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ON KAISER CLASS OF UNARS IN EXPANDED SIGNATURE

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Abstract. The present paper is connected with studying properties of Jonsson theories of an unar. The main idea is to study a structure of the signature with one unary functional symbol by expanding it with both a new constant symbol and a unary predicate symbol. We construct the semantic Jonsson quasivariety using the semantic models of enriched Jonsson primitives of unars and consider its Jonsson spectrum, divided by cosemanticness relation onto factor-set and the obtained factor-set is divided by new equivalence relation with regard to Kaiser class. Additionally, we consider the notion of normal Jonsson theory and prove that the theory of all unars is normal, and obtain new results concerning components of unars and the Kaiser class of their theories.

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1 Introduction

The research concerning complete theories of unars is connected with the works of Yu.Ye. Shishmarev [31], A.A. Ivanov [11, 12], A.N. Ryaskin [27, 28], L. Marcus [16, 17]. Unary algebras as structures were studied in the works of S. Burris [5, 6], G.E. Puninskii [25], and A.W. Miller [19].

Recent papers in Eurasian Mathematical Journal also show that questions related to algebraic methods and model-theoretic structures remain active in different but connected directions. In particular, generalized reduction algorithms were considered in [2], while algebras of binary formulas for weakly circularly minimal theories with equivalence relations were studied in [14]. These works illustrate the broader context in which algebraic and logical methods interact.

Since Jonsson theories are, generally speaking, not complete theories, it is interesting to study Jonsson unars and consider their general problems in terms of Model Theory (for example using results from [26, 30, 18, 24, 3]), as well as in the framework of universal algebra, concerning the nature of unars.

The study of unars in terms of Jonsson theories started from the works of T.G. Mustafin and A.R. Yeshkeyev [38, 37]. The authors considered the properties of Jonsson universals and Jonsson primitives, as well as obtained the definition of characteristics of their semantic models. In [40] the authors expanded the signature of unars by a new constant symbol and one unary predicate, that distinguished an existentially closed Jonsson unar and obtained some properties of the enriched Jonsson universals and primitives. The main idea for the present paper was to naturally continue using this technique in terms of constructing Jonsson spectrum of semantic Jonsson quasivariety of unars.

It is important to note, that the concept of double factorization was first introduced by A.R. Yeshkeyev in [39]. However, in [39] the double factorization was used in terms of cosemanticness relation and Jonsson semantic and Jonsson syntactic similarity. In order to distinct, we will use the denotation D1F for the double factorization in the present paper - in terms of cosemanticness relation and K_T-equivalence.

More information on Jonsson theories and their properties one can find in [35, 13, 22, 1]. For the convenience of readers, we recall the basic terminology used in the paper. A first-order language, or signature, is understood as a collection of non-logical symbols, such as constant, function and relation symbols. A theory is a set of first-order sentences in a fixed language. A unar is a structure whose signature contains one unary functional symbol; in this paper we work in the signature $\sigma = \{f^1\}$. For standard model-theoretic terminology see, for example, [32]; for unars in the signature of one unary function see [31, 28].

2 Jonsson theories of unars

Throughout this paper a unar is understood as a structure of the signature $\sigma = \{f^1\}$, where f is a unary functional symbol; equivalently, it is a model with one unary operation. We first recall some important notions and definitions concerning model-theoretic properties of Jonsson theories, starting with the definition of Jonsson theory itself.

Definition 1. [35]A theory T is said to be Jonsson, if it satisfies the following conditions:
1) T has at least one infinite model;
2) T is $\forall\exists$-axiomatisable;
3) T has joint embedding property (JEP);
4) T has amalgamation property (AP).

The existence of at least one infinite model is a strong demand, this follows from the first condition. The second condition is syntactic and it coincides with property of inductiveness, i.e., the chain of any models of this theory will be a model of this theory as well. The third and fourth are very important properties of AP and JEP (more information on mentioned properties one can find in [35]) that play a big role in algebras having a syntactic property (for example, expressibility by sentences from a specific fragment of first-order logic such as equational, Horn, universal, $\forall\exists$, existential, positive-primitive, etc.). Let us recall the definition of an inductive theory.

Definition 2. [4] Let L be a first-order language. A theory T in the language L is called inductive (or a $\forall\exists$-theory) if it is axiomatizable by a set of $\forall\exists$-sentences; that is, it is equivalent to a set of sentences of the form $\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m \ \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, where ψ is a quantifier-free L-formula (contains no quantifiers) and $m, n \geq 0$.

More detailed information on inductive theories can be found in [32, 8, 33, 10].

Definition 3. [23] Let T be a Jonsson theory. A model $\mathfrak{C}_T$ of power $2^{|\omega|}$ is said to be a semantic model of the theory T if $\mathfrak{C}_T$ is a ω^+-homogeneous ω^+-universal model of the theory T.

If a Jonsson theory has several semantic models then they are all equal up to isomorphism. The following criterion is crucial.

Theorem 2.1. [23] *Inductive theory T is Jonsson if and only if it has a semantic model $\mathfrak{C}_T$.*

It is easy to see, that if a theory is Jonsson then it has a semantic model and vice versa. The next definition of the center of a Jonsson theory was introduced by T.G. Mustafin.

Definition 4. [35]The elementary theory of a semantic model of a Jonsson theory T is called the center of this theory. It is denoted by T^*, i.e. $Th(\mathfrak{C}_T) = T^*$.

It is important to note, that the center of a Jonsson theory is a complete theory by definition. However, we cannot claim that for any Jonsson theory its center is a Jonsson theory too. For example, group theory is a Jonsson theory, but its center is not a Jonsson theory. But in the case of perfect Jonsson theories, such as, for example, theories of abelian groups, this statement is true. More detailed information can be found in [35].

The present work is associated with the Jonsson universal of unars in the frame of first-order language L of the signature $\sigma = \{f^1\}$, where f is a unary functional symbol. Let us recall some important concepts.

Denotation 1. [38]Let ∇ be $\Pi_1 \cup \Sigma_1$, i.e. ∇ is the set of all universal or existential L-formulas.

Thus, ∇ is the set of all formulas that have in prenex normal form either one universal quantifier or one existential quantifier.

Definition 5. [38] 1) If $T = T_\forall$, i.e. the theory is equal to its universal logical consequences, then T is said to be universal;

2) If $T = T_\nabla$, i.e. the theory is equal to its universal or existential logical consequences, then the theory T is called primitive.

The connection between two Jonsson universals with regards to their centers and semantic models is presented in the form of the following proposition.

Proposition 2.1. [38] *Let $T_{\forall_1}, T_{\forall_2}$ be Jonsson universals. Then the following conditions are equivalent:*

1) $T_{\forall_1} = T_{\forall_2}$;

2) $\mathfrak{C}_{T_{\forall_1}} \simeq \mathfrak{C}_{T_{\forall_2}}$;

3) $T^*_{\forall_1} = T^*_{\forall_2}$.

Here $\mathfrak{C}_{T_{\forall_1}}$ and $\mathfrak{C}_{T_{\forall_2}}$ are semantic models, $T^*_{\forall_1}$ and $T^*_{\forall_2}$ are the centers of Jonsson universals $T_{\forall_1}, T_{\forall_2}$ respectively.

Let $\mathfrak{A}$ be some unar, i.e. a model of signature $\sigma = \{f^1\}$, where f is a unary functional symbol. Since each model of a Jonsson theory embeds into its semantic model by Definition 3, the following fact is true for Jonsson universals of unars. Jonsson universals of unars, considering Definition 5, are theories consisting of universal sentences in terms of $\sigma = \{f^1\}$.

Lemma 2.1. [38] *For any unar $\mathfrak{A}$ the following holds*

$$\mathfrak{A} \text{ is a model of } T_\forall \Leftrightarrow \mathfrak{A} \text{ embeds in } \mathfrak{C}_{T_\forall},$$

where $\mathfrak{C}_{T_\forall}$ is a semantic model of a Jonsson universal of unars $T_\forall$.

In what follows, the relation "$\mathfrak{A}$ is a model of $T_\forall$" will be denoted by the standard satisfaction symbol $\models$. Thus, instead of writing that a structure $\mathfrak{A}$ is a model of a theory $T_\forall$, we will write $\mathfrak{A} \models T_\forall$.

Let $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$, $n \in \omega$. Elements $a, b \in \mathfrak{A}$ are called $\mathfrak{A}$-connected in $X \subseteq \mathfrak{A}$ if there exist natural numbers m and n such that $f^m(a) = f^n(b)$ and $f^0(a), ..., f^m(a), f^0(b), ..., f^n(b) \in X$.

A set $X \subseteq \mathfrak{A}$ is called $\mathfrak{A}$-connected if any two elements from X are $\mathfrak{A}$-connected. A subsystem $\mathfrak{B} \subseteq \mathfrak{A}$ whose universe is the maximal $\mathfrak{A}$-connected subset of the universe of $\mathfrak{A}$ is called a component in $\mathfrak{A}$.

Let us recall some important definitions.

Denotation 2. [38]For any $a \in \mathfrak{C}_{T_\forall}$ let

$$\chi(a) = \begin{cases} \omega, \text{if } f^n(a) \neq f^k(a), \text{for any } n < k < \omega \\ \langle n, m\rangle, \text{if } \langle n, m\rangle = min\{\langle n, m\rangle : f^n(a) = f^{n+m}(a)\}. \end{cases}$$

Definition 6. [38]A set $\{a_1, ..., a_m\}$ of elements of $\mathfrak{C}_{T_\forall}$ is called an m-cycle, if $a_i \neq a_j, f(a_i) = a_{i+1}$ for all $1 \leq i < j \leq m$ and $f(a_m) = a_1$.

Denotation 3. [38]For any $a \in \mathfrak{C}_{T_\forall}$ let

$$k(a) = |\{b \in \mathfrak{C}_{T_\forall} : f(b) = a\}|$$

i.e. $k(a)$ is the cardinality of the set of all preimages of a by f.

Probably one of the most important definitions is Definition 7. It defines the characteristic of a semantic model of a Jonsson universal of unars.

Definition 7. [38] A tetrad $(\Omega, \nu, \mu, \varepsilon)$ is said to be the characteristic $\mathfrak{C}_{T_\forall}$ and denoted by $char(\mathfrak{C}_{T_\forall})$, if

$\Omega = \{\chi(a) : a \in \mathfrak{C}_{T_\forall}\}$,

$\nu : \omega \setminus \{0\} \to \omega \cup \{\infty\}$ such that for any $m > 0$,

$$\nu(m) = \begin{cases} k, \text{if the number of } m\text{-cycles in } \mathfrak{C}_{T_\forall} \text{ is finite and equals } k, \\ \infty, \text{otherwise;} \end{cases}$$

$\mu : \Omega \to \omega \cup \{\infty\}$ such that if $\alpha \in \Omega$ and $\alpha \in \chi(a)$, then $\mu(\alpha) = k(a)$, if $k(a) < \omega$ and $\mu(\alpha) = \infty$, if $k(a) = |\mathfrak{C}_{T_\forall}|$;

$$\varepsilon = \begin{cases} 0, \text{if } |\{a \in \mathfrak{C}_{T_\forall} : \chi(a) = \omega\}| = 0, \\ \infty, \text{otherwise.} \end{cases}$$

Let us give a more detailed look at Definition 7.

1) Ω is the set that consists of $\chi(a), a \in \mathfrak{C}_{T_\forall}$, i.e. a is an element of the universe of a semantic model, i.e., $\chi(a) = \omega$ if $f^n(a) \neq f^k(a)$, n and k indicate the number of acts on element a by a function f, for any $n < k < \omega$, both n and k are finite numbers, $k > n$. $\chi(a) = \langle n, m\rangle$, $\langle n, m\rangle$ is the minimal ordered pair such that $f^n(a) = f^{n+m}(a)$. Hence, Ω expresses whether there are finite or infinite cycles in the semantic model.

2) ν is a correspondence from the set ω without 0 into the set that consists of ω or $\{\infty\}$. The correspondence holds as follows: for any $m > 0$,

$$\nu(m) = \begin{cases} k, \text{if the number of } m\text{-cycles in } \mathfrak{C}_{T_\forall} \text{ is finite and equals } k, \\ \infty, \text{otherwise.} \end{cases}$$

That is, the correspondence $\nu(m)$ expresses the number of m-cycles, where m is the number of elements in the cycle.

3) μ is a correspondence from the set $\Omega = \{\chi(a) : a \in \mathfrak{C}_{T_\forall}\}$ into the set that consists of ω or $\{\infty\}$. The correspondence holds as follows: if $\alpha \in \Omega$ and $\alpha \in \chi(a)$, then $\mu(\alpha) = k(a)$, if $k(a) < \omega$ and $\mu(\alpha) = \infty$, then $k(a) = |\mathfrak{C}_{T_\forall}|$. Since we are working in the frame of [23], $|\mathfrak{C}_{T_\forall}| = 2^\omega$.

$k(a)$ is the power of the set that consists of all roots of element $b \in \mathfrak{C}_{T_\forall}$ such that $f(b) = a$, $a \in \mathfrak{C}_{T_\forall}$. The properties and dependencies between the concepts of $k(a)$ and $\chi(a)$ were studied in [38].

4) ε can vary between two values: 0 and ∞. It depends on the power of the set that consists of such elements $a \in \mathfrak{C}_{T_\forall}$, that $\chi(a) = \omega$. If no such elements exists the power of this set will be equal to 0, i.e. $\varepsilon = 0$, if there are such elements, then $\varepsilon = \infty$. ε expresses the existence of cycles in the semantic model.

The following properties follow immediately from Definition 7.

Lemma 2.2. [38] *If* $char(\mathfrak{C}_{T_\forall}) = (\Omega, \nu, \mu, \varepsilon)$, *then*

$1°$. $\varnothing \neq \Omega \subseteq \{\omega\} \cup (\omega \times \omega)$;

$2°$. *For any* $\langle n, m\rangle \in \Omega$ *and* $0 \le k < n$ *we have* $\langle k, m\rangle \in \Omega$;

$3°$. $\nu(m) > 0 \Leftrightarrow (0, m) \in \Omega$;

$4°$. $\omega \in \Omega \Leftrightarrow \varepsilon = \infty$;

$5°$. $|\Omega| = \omega \Rightarrow \omega \in \Omega$;

$6°$. $\langle n, m\rangle \in \Omega \Rightarrow \langle n+1, m\rangle \notin \Omega \Leftrightarrow \mu(\langle n, m\rangle) = 0$;

$7°$. *If* $\omega \notin \Omega$ *and* Ω *is finite, there is* $m < \omega$ *such that* $\nu(m) = \infty$ *or there are* $n, m < \omega$ *such that* $\langle n, m\rangle \in \Omega$ *and* $\mu(\langle n, m\rangle) = \infty$;

$8°$. *If* $|\Omega| = \omega$ *we have either* $\mu(\omega) \ge k$, *if there are finite* k, l *such that* $k = max\{\mu(\langle n, m\rangle) \in \Omega, n + m \ge l\}$ *or if such* k, l *do not exist, then* $\mu(\omega) = \infty$.

The characteristic $\mathfrak{C}_{T_\forall}$ is considered to be an invariant of the Jonsson universal of unars. It is important to note that in [38, 37] T.G. Mustafin and A.R. Yeshkeyev obtained the result that shows that for any arbitrary characteristics there is a universal of unars.

3 On normal Jonsson theory of unars

For this section we need to consider the definition of a perfect Jonsson theory.

Definition 8. [35] A Jonsson theory T is called perfect if its semantic model $\mathfrak{C}_T$ is ω^+-saturated.

The example of a Jonsson theory that is not perfect is a group theory.

Theorem 3.1. [35] *Let* T *be an arbitrary Jonsson theory. Then the following conditions are equivalent:*

1) T *is perfect;*

2) T^* *is the model completion of* T.

Theorem 3.2. [37] *Let* T *be a Jonsson universal of unars,* T^* *its center. Then*

1) T^* *is the model completion of* T*;*

2) T^* *allows quantifier elimination (i.e. submodel complete);*

3) T^* *is* ω*-stable.*

By aforementioned Theorems 3.1 and 3.2 the universal of unars is a perfect Jonsson theory. This fact was proved in work [37].

Let $\mathbb{T}_U$ be the theory of all unars. It is known that $\mathbb{T}_U$ is an empty theory (the set of axioms of this theory is empty), i.e. it is a set of all sentences in the language L of the signature $\sigma = \{f^1\}$, where f is a unary functional symbol. It was proved in [40] that the theory of all unars is a Jonsson theory. Let $\mathfrak{C}_{\mathbb{T}_U}$ be its semantic model.

According to Definition 3 of a semantic model of a Jonsson theory and Definition 7 of $char(\mathfrak{C}_{T_\forall})$, $\mathfrak{C}_{\mathbb{T}_U}$ (semantic model of theory of all unars) is a concatenation of components.

In [40] the following lemma was proved:

Lemma 3.1. [40] *Let* $\mathbb{T}_U$ *be the Jonsson theory of all unars, and let* M *be its model. Then* M *is a component of the theory* $\mathbb{T}_U$ *if and only if* $M \in E_{\mathbb{T}_U}$*, where* $E_{\mathbb{T}_U}$ *is the class of all existentially closed models of the theory* $\mathbb{T}_U$.

Hence, any component of the theory of all unars is an existentially closed model. Since the theory of all unars $\mathbb{T}_U$ is a universal theory [40], according to Theorem 3.2 $\mathbb{T}_U$ is a perfect Jonsson theory.

Lemma 3.2. [35] *If* T *is perfect, then the class of all* Σ_1*-closed models* E_T *coincides with* $Mod\ (T^*)$.

Moreover, the following statement holds.

Theorem 3.3. [35] *If T is perfect, then T^* is a Jonsson theory.*

Thus, by Lemma 3.2 and Theorem 3.3, $Th(\mathfrak{C}_{\mathbb{T}_U}) = \mathbb{T}_U^*$ is a complete perfect Jonsson theory and $Mod(\mathbb{T}_U^*) = E_{\mathbb{T}_U}$.

In order to give the definition of a normal Jonsson theory, we need to consider first two important notions of a cosemanticness relation and a Kaiser class.

Definition 9. [35] Let T_1 and T_2 be Jonsson theories, C_{T_1} and C_{T_2} be their semantic models, respectively. T_1 and T_2 are said to be cosemantic Jonsson theories (denoted by $T_1 \bowtie T_2$), if $C_{T_1} = C_{T_2}$.

Definition 10. Let T be a Jonsson theory. The class of all models $K_T \subseteq Mod(T)$ such that $K_T = \{\mathfrak{A} \mid \mathfrak{A} \in Mod(T)$ and $Th_{\forall\exists}(\mathfrak{A})$ is a Jonsson theory$\}$ is called the Kaiser class of T.

In the framework of studying the theories of all unars the following statement holds:

Lemma 3.3. *Let $\mathbb{T}_U$ be a theory of all unars, $\mathbb{T}_U^*$ be its center. If $E_{\mathbb{T}_U}$ is its class of existentially closed models, $K_{\mathbb{T}_U}$ is its Kaiser class, and $K^*_{\mathbb{T}_U}$ is the Kaiser class of $\mathbb{T}_U^*$, then $K_{\mathbb{T}_U} \supseteq E_{\mathbb{T}_U}$ and $K^*_{\mathbb{T}_U} = E_{\mathbb{T}_U}$.*

Proof. The theory of all unars is a perfect Jonsson theory and $Mod(\mathbb{T}_U^*) = E_{\mathbb{T}_U}$. The statement of the lemma follows from the following theorem.

Theorem 3.4. [36] *Let T be a Jonsson theory. Then, for any model $A \in E_T$, the theory $T^0(A) = Th_{\forall\exists}(A)$ is Jonsson.*

It is trivial, that $K^*_{\mathbb{T}_U} = E_{\mathbb{T}_U}$ by Theorem 3.4 $Mod(\mathbb{T}_U) = Inf(\mathbb{T}_U) \cup K_{\mathbb{T}_U} \cup Fin(\mathbb{T}_U)$, where $Inf(\mathbb{T}_U)$ is the set all of such infinite models $\mathfrak{A}$ that $Th_{\forall\exists}(\mathfrak{A})$ is not a Jonsson theory (such theory does not have AP property), and $Fin(\mathbb{T}_U)$ is the set of all finite models of the theory $\mathbb{T}_U$. The existence of such models is guaranteed by the fact that Jonsson theories are, generally speaking, not complete theories. By Theorem 3.4 $E_{\mathbb{T}_U} \notin Inf(\mathbb{T}_U)$, and, obviously, $E_{\mathbb{T}_U} \notin Fin(\mathbb{T}_U)$, therefore, it is evident that $K_{\mathbb{T}_U} \supseteq E_{\mathbb{T}_U}$. □

Theorem 3.4 shows that a Kaiser class always exists for any Jonsson theory. Since by Definition 10 E_T (the class of all existentially closed models of Jonsson theory T) will always be a subset of K_T.

Theorem 3.5. *Let $\mathbb{T}_U$ be the Jonsson theory of all unars, and let $\mathfrak{M}$ be its model. Then $\mathfrak{M}$ is a component of the theory $\mathbb{T}_U$ if and only if $\mathfrak{M} \in K_{\mathbb{T}_U}$, where $K_{\mathbb{T}_U}$ is the Kaiser class of the theory $\mathbb{T}_U$.*

Proof. The proof follows from Lemma 3.1 and Theorem 3.4. Since $K_{\mathbb{T}_U} \supseteq E_{\mathbb{T}_U}$ for the theory of all unars, it is evident that for any component $\mathfrak{M}$ of the theory $\mathbb{T}_U$: $\mathfrak{M} \in K_{\mathbb{T}_U}$ and, moreover, by the definition of Kaiser class (Definition 10), $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory. □

Definition 11. A Jonsson theory T is called a normal theory if for any $\mathfrak{M} \in K_T : \mathfrak{M}^0 = Th_{\forall\exists}(\mathfrak{M})$, and $\mathfrak{C}_{\mathfrak{M}^0}$ is an existentially closed submodel of $\mathfrak{C}_T$ ($\mathfrak{C}_{\mathfrak{M}^0}$ is a semantic model of $\mathfrak{M}^0$, $\mathfrak{C}_T$ is a semantic model of T).

As an example of a normal Jonsson theory we can consider the theory of all abelian groups, more detailed explanation one can find in [39].

Theorem 3.6. *The theory of all unars $\mathbb{T}_U$ is a normal Jonsson theory.*

Proof. Let us consider the theory of all unars $\mathbb{T}_U$ and its semantic model $\mathfrak{C}_{\mathbb{T}_U}$. $\mathfrak{M} \in K_{\mathbb{T}_U}$ means that $\mathfrak{M} \in Mod(\mathbb{T}_U)$ and $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory, where $K_{\mathbb{T}_U}$ is a Kaiser class of the theory of all unars. Since by Theorem 3.2 a Jonsson universal of unars admits quantifier elimination, $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory. Since it is a Jonsson theory by Theorem 2.1 it has a semantic model. Let $\mathfrak{M}^0 = Th_{\forall\exists}(\mathfrak{M})$ and $\mathfrak{C}_{\mathfrak{M}^0}$ be a semantic model of $\mathfrak{M}^0$ by virtue of cosemanticness relation (Definition 9).

Let us consider $\mathfrak{C}_{\mathfrak{M}^0}$ and $\mathfrak{C}_{\mathbb{T}_U}$. Here, we can use the following lemma.

Lemma 3.4. [35] *The semantic model C_T of a Jonsson theory T is T-existentially closed.*

$\mathfrak{C}_{\mathfrak{M}^0} \in Mod(\mathbb{T}_U)$ holds since any L-structure is a model of the theory of all unars due to it being an empty theory. It means that since $\mathfrak{C}_{\mathfrak{M}^0} \in Mod(\mathbb{T}_U)$ and $\mathfrak{C}_{\mathbb{T}_U}$ by Lemma 3.4 is an existentially closed model, then $\exists x\varphi(x,a)$, where $\varphi(x,a)$ is a formula of the language L, $a \in \mathfrak{C}_{\mathfrak{M}^0}$, from the fact that $\mathfrak{C}_{\mathbb{T}_U} \models \exists x\varphi(x,a)$ follows that there exists such $a' \in \mathfrak{C}_{\mathfrak{M}^0}$ that $\mathfrak{C}_{\mathfrak{M}^0} \models \exists x\varphi(x,a')$. It is evident that by the definition of characteristic of semantic models of all unars (Definition 7) and the fact that the theory of all unars is a perfect Jonsson theory, therefore, $\mathfrak{C}_{\mathbb{T}_U}$ is an ω^+-saturated model (Definition 8), this expression holds.

□

4 On double factorization and permissible expansion of the signature of unars

Let K be a class of models of a fixed signature σ. Then we can consider Jonsson spectrum for K defined as follows.

Definition 12. [36] The set $JSp(K)$ of Jonsson theories of the signature σ, where

$$JSp(K) = \{\Delta \mid \Delta \text{ is a Jonsson theory and } K \subseteq Mod(\Delta)\},$$

is called the Jonsson spectrum for the class K.

A more detailed look at the Jonsson spectrum one can find in [36]. Now, considering this set we can introduce various equivalence relations on it and obtain its factor-sets. Let us give the definition of the following binary relation.

Definition 13. Let T_1 and T_2 be Jonsson theories of the considered language L. T_1 and T_2 are called K_T-equivalent ($T_1 \tilde{\bowtie} T_2$) if $K_{T_1} = K_{T_2}$.

We can consider the cosemanticness relation on Jonsson spectrum $JSp(K)$ and obtain a partition of $JSp(K)$ onto disjoint equivalence classes with coinciding semantic models. We get a factor-set, denoted as $JSp(K)_{/\bowtie}$. We introduce the K_T-equivalent relation on the factor-set and obtain its double factorization $JSp(K)/_{D1F}$.

Let K be a class of quasivariety in the sense of [15] of a first-order language L, $L_0 \subset L$, where L_0 is the set of all sentences of the language L, i.e. $L_0 = L/L_{Fm}$, here L_{Fm} is the set of all formulas with at least one free variable. We consider the elementary theory $Th(K)$ of such class K, by adding to $Th(K)$ such $\forall\exists$ sentences of the language L that are not contained in the $Th(K)$. We can consider the set of Jonsson theories $J(Th(K))$ defined as follows.

Denotation 4. [36]A set $J(Th(K)) = \{\Delta \mid \Delta \text{ is a Jonsson theory, } \Delta = Th(K) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(L_0)$ and $\varphi^i \notin Th(K)$, $i = 0$ or $i = 1$ (i.e. it is a formula or its negation), $Th(K)$ is an elementary theory of the class of quasivariety K, $\forall\exists(L_0)$ is the set of all $\forall\exists$ sentences of first-order language L.

Obviously, every Δ has its own semantic model. Let us consider the set of all such semantic models and denote it as $J\mathbb{C}$.

Denotation 5. [36]The set $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)), \mathfrak{C}_\Delta$ is a semantic model of $\Delta\}$.

We will call the set $J\mathbb{C}$ semantic Jonsson quasivariety of class K if its elementary theory $Th(J\mathbb{C})$ is a Jonsson theory. A thorough study on semantic Jonsson quasivarieties and their Jonsson spectra can be found in [36].

Let us consider the first-order language L of the signature $\sigma = \{f^1\}$, where f is a unary functional symbol and expand it by symbols of a new constant c and a unary predicate P^1. More on expansion of a signature can be found in [9, 34, 21, 29, 20, 7].

Let $\sigma'' = \sigma \cup \sigma'$, where $\sigma = \{f^1\}$, $\sigma' = (P^1, c)$. We consider the theory $\overline{T}_\forall$ in the new expanded signature σ'' as follows [40]:

$$\overline{T}_\forall = T_\forall \cup Th_\forall(\mathfrak{C}_{T_\forall}, a)_{a \in P^1(\mathfrak{C}_{T_\forall}) \cup P^1(c)} \cup \{P^1, \subseteq\} \cup P^1(c).$$

Here, P^1 is a new unary predicate symbol, $\{P^1, \subseteq\}$ is an infinite set of sentences, which are expressing the fact that in $\mathfrak{C}_{T_\forall}$ the predicate P^1 distinguishes an existentially closed submodel of $\mathfrak{C}_{T_\forall}$, i.e. $P^1(\mathfrak{C}_{T_\forall}) = \mathfrak{M}, \mathfrak{M} \in K_{T_\forall}$, $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory, and $K_{T_\forall}$ is a Kaiser class of theory $T_\forall$. Let us denote the center of $\overline{T}_\forall$ as follows:

$$\overline{T}^*_\forall = Th(\overline{\mathfrak{C}}_{T_\forall}) = Th(\mathfrak{C}_{T_\forall}, c, a)_{c, a \in P^1(\mathfrak{C}_{T_\forall})}$$

It was proved in [40], that the theory which was constructed in such a way is Jonsson and, moreover, it is a perfect Jonsson theory.

Definition 14. [36] A Jonsson theory is said to be *hereditary* if, in any of its permissible expansions, it preserves its Jonssonness.

Therefore, the following theorem was proved.

Theorem 4.1. [40] *If a Jonsson theory of unars $T_\forall$ is a perfect Jonsson theory and $\overline{T}_\forall$ is its hereditary expansion, then $\overline{T}_\forall$ is also a perfect Jonsson theory of unars.*

Moreover, it turns out, that the theory $\overline{T}_\forall$ will also be a normal theory. The normality property of a Jonsson theory is preserved in an enriched Jonsson theory if the expansion of the signature is hereditary.

Theorem 4.2. *If a Jonsson theory of unars $T_\forall$ is a normal perfect Jonsson theory and $\overline{T}_\forall$ is its hereditary expansion, then $\overline{T}_\forall$ is also a normal perfect Jonsson theory of unars.*

Proof. By Theorem 4.1 $\overline{T}_\forall$ is a perfect Jonsson theory of unars, and $\overline{\mathfrak{C}}_{T_\forall}$ (semantic model of enriched theory of all unars in the expanded signature) is a concatenation of components. A normality property of the Jonsson theory (Definition 11) means that any semantic model $\mathfrak{C}_{\mathfrak{M}^0}$ ($\mathfrak{M}^0 = Th_{\forall\exists}(\mathfrak{M})$) is an existentially closed submodel of $\overline{\mathfrak{C}}_{T_\forall}$. Moreover, any $P^1(\mathfrak{C}_{T_\forall}) = \mathfrak{M}, \mathfrak{M} \in K_{T_\forall}$ distinguished by the unary predicate is a component by virtue of Lemma 3.1. Since $\overline{\mathfrak{C}}_{T_\forall}$ is a concatenation of components by Definition 3 and Definition 7, this property of normality will be preserved in the permissible expansion introduced above, therefore, $\overline{T}_\forall$ is a normal perfect Jonsson theory of unars. □

Lemma 4.1. *If $T_\forall$ is a normal perfect Jonsson universal of unars and $\overline{T}_\forall$ is its hereditary expansion, then $\overline{T}^*_\forall$ is a perfect Jonsson theory.*

Proof. Let us consider a Kaiser class $K_{T_\forall}$ and a class of existentially closed models $E_{T_\forall}$ of the theory $T_\forall$. By virtue of Lemma 3.3 $E_{T_\forall} \subseteq K_{T_\forall}$. Since $T_\forall$ is a perfect Jonsson theory, its center $T^*_\forall$ $(Th(\mathfrak{C}_{T_\forall}))$ is a complete perfect Jonsson theory and $E_{T_\forall} = Mod(T^*_\forall)$. Therefore, $Mod(T^*_\forall) \subseteq K_{T_\forall}$. This means that any $\mathfrak{A} \in Mod(T^*_\forall)$ is an existentially closed model and its $Th_{\forall\exists}(\mathfrak{A})$ theory is a Jonsson theory. Hence, $T^*_\forall$ is model complete by Theorem 3.2. Considering the expansion of $T^*_\forall$ as above, and the fact that $T_\forall$ admits quantifier elimination, $\overline{T}^*_\forall$ is model complete as well. Hence, $\overline{T}^*_\forall$ is a perfect Jonsson theory. □

We consider the theory $\overline{T}_\nabla$ in the new expanded signature σ'' as follows [40]:

$$\overline{T}_\nabla = T_\nabla \cup Th_\nabla(\mathfrak{C}_{T_\nabla}, a)_{a \in P^1(\mathfrak{C}_{T_\nabla}) \cup P^1(c)} \cup \{P^1, \subseteq\} \cup P^1(c).$$

Here P^1 is a new unary predicate symbol, $\{P^1, \subseteq\}$ is an infinite set of sentences, which are expressing the fact that in the semantic model $\mathfrak{C}_{T_\nabla}$ of T_∇ the predicate P^1 distinguishes an existentially closed submodel of $\mathfrak{C}_{T_\nabla}$, i.e. $P^1(\mathfrak{C}_{T_\nabla}) = \mathfrak{M}, \mathfrak{M} \in K_{T_\nabla}$, $Th_{\forall\exists}(\mathfrak{M})$ is a Jonsson theory, K_{T_∇} is a Kaiser class of theory T_∇. Let us denote the center of $\overline{T}_\nabla$ as follows:

$$\overline{T}^*_\nabla = Th(\overline{\mathfrak{C}}_{T_\nabla}) = Th(\mathfrak{C}_{T_\nabla}, c, a)_{c, a \in P^1(\mathfrak{C}_{T_\nabla})}$$

Let us consider the following notions in the new expanded signature σ''. Let us start from Denotation 4.

Denotation 6. A set $J(Th(\overline{K})) = \{\overline{\Delta} \mid \overline{\Delta}$ is a Jonsson primitive of unars, $\overline{\Delta} = Th(\overline{K}) \cup \{\varphi^i\}\}$, where $\varphi^i \in \forall\exists(\overline{L}_0)$ and $\varphi^i \notin Th(\overline{K})$, $i = 0$ or $i = 1$ (i.e. it is a formula or its negation), $Th(\overline{K})$ is an elementary theory of the class of the quasivariety of unars $\overline{K}$ in the new expanded signature σ'', $\forall\exists(\overline{L}_0)$ is the set of all $\forall\exists$ sentences of first-order language $\overline{L}$ of the signature σ''.

Definition 15. A set $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla} = \{\overline{\mathfrak{C}}_\Delta \mid \Delta \in J(Th(\overline{K})), \overline{\mathfrak{C}}_\Delta$ is a semantic model of $\overline{\Delta}\}$ is called a semantic Jonsson quasivariety of Jonsson primitives of unars in a signature σ'' if its elementary theory $Th(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})$ is a Jonsson theory.

Let us consider a Jonsson spectrum of semantic Jonsson quasivariety of unars in the new expanded signature as follows.

Definition 16. The set $JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})$ of all Jonsson theories of signature σ'', where

$$JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}) = \{\overline{T}_\nabla \mid \overline{T}_\nabla \text{ is a Jonsson primitive of unars and } J\overline{\mathbb{C}}_{\mathbb{T}_\nabla} \subseteq Mod(\overline{T}_\nabla)\},$$

is called the Jonsson spectrum for the class $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}$, where $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}$ is a semantic Jonsson quasivariety of Jonsson primitives of unars in the signature σ''.

Let us introduce the double factorization on the set $JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})$ and obtain such disjoint equivalence classes $[[\overline{T}_\nabla]] \in JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})/_{D1F}$. $[[\overline{T}_\nabla]]$ is an equivalence class of Jonsson primitives of unars in the new expanded signature, containing theories that have coinciding centers, semantic models and Kaiser classes, as well as the class of models of such theories is equal to or contains semantic Jonsson quasivariety of primitives of unars $J\overline{\mathbb{C}}_{\mathbb{T}_\nabla}$ (Definition 15).

Let $[[\overline{T}_\nabla]] \in JSp(J\overline{\mathbb{C}}_{\mathbb{T}_\nabla})/_{D1F}$, where $[[\overline{T}_\nabla]]$ is the class described above. Let $[[\overline{T}_\nabla]]^*$ be its center, $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$ be its semantic model.

Theorem 4.3. *Let* $[[\overline{T}_\nabla]]$ *be a double factorization class (D1F), and let* $[[\overline{T}_\nabla]]^*$ *be its center. Then,* $[[\overline{T}_\nabla]]_\forall = [[\overline{T}_\nabla]]^*_\forall$.

Proof. It is obvious that $[[\overline{T}_\nabla]]$ is a proper subset or equals $[[\overline{T}_\nabla]]^*$, hence, $[[\overline{T}_\nabla]]_\forall$ is also a proper subset or equals $[[\overline{T}_\nabla]]^*_\forall$ (this follows from the definition of universals and primitives). Suppose the opposite: let us assume that there is such a sentence φ in the complement of $[[\overline{T}_\nabla]]_\forall$ to $[[\overline{T}_\nabla]]^*_\forall$ that $\varphi \in [[\overline{T}_\nabla]]^*_\forall \setminus [[\overline{T}_\nabla]]_\forall$. Let $\varphi = \forall \bar{x}\, \psi(\bar{x})$ be a universal sentence. Since $[[\overline{T}_\nabla]] \supseteq [[\overline{T}_\nabla]]_\forall$ and $\varphi \notin [[\overline{T}_\nabla]]_\forall$, $[[\overline{T}_\nabla]] \vdash \varphi$ does not hold, then it follows that a theory $[[\overline{T}_\nabla]] \cup \{\neg\varphi\}$ is a consistent theory.

If the theory is consistent, then it has a model. Let $\mathfrak{A} \models [[\overline{T}_\nabla]] \cup \{\neg\varphi\}$. Then, from the fact that φ is a universal sentence follows $\neg\varphi \equiv \exists\bar{x}\,\neg\psi(\bar{x})$ and the following is true for the model $\mathfrak{A}$: $\mathfrak{A} \models \exists\bar{x}\,\neg\psi(\bar{x})$, $\mathfrak{A} \models [[\overline{T}_\nabla]]$. Due to the ω^+-universality of the model $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$, we can assume that $\mathfrak{A} \subseteq \overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$, where $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]}$ is the semantic model of class $[[\overline{T}_\nabla]]$ and $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models [[\overline{T}_\nabla]]^*$, since $[[\overline{T}_\nabla]]^*$ is $Th(\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]})$ by the definition of the center of a Jonsson theory. Let $\bar{a} \in \mathfrak{A}$ be such that $\mathfrak{A} \models \neg\psi(\bar{a})$. Since the formula $\neg\psi(\bar{x})$ contains no quantifiers, $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models \neg\psi(\bar{a})$. However, since $\varphi \in [[\overline{T}_\nabla]]^*$ and $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models [[\overline{T}_\nabla]]^*$, we have $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models \varphi$, that is, $\overline{\mathfrak{C}}_{[[\overline{T}_\nabla]]} \models \forall\bar{x}\,\psi(\bar{x})$.

This leads to a contradiction. $\square$

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