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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
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Kazakhstan

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INEQUALITIES FOR WEIGHTED HARDY OPERATORS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACE WITH $0 < p(x) < 1$

S.A. Bendaoud, A. Senouci

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Abstract. Weighted inequalities are proved for the weighted Hardy operators and the weighted dual of the classical Hardy operator acting from one weighted variable exponent Lebesgue space $L_{p(.),\omega_1}(0,\infty)$ to another weighted variable exponent Lebesgue space $L_{p(.),\omega_2}(0,\infty)$ for $0 < p(x) \leq q(x) < 1$.

1 Introduction

Function spaces of variable integrability appeared in the work by Orlicz [8] already in 1931, but the recent interest in there spaces is based on the paper of Kovačik and Râkosnik [7] together with applications to modeling electrorheological fluids [9]. A fundamental breakthrough concerning spaces of variable integrability was the observation that, under certain regularity assumptions on $p(.)$, the Hardy-Littlewood maximal operator is bounded on $L_{p(.)}(\mathbb{R}^n)$, see [6].

The aim of this paper is to obtain weighted inequalities for the weighted Hardy operator and the weighted dual of the classical Hardy operator acting from one weighted variable exponent Lebesgue space to another weighted variable exponent Lebesgue space for $0 < p(x) < 1$, for non-negative Lebesgue measurable functions on $(0,\infty)$ satisfying a certain weak condition of monotonicity type.

It is well known that for L_p-spaces with $0 < p < 1$ the Hardy inequality is not satisfied for arbitrary non-negative measurable functions, but is satisfied for non-negative monotone functions. Moreover in [4], pp. 90-91, the sharp constant in the Hardy-type inequality for non-negative non-increasing functions was found (see [5] for more details). Later the monotonicity was replaced by a weaker condition (see [11]), in particular, the following statements were proved.

Let for $x > 0$, $f \in L_1^{loc}(0,\infty)$,

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt.$$

Theorem 1.1. *Let $x > 0$, $0 < p < 1$, and $\alpha < 1 - \frac{1}{p}$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ and satisfies for some $M > 0$ the inequality*

$$f(x) \leq \frac{M}{x}\Big(\int_0^x f^p(y)y^{p-1}dy\Big)^{\frac{1}{p}}, \tag{1.1}$$

for all $x > 0$, then

$$\|x^\alpha (Hf)(x)\|_{L_p(0,\infty)} \leq N\|t^\alpha f(t)\|_{L_p(0,\infty)}, \tag{1.2}$$

where

$$N = M^{1-p} p^{1-\frac{1}{p}} \left(1 - \alpha - \frac{1}{p}\right)^{-\frac{1}{p}}. \tag{1.3}$$

Moreover, the constant N is sharp.

Later inequality (1.2) was extended for the weighted Hardy operator (for more details see [1]). Namely the following statements were proved there.

Let ω denote a weight function on $(0,\infty)$, i.e. a positive Lebesgue measurable function on $(0,\infty)$. For $0 < p < \infty$ the weighted space $L_{p,\omega}(0,\infty)$ is the space of all real-valued functions with finite quasi-norm

$$\|f\|_{L_{p,\omega}(0,\infty)} = \Big(\int_0^\infty |f(x)|^p \omega(x)dx\Big)^{\frac{1}{p}}.$$

The weighted Hardy operator is defined by

$$(H_\omega f)(x) = \frac{1}{W(x)} \int_0^x f(t)\omega(t)dt, \quad x > 0,$$

where $0 < W(x) = \int_0^x \omega(t)dt < \infty$ for all $t > 0$.

Note that for $\omega(t) \equiv 1$, the operator H_ω is the usual Hardy operator

$$(Hf)(x) = \frac{1}{x} \int_0^x f(t)dt.$$

Lemma 1.1. *Let $0 < p < 1$, $c_1 > 0$, $A > 0$, ω be a weight function on $(0,\infty)$ satisfying the condition*

$$\omega(t) \le c_1 \omega(y) \quad \text{for} \quad 0 < y < t < \infty. \tag{1.4}$$

If f is a non-negative Lebesgue measurable function on $(0,\infty)$ such that for almost all $0 < t < \infty$,

$$f(t) \le A\Big(\int_0^t \omega(y) y^{p-1} dy\Big)^{-\frac{1}{p}} \Big(\int_0^t f^p(y)\omega(y) y^{p-1} dy\Big)^{\frac{1}{p}}, \tag{1.5}$$

then for all $x > 0$

$$(H_\omega f)(x) \le \frac{c_2}{x\omega(x)^{\frac{1}{p}}} \Big(\int_0^x f^p(y)\omega(y) y^{p-1} dy\Big)^{\frac{1}{p}}, \tag{1.6}$$

where $c_2 = p^{\frac{1}{p}} A^{1-p} c_1^{\frac{2}{p}-1}$.

Remark 1. If in Lemma 1.1 $\omega = 1$, then inequality (1.5) takes form (1.1) with $M = Ap^{\frac{1}{p}}$ and $c_1 = 1$, consequently $c_2 = p^{\frac{1}{p}} A^{1-p}$ (see[1]).

Remark 2. If f is a non-increasing function on $(0,\infty)$, then (1.5) holds with $A = 1$ (see [1]).

Theorem 1.2. *Let $0 < p < 1$, $c_1 > 0$, $A > 0$, ω be a weight function on $(0,\infty)$ satisfying condition (1.4), and $\alpha < 1 - \frac{1}{p}$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ satisfying (1.5), then*

$$\|x^\alpha (H_\omega f)(x)\|_{L_{p,\omega}(0,\infty)} \le D\|t^\alpha f(t)\|_{L_{p,\omega}(0,\infty)}, \tag{1.7}$$

where

$$D = A^{1-p} c_1^{\frac{2}{p}-1} \left(1 - \alpha - \frac{1}{p}\right)^{-\frac{1}{p}}. \tag{1.8}$$

In Bandaliev's paper [3], two weighted inequalities were proved for the classical Hardy operator acting from one weighted variable exponent Lebesgue space to another weighted variable exponent Lebesgue space for non-negative monotone functions defined on $(0,\infty)$. In this work weighted inequalities are proved for the weighted Hardy operator and the weighted dual of the classical Hardy operator. In particular if $\omega(x) = 1$ and f is a non-increasing function, we obtain Bandaliev's results (see Theorem 2.2) and some corollaries for $p(x) = p = const$.

2 Preliminaries

Let H_w^* be the dual of the operator H_ω in $L_2(0,\infty)$. Then for any $f,g \in L_2(0,\infty)$

$$\int_0^\infty \Big(\frac{1}{W(x)}\int_0^x f(t)\omega(t)dt\Big)g(x)dx = \int_0^\infty \Big(\int_t^\infty \frac{g(x)}{W(x)}dx\Big)f(t)\omega(t)dt$$
$$= \int_0^\infty \omega(t)(H^*g)(x)f(t)dt$$
$$= \int_0^\infty \omega(t)\Big(\int_t^\infty \frac{g(x)}{W(x)}dx\Big)f(t)dt.$$

Hence the equality $(H_\omega f, g)_{L_2(0,\infty)} = (f, H_\omega^* g)_{L_2(0,\infty)}$ is satisfied for the operator H_ω^* defined by

$$\Big(H_\omega^* f\Big)(x) = \omega(x)\int_x^\infty \frac{g(t)}{W(t)}dt, \quad x>0.$$

Lemma 2.1. *Let $0<p<1$, $B>0$, ω be a weight function on $(0,\infty)$ such that for all $x>0$ $\int_0^x \omega(y)dy < \infty$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ such that for almost all $0<x<\infty$*

$$\int_x^\infty f^p(y)\omega(y)y^{p-1}dy < \infty$$

and

$$f(x) \le \frac{B}{x}\Big(\int_x^\infty f^p(y)\omega(y)y^{p-1}dy\Big)^{1/p}\omega(x)^{\frac{1}{1-p}}\Big(\int_0^x \omega(y)dy\Big)^{\frac{1}{1-p}}. \tag{2.1}$$

Then for $r>0$

$$(H_\omega^* f)(r) \le c_3\omega(r)\Big(\int_r^\infty f^p(y)\omega(y)y^{p-1}dy\Big)^{1/p}, \tag{2.2}$$

where $c_3 = pB^{1-p}$.

Proof. By (2.1) it follows that

$$x^{1-p}f(x)^{1-p} \le B^{1-p}\Big(\int_x^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}-1}\omega(x)\int_0^x \omega(y)dy.$$

Hence

$$\frac{f(x)}{W(x)} \le B^{1-p}\Big(\int_x^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}-1}\omega(x)f(x)^p x^{p-1}$$
$$= pB^{1-p}(-1)\Big[\Big(\int_x^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}}\Big]'.$$

Integrating over (r,∞) we obtain

$$\int_r^\infty \frac{f(x)}{W(x)}dx$$
$$\le pB^{1-p}\lim_{b\to+\infty}\Bigg(\Big(\int_r^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}} - \Big(\int_b^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}}\Bigg)$$
$$\le pB^{1-p}\Big(\int_r^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}},$$

hence

$$\left(H_\omega^* f\right)(r) = \omega(r)\int_r^\infty \frac{f(x)}{W(x)}dx \le pB^{1-p}\omega(r)\Big(\int_r^\infty f(y)^p\omega(y)y^{p-1}dy\Big)^{\frac{1}{p}}.$$

If $\omega(x) = 1$ in (2.1) and (2.1), then we have the following corollary. □

Corollary 2.1. Let $0 < p < 1$. If f is non-negative Lebesque measurable function on $(0, \infty)$ such that for all $0 < x < \infty$ $\int_x^\infty f^p(y)y^{p-1}dy < \infty$ and for some $B > 0$ the inequality

$$f(x) \le \frac{B}{x^{p'}}\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}}, \tag{2.3}$$

is satisfied, then for $r > 0$

$$(H^* f)(r) \le c_3\left(\int_r^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}}, \tag{2.4}$$

where $c_3 = pB^{1-p}$ and p' is the conjugate exponent of p.

Remark 3. Inequality (2.3), (2.4) respectively, are analogues of inequality (1.1) and inequality (2.2) in [7] for the dual of the classical Hardy operator.

Theorem 2.1. *Let $0 < p < 1$, $x > 0$ and $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$. If f is a non-negative Lebesgue measurable function on $(0, \infty)$ and satisfies (2.3), then*

$$\|\delta^\alpha (H^* f)(\delta)\|_{L_p(0,\infty)} \le c_4\|y^{\alpha+1}f(y)\|_{L_p(0,\infty)} \tag{2.5}$$

where $c_4 = pB^{1-p}(\alpha p + 1)^{-\frac{1}{p}}$.

Proof.

$$K_1 = \|\delta^\alpha\left(H^* f\right)(\delta)\|_{L_p(0,\infty)} = \left[\int_0^\infty \delta^{\alpha p}\left(H^* f\right)^p(\delta)d\delta\right]^{\frac{1}{p}}$$

$$= \left[\int_0^\infty \delta^{\alpha p}\left(\int_\delta^{+\infty}\frac{f(y)}{y}dy\right)^p d\delta\right]^{\frac{1}{p}}.$$

Then by (2.4) it follows that

$$K_1 \le \left[\int_0^\infty \delta^{\alpha p}c_3^p\left(\int_\delta^\infty f^p(y)y^{p-1}dy\right)d\delta\right]^{\frac{1}{p}}$$

$$= c_3\left[\int_0^\infty f^p(y)y^{p-1}\left(\int_0^y \delta^{\alpha p}d\delta\right)dy\right]^{\frac{1}{p}}$$

$$= pB^{1-p}(\alpha p + 1)^{-\frac{1}{p}}\|y^{\alpha+1}f(y)\|_{L_p(0,\infty)}.$$

□

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Suppose that p is a Lebesgue measurable function on Ω such that $0 < \underline{p} \le p(x) \le \overline{p} < \infty$, $\underline{p} = ess\ \inf_{x\in\Omega} p(x)$, $\overline{p} = ess\ \sup_{x\in\Omega} p(x)$ and w is a weight function on Ω, i.e. is a non-negative, almost everywhere (a.e) positive function on Ω.

Definition 1. By $L_{p(x),\omega}(\Omega)$ we denote the set of all measurable function f on Ω such that

$$I_{p,\omega}(f)=\int_{\Omega}(|f(x)|\omega(x))^{p(x)}dx<\infty. \tag{2.6}$$

Note that the expression

$$\|f\|_{L_{p(.),\omega}(\Omega)}=\inf\{\lambda>0;\ \int_{\Omega}\Big(\frac{|f(x)|\omega(x)}{\lambda}\Big)^{p(x)}dx\le 1\}, \tag{2.7}$$

defines a quasi-norm on $L_{p(x),\omega}(\Omega)$. $L_{p(x),\omega}(\Omega)$ is a quasi-Banach space equipped with this quasi-norm (see [7] and [10]).

In [3] the following Corollary was proved.

Corollary 2.2. *Let* $0<\underline{p}\le p(x)\le q(x)\le\overline{q}<\infty$ *and* $r(x)=\frac{p(x)q(x)}{q(x)-p(x)}$. *Suppose that* ω_1 *and* ω_2 *are weight functions in* Ω *satisfying the condition:*

$$\Big\|\frac{\omega_1}{\omega_2}\Big\|_{L_{r(.)}(\Omega)}<\infty.$$

Then the inequality

$$\|f\|_{L_{p(.),\omega_1}(\Omega)}\le(A_1+B_1+\|\chi_{\Omega_2}\|_{L_\infty(\Omega)})^{1/\underline{p}}\Big\|\frac{\omega_1}{\omega_2}\Big\|_{L_{r(.)}(\Omega)}\|f\|_{L_{q(.),\omega_2}(\Omega)}, \tag{2.8}$$

holds for every $f\in L_{q(x),\omega_2}(\Omega)$, *where*

$$\Omega_1=\{x\in\Omega:\ p(x)<q(x)\},\qquad \Omega_2=\{x\in\Omega:\ p(x)=q(x)\},$$
$$A_1=\sup_{x\in\Omega_1}\frac{p(x)}{q(x)},\qquad B_1=\sup_{x\in\Omega_1}\frac{q(x)-p(x)}{q(x)}.$$

The following lemma is known (see [2]).

Lemma 2.2. *Let* $1\le\underline{p}\le p(x)\le q(y)\le\overline{q}<\infty$; *for all* $x\in\Omega_1\subset\mathbb{R}^n$ *and* $y\in\Omega_2\subset\mathbb{R}^m$. *If* $p\in C(\Omega_1)$, *then the inequality*

$$\Big\|\|f\|_{L_{p(.)}(\Omega_1)}\Big\|_{L_{q(.)}(\Omega_2)}\le C_{p,q}\Big\|\|f\|_{L_{q(.)}(\Omega_2)}\Big\|_{L_{p(.)}(\Omega_1)} \tag{2.9}$$

is valid, where

$$C_{p,q}=\Big(\|\chi_{\Delta_1}\|_\infty+\|\chi_{\Delta_2}\|_\infty+\frac{\overline{p}}{\underline{q}}+\frac{\underline{p}}{\overline{\overline{q}}}\Big)(\|\chi_{\Delta_1}\|_\infty+\|\chi_{\Delta_2}\|_\infty), \tag{2.10}$$

$$\underline{q}=ess\ \inf_{\Omega_2}q(x)\qquad \overline{q}=ess\ \sup_{\Omega_2}q(x),$$

$$\Delta_1=\{(x,y)\in\Omega_1\times\Omega_2;\ p(x)=q(x)\ \},\qquad \Delta_2=\Omega_1\times\Omega_2/\Delta_1,$$

and $C(\Omega_1)$ *is the space of continuous functions in* Ω_1 *and* $f:\ \Omega_1\times\Omega_2\to\mathbb{R}$ *is any measurable function such that* $\Big\|\|f\|_{L_{q(.)}(\Omega_2)}\Big\|_{L_{p(.)}(\Omega_1)}<\infty$.

The following theorem is proved in [3].

Theorem 2.2. *Let $x \in (0,\infty)$; $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, and f be a non-negative and non-increasing function defined on $(0,\infty)$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then for any $f \in L_{p(x),\omega_1}(0,\infty)$ the inequality

$$\|Hf\|_{L_{q(.),\omega_2}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} d_p \left\| \frac{t^{1/\overline{p}'} \|\frac{\omega_2}{x}\|_{L_{q(.)}(t,\infty)}}{\omega_1} \right\|_{L_{r(.)}(0,\infty)} \|f\|_{L_{p(.),\omega_1}(0,\infty)}, \tag{2.11}$$

holds, where

$$C_{p,q} = \left(\|\chi \triangle_1\|_{L_\infty(0,\infty)} + \|\chi \triangle_2\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right) \Big(\|\chi S_1\|_{L_\infty(0,\infty)} + \|\chi S_2\|_{L_\infty(0,\infty)} \Big),$$

$S_1 = \{x \in (0,\infty): \ p(x) = \underline{p}\}$, $S_2 = (0,\infty)\backslash S_1$ *and* $d_p = \left(1 - \frac{\overline{p}-\underline{p}}{\overline{p}} + \|\chi_{S_1}\|_{L_\infty(0,\infty)}\right)^{\frac{1}{\underline{p}}}$.

3 Main results

We consider the weighted Hardy operator

$$(H_\omega f)(x) = \frac{1}{W(x)} \int_0^x f(t)\omega(t)dt.$$

Theorem 3.1. *Let $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $\alpha < 1 - \frac{1}{\underline{p}}$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ and f be a non-negative Lebesgue measurable function defined on $(0,\infty)$ satisfying inequality (1.5) with p replaced by $\underline{p}$ and ω be a weight function satisfying condition (1.4). Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then for any $f \in L_{p(x),\omega_1}(0,\infty)$ the inequality

$$\|(H_\omega f)(x)\|_{L_{q(.),\omega_2}(0,\infty)}$$

$$\le c_2 C_{p,q} d_p \left\| \frac{\omega^{1/\underline{p}} y^{1/\underline{p}'} \|\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\|_{L_{q(.)}(y,\infty)}}{\omega_1} \right\|_{L_{r(.)}(0,\infty)} \|f\|_{L_{p(.),\omega_1}(0,\infty)}, \tag{3.1}$$

holds, where $c_2 = \underline{p}^{\frac{1}{\underline{p}}} c_1^{\frac{2}{\underline{p}}-1} A^{1-\underline{p}}$.

Proof. By applying Lemma 1.1, we obtain

$$\begin{aligned}
\|H_\omega f\|_{L_{q(.),\omega_2}(0,\infty)} &= \|\omega_2 H_\omega f\|_{L_{q(.)}(0,\infty)} \\
&\le \left\| \frac{c_2\omega_2(x)}{x\omega^{1/\underline{p}}(x)} \Big(\int_0^x f^{\underline{p}}(y)\omega(y) y^{\underline{p}-1} dy \Big)^{1/\underline{p}} \right\|_{L_{q(.)}(0,\infty)} \\
&= c_2 \left\| \frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)} \Big(\int_0^x f^{\underline{p}}(y)\omega(y) y^{\underline{p}-1} dy \Big)^{1/\underline{p}} \right\|_{L_{q(.)}(0,\infty)}.
\end{aligned}$$

Let $\quad I_1 = \left\| \frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)} \Big(\int_0^x f^{\underline{p}}(y)\omega(y) y^{\underline{p}-1} dy \Big)^{1/\underline{p}} \right\|_{L_{q(.)}(0,\infty)}$, then

$$I_1 = \left\| \Big(\int_0^\infty \Big[f^{\underline{p}}(y)\omega(y) \Big] \chi_{(0,x)}(y) \Big[\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)} \Big]^{\underline{p}} y^{\underline{p}-1} dy \Big)^{1/\underline{p}} \right\|_{L_{q(.)}(0,\infty)}$$

$$= \Big\| \int_0^\infty \Big[f^{\underline{p}}(y)\omega(y)\Big]\chi_{(0,x)}(y)\Big[\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big]^{\underline{p}} y^{\underline{p}-1}dy\Big\|^{1/\underline{p}}_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)}$$

$$= \Big\|\|[f^{\underline{p}}(y)\omega(y)]\chi_{(0,x)}(y)\Big[\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big]^{\underline{p}} y^{\underline{p}-1}\|_{L_1(0,\infty)}\Big\|^{1/\underline{p}}_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)}.$$

Next, by applying Lemma 2.2, we get

$$I_1 \le C_{p,q}\Big(\int_0^\infty \Big\|[f^{\underline{p}}(y)\omega(y)]\chi_{(0,x)}(y)\Big[\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big]^{\underline{p}} y^{\underline{p}-1}\Big\|_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)} dy\Big)^{1/\underline{p}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y)\omega(y)y^{\underline{p}-1}\Big\|\chi_{(0,x)}(y)\Big[\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big]^{\underline{p}}\Big\|_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)} dy\Big)^{1/\underline{p}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y)\omega(y)y^{\underline{p}-1}\Big\|\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big\|^{\underline{p}}_{L_{q(.)}(y,\infty)} dy\Big)^{1/\underline{p}}$$

$$= C_{p,q}\Big\|f(y)\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\Big\|\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big\|_{L_{q(.)}(y,\infty)}\Big\|_{L_{\underline{p}}(0,\infty)}.$$

Let $\quad I_2 = \Big\|f(y)\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\Big\|\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\Big\|_{L_{q(.)}(y,\infty)}\Big\|_{L_{\underline{p}}(0,\infty)}$,
then by applying Corollary 2.2, we obtain

$$I_2 \le d_p\Big\|\frac{\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\|_{L_{q(.)}(y,\infty)}}{\omega_1}\Big\|_{L_{r(.)}(0,\infty)}\|f\|_{L_{p(.),\omega_1}(0,\infty)},$$

consequently

$$\|H_\omega f\|_{L_{q(.),\omega_2}(0,\infty)} \le c_2 C_{p,q} d_p\Big\|\frac{\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\frac{\omega_2(x)}{x\omega^{1/\underline{p}}(x)}\|_{L_{q(.)}(y,\infty)}}{\omega_1}\Big\|_{L_{r(.)}(0,\infty)}\|f\|_{L_{p(.),\omega_1}(0,\infty)}.$$

For the dual operator H^*_ω, we have the following theorem. □

Theorem 3.2. *Let $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $\alpha < 1 - \frac{1}{\underline{p}}$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ and f be a non-negative Lebesgue measurable function defined on $(0,\infty)$ satisfying inequality (2.1) with p replaced by $\underline{p}$ and w be a weight function satisfying conditions of Lemma 2.1 Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then for any $f \in L_{p(x),\omega_1}(0,\infty)$ the inequality

$$\|(H^*_\omega f)(x)\|_{L_{q(.),\omega_2}(0,\infty)}$$
$$\le c_3 C_{p,q} d_p\Big\|\frac{\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\omega_2(x)\omega(x)\|_{L_{q(.)}(0,y)}}{\omega_1}\Big\|_{L_{r(.)}(0,\infty)}\|f\|_{L_{p(.),\omega_1}(0,\infty)} \tag{3.2}$$

holds, where $c_3 = \underline{p}B^{1-\underline{p}}$.

Proof. By applying Lemma 1.1, we get

$$\|H^*_\omega f\|_{L_{q(.),\omega_2}(0,\infty)} = \|\omega_2 H^*_\omega f\|_{L_{q(.)}(0,\infty)}$$
$$\le c_3\Big\|\omega_2(x)\omega(x)\Big(\int_x^\infty f^{\underline{p}}(y)\omega(y)y^{\underline{p}-1}dy\Big)^{1/\underline{p}}\Big\|_{L_{q(.)}(0,\infty)}.$$

Let $\quad J_1 = \Big\|\omega_2(x)\omega(x)\Big(\int_x^\infty f^{\underline{p}}(y)\omega(y)y^{\underline{p}-1}dy\Big)^{1/\underline{p}}\Big\|_{L_{q(.)}(0,\infty)}$,

hence

$$J_1 = \Big\|\Big(\int_0^\infty \Big[f^{\underline{p}}(y)\omega(y)\Big]\chi_{(x,\infty)}(y)\Big[\omega_2(x)\omega(x)\Big]^{\underline{p}}y^{\underline{p}-1}dy\Big)^{1/\underline{p}}\Big\|_{L_{q(.)}(0,\infty)}$$

$$= \Big\|\int_0^\infty \Big[f^{\underline{p}}(y)\omega(y)\Big]\chi_{(x,\infty)}(y)\Big[\omega_2(x)\omega(x)\Big]^{\underline{p}}y^{\underline{p}-1}dy\Big\|^{1/\underline{p}}_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)}$$

$$= \Big\|\|[f^{\underline{p}}(y)\omega(y)]\chi_{(x,\infty)}(y)\Big[\omega_2(x)\omega(x)\Big]^{\underline{p}}y^{\underline{p}-1}\|_{L_1(0,\infty)}\Big\|^{1/\underline{p}}_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)}.$$

Now, by applying Lemma 2.2, we obtain

$$J_1 \le C_{p,q}\Big(\int_0^\infty \Big\|[f^{\underline{p}}(y)\omega(y)]\chi_{(x,\infty)}(y)\Big[\omega_2(x)\omega(x)\Big]^{\underline{p}}y^{\underline{p}-1}\Big\|_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)}dy\Big)^{1/\underline{p}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y)\omega(y)y^{\underline{p}-1}\Big\|\chi_{(x,\infty)}(y)\Big[\omega_2(x)\omega(x)\Big]^{\underline{p}}\Big\|_{L_{\frac{q(.)}{\underline{p}}}(0,\infty)}dy\Big)^{1/\underline{p}}$$

$$= C_{p,q}\Big(\int_0^\infty f^{\underline{p}}(y)\omega(y)y^{\underline{p}-1}\Big\|\omega_2(x)\omega(x)\Big\|^{\underline{p}}_{L_{q(.)}(0,y)}dy\Big)^{1/\underline{p}}$$

$$= C_{p,q}\Big\|f(y)\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\omega_2(x)\omega(x)\|_{L_{q(.)}(0,y)}\Big\|_{L_{\underline{p}}(0,\infty)}.$$

Finally, applying Corollary 2.2, we get

$$\Big\|f(y)\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\omega_2(x)\omega(x)\|_{L_{q(.)}(0,y)}\Big\|_{L_{\underline{p}}(0,\infty)}$$

$$\le d_p\Big\|\frac{\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\omega_2(x)\omega(x)\|_{L_{q(.)}(0,y)}}{\omega_1}\Big\|_{L_{r(.)}(0,\infty)}\|f\|_{L_{p(.),\omega_1}(0,\infty)}.$$

Thus

$$\|H^*_\omega f\|_{L_{q(.),\omega_2}(0,\infty)} \le c_3C_{p,q}d_p\Big\|\frac{\omega^{1/\underline{p}}(y)y^{1/\underline{p}'}\|\omega_2(x)\omega(x)\|_{L_{q(.)}(0,y)}}{\omega_1}\Big\|_{L_{r(.)}(0,\infty)}\|f\|_{L_{p(.),\omega_1}(0,\infty)}.$$

Taking in account Remark 1., Remark 2. and replacing (1.5) by (1.1) in the proof of Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $\alpha < 1 - \frac{1}{\underline{p}}$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ and f be a non-negative Lebesgue measurable function defined on $(0,\infty)$ satisfying inequality (1.1). Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$. Then for any $f \in L_{p(x),\omega_1}(0,\infty)$ the inequality*

$$\|Hf\|_{L_{q(.),\omega_2}(0,\infty)} \le KC_{p,q}d_p\Big\|\frac{y^{1/\underline{p}'}\|\frac{\omega_2(x)}{x}\|_{L_{q(.)}(y,\infty)}}{\omega_1}\Big\|_{L_{r(.)}(0,\infty)}\|f\|_{L_{p(.),\omega_1}(0,\infty)}, \tag{3.3}$$

holds, where $K = \underline{p}^{\frac{1}{\underline{p}}}A^{1-\underline{p}}$.

Remark 4. If f is non-negative non-increasing function on $(0,\infty)$, inequality (1.1) is satisfied with $M = p^{\frac{1}{p}}$ and in corollary 3.1 $K = \underline{p}^{\frac{1}{\underline{p}}}$, consequently we obtain inequality (2.11) of Theorem 2.2.

We consider the operator (H^*f) with $\omega(x) = 1$, we get

$$(H^*f)(x) = \int_x^\infty \frac{f(y)}{y} dy.$$

Replacing (2.2) by (2.4) in the proof of Theorem 3.2, we obtain the following corollary for the operator H^*f.

Corollary 3.2. *Let $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $\alpha < 1 - \frac{1}{\underline{p}}$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ and f be a non-negative Lebesgue measurable function satisfying inequality (2.4). Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then for any $f \in L_{p(x),\omega_1}(0,\infty)$ the inequality

$$\|H_1^* f\|_{L_{q(.),\omega_2}(0,\infty)}$$

$$\le pB^{1-\underline{p}} C_{p,q} d_p \left\| \frac{y^{1/p'} \|\omega_2(x)\|_{L_{q(.)}(0,y)}}{\omega_1} \right\|_{L_{r(.)}(0,\infty)} \|f\|_{L_{p(.),\omega_1}(0,\infty)}, \tag{3.4}$$

holds, where $B, C_{p,q}$, d_p are respectively the constants in Corollary 2.1 and Corollary 3.1.

Remark 5. Note that Corollary 3.1 in the case where f is non-negative non-increasing function and $p(x) = q(x) = \underline{p} = const$ with $\omega_1(x) = \omega_2(x) = x^\alpha$ was proved in [5] with sharp constant in Hardy type inequality.

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Abed Sidahmed Bendaoud, Abdelkader Senouci
Departement of Mathematics
Ibn Khaldoun University
Tiaret, Algeria
E-mail: abed.bendaoud@yahoo.fr, kamer295@yahoo.fr

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