbb{Q}_+ * x$ an *the additive completion* of $\mathbb{Q}_+ * x$ and denote it by $[\mathbb{Q}_+ * x]$. Here, like the scalar case, we identify the sections of first kind with the same extremal element and define addition and a linear order in $[\mathbb{Q}_+ * x]$ with the help of upper classes of the sections. Finally, set

$$\Big[X\Big] = \bigcup_{x \in X} [\mathbb{Q}_+ * x]$$

and call it an *additive completion* of X.

Next, the classical Dedekind theorem (see [4]) states that all Dedekind sections in $\mathbb{R}$ are of the first kind, only. But from the uniqueness conditions (2.2)–(2.3) follows also an isomorphism between $\mathbb{Q}_+$ and $\mathbb{Q}_+ * x$: $(r_1 \neq r_2) \Rightarrow (r_1 * x \neq r_2 * x)$, whence an isomorphism $\mathbb{R}_+$ and $\mathbb{R}_+ * x$ follows as well. So, Dedekind theorem remains valid for (UD)-semigroups.

Theorem 2.4. *If X is a (UD)-semigroup, then its additive completion $[X]$ is additively complete, i.e. $[[X]] \cong [X]$.*

Let us extract a rather wide class of additively complete (UD)-semigroups, i.e. (UD)-semigroups that satisfy condition $[[X]] \cong [X]$.

Theorem 2.5. *Let $(X, +, \cdot)$ be a (UD)-cone. Suppose that such a continuous and strongly increasing function $\varphi :\ \mathbb{Q}_+ \to \mathbb{R}_+$ exists that*

$$\forall x \in X,\ \forall r \in \mathbb{Q}_+ :\ r * x = \varphi(r) \cdot x;$$

i.e. the cone X is additively homogeneous. Then X is additively complete and we can set

$$\alpha * x = \varphi(\alpha) \cdot x\ (\alpha \in \mathbb{R}_+,\ x \in X).$$

Let us give a concrete example.

Example 3. Let $\mathbb{R}_+^{(p)}$ be $\mathbb{R}_+$ equipped with the addition

$$x_1 \oplus x_2 = (x_1^p + x_2^p)^{1/p}.$$

and with the usual multiplication. Then the additive multiplication in X takes form $\alpha * x = \alpha^{1/p} \cdot x$.

Finally, let us formulate the main result of this section.

Theorem 2.6. *Let X be a (UD)-semigroup. Then the following statements are valid:*

(i) the addition completion $([X], +, *)$ forms a convex cone;

(ii) the canonical embedding $X \hookrightarrow [X]$ $(x \mapsto [x])$ is injective and additive;

(iii) if moreover, X is additively complete, then the isomorphism $[X] \cong X$ holds.

Remark 3. By summing the preceding results, let us emphasize that each section of the second kind in $\mathbb{Q}_+ * x$ can be identified with the product $\alpha * x$, where α is an irrational number generated by the corresponding section $A_1|A_2$ in $\mathbb{Q}$.

But in view of Theorem 2.6, if X is an additively complete (UD)-semigroup then all sections of the sets $\mathbb{Q}_+ * x$ $(x \in X)$ are of the first kind. Thus, in this case all "additive rays" in X can be identified with the sets $\mathbb{R}_+ * x$, i.e. up to isomorphism, X contains all products $\alpha * x$ $(\alpha \in \mathbb{R}_+,\ x \in X)$.

2.2 Additive Hamel basis in a (UD)-semigroup

By using results of Subsection 2.1, we extend here the main results of Section 1 to the case of a uniquely divisible semigroup. So, let $(X, +)$ be a uniquely divisible semigroup, $([X], +, *)$ be its additive completion.

By virtue of Theorem 1.1, it is possible to choose some sublinear Hamel basis $[H] = \{\gamma_i * h_i\}_{i \in I}$ in $[X]$, where $h_i \in X$. In particular, for any $x \in X$ an expansion

$$x + \sum_{k=1}^{n} \alpha_k * (\gamma_k * h_{i_k}) = \sum_{k=1}^{n} \beta_k * (\gamma_k * h_{i_k}),$$

takes place. From here, by denoting $\lambda_k = \alpha_k \gamma_k$ and $\mu_k = \beta_k \gamma_k$ it follows

$$x + \sum_{k=1}^{n} \lambda_k * h_{i_k} = \sum_{k=1}^{n} \mu_k * h_{i_k}. \tag{2.4}$$

Thus, the system $H = \{h_{i_k}\}$ forms an additive basis in X, under the condition that the products $\lambda_k * h_{i_k}$ and $\mu_k * h_{i_k}$ separately can be not from X. Obviously that in the case of additively complete X all summands in 2.4 are from X. Let us formulate the obtained result.

Theorem 2.7. *If X is a (UD)-semigroup then such an additive Hamel basis $H = \{h_i\}_{i \in I}$ in X exists that each element $x \in X$ allows bi-expansion of the type*

$$x + \sum_{k=1}^{n} \lambda_k * h_{i_k} = \sum_{k=1}^{n} \mu_k * h_{i_k}, \tag{2.5}$$

where the summands are from $[X]$. In the case of an additively complete X all summands in 2.5 belong to X.

At last, by combining statements 1.4 and 1.1 with Theorem 1.1, we come to results on embedding of a (UD)-semigroup. Here we also take into account that the presence of the "cancellation law" $((x+y=x+z) \Rightarrow (y=z))$ in X implies the same in $[X]$.

Theorem 2.8. *If a (UD)-semigroup X satisfies (CL), then X allows additive injective embedding into some linear space E.*

Corollary 2.1. *Under the conditions of Theorem 2.8, a semigroup kernel $Ker^{-}(X)$ allows an additive injection into such a subspace of E that is maximal subspace which is contained in $[X]$.*

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