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THE VARIATIONAL APPROACH TO TIME DISCRETIZATION OF BIRKHOFF'S EQUATIONS FOR INFINITE-DIMENSIONAL SYSTEMS

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Abstract. Difference methods are widely used for the numerical solution of problems in mechanics and physics. When constructing discrete analogues, it is important to preserve the basic properties of the original differential problem. The main goal of this work is to discretize a system of equations of the form $C(x,t,u)\,u_t + E(x,t,u_\alpha) = 0$, based on its functional — the Hamiltonian action. Necessary and sufficient conditions for potentiality with respect to a given bilinear form are obtained. The Hamiltonian action for this system is constructed and its representation in the form of Birkhoff's equations for infinite-dimensional systems is obtained. By approximating the constructed functional by its discrete analogue, a discrete-time analogue of Birkhoff's equations is obtained based on the variational principle. Theoretical results are illustrated by an example of a wave equation with axial symmetry.

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1 Introduction and problem statement

Let the configuration of an infinite-dimensional potential system be defined by the vector function $u(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^{2n}(x,t))^{\mathbb{T}}$, $(x,t) \in Q_T = \Omega \times (0,T)$, Ω be a bounded domain in $\mathbb{R}^m$ with a piecewise-smooth boundary $\partial\Omega$.

Consider the following system of equations:

$$N(u) \equiv C(x,t,u)u_t + E(x,t,u_\alpha) = 0, \tag{1.1}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, α_i (for $i = \overline{1,m}$) are non-negative integers, $|\alpha| = \sum_{i=1}^m \alpha_i$, $|\alpha| = \overline{0,s}$; $C(x,t,u)$ is a given matrix $[C_{ik}(x,t,u)]_{2n\times 2n}$, $E(x,t,u_\alpha) = (E_1(x,t,u_\alpha), E_2(x,t,u_\alpha), \ldots, E_{2n}(x,t,u_\alpha))^{\mathbb{T}}$ is a given vector function, and $u = (u^1, \ldots, u^{2n})$ is the unknown vector function. Here $u_t^i = \dfrac{\partial u^i}{\partial t}$ for $i = \overline{1,2n}$, and $u_\alpha = D_\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_m^{\alpha_m}}$.

Moreover, $C_{ik} : \overline{\Omega} \times [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $E_i : \overline{\Omega} \times [0,T] \times \mathbb{R}^q \to \mathbb{R}$ are given smooth functions, where q is the length of the vector $\{u_\alpha\}$, $|\alpha| = \overline{0,s}$ and $\overline{\Omega} = \partial\Omega \cup \Omega$.

We will consider system of equations (1.1) on the set

$$\begin{aligned} D(N) = \Big\{ u \in U = (U^1, \ldots, U^{2n})^{\mathbb{T}} : u^i \in U^i = C^{2s,1}_{x,t}(\overline{\Omega} \times [0,T]) : u^i\big|_{t=0} = \varphi_0^i(x), \\ u^i\big|_{t=T} = \varphi_1^i(x), \left.\frac{\partial^\nu u^i}{\partial n_x^\nu}\right|_{\Gamma_T} = \psi_\nu^i(x,t), i = \overline{1,2n}, |\nu| = \overline{0,s-1} \Big\}, \end{aligned} \tag{1.2}$$

where $\Gamma_T = \partial\Omega \times (0,T)$, n_x is the outward normal to $\partial\Omega$; φ_0^i, φ_1^i, $\psi_\nu^i(x,t)$ are given smooth functions.

Note that (1.1) is a generalization of system of equations of the form

$$C(t,u)\dot{u}(t) + E(t,u) = 0, \tag{1.3}$$

where $\dot{u}(t) = \dfrac{du}{dt}$.

In [4] it was proved that (1.1) admits a classical variational formulation if and only if it can be represented in the form of Birkhoff's equations [1]. In this case, its Hamiltonian action [8] was constructed.

In [5] the potentiality of a discrete system obtained from equations of form (1.3) with continuous time was investigated. Necessary and sufficient conditions for potentiality with respect to a given bilinear form were provided. An algorithm for constructing the corresponding functional, the discrete analogue of the Hamiltonian action, was outlined.

The main goal of this work is to construct a discrete-time analogue of system (1.1) based on its Hamiltonian action.

2 Necessary and sufficient conditions for potentiality

Let $V = (V^1, V^2, \dots, V^{2n}) : V^i = C(\overline{\Omega} \times [0,T]), i = \overline{1,2n}$. Define a bilinear form $\langle \cdot,\cdot \rangle : V \times U \to \mathbb{R}$ as follows:

$$\langle v, g \rangle = \int\limits_0^T \int\limits_\Omega \sum_{i=1}^{2n} v^i g^i dx dt. \tag{2.1}$$

Following [2, 6], we say that problem (1.1), (1.2) admits a direct variational formulation with respect to (2.1) if there exists a differentiable Gâteaux functional $F_N : D(N) \to \mathbb{R}$ such that its differential has the form:

$$\delta F_N[u,h] = \langle N(u), h \rangle, \quad \forall u \in D(N), \forall h \in D(N'_u).$$

Here, $D(N'_u)$ is the domain of the Gâteaux derivative N'_u of the operator N at a point $u \in D(N)$. In this case, it is also said that the operator N is potential on $D(N)$ with respect to bilinear form (2.1).

The criterion for potentiality of N on the convex set $D(N)$ is the symmetry condition [2, 6]

$$\langle N'_u h, g \rangle = \langle N'_u g, h \rangle, \quad \forall u \in D(N), \forall h, g \in D(N'_u). \tag{2.2}$$

When this condition is satisfied, the desired functional F_N — the Hamiltonian action — can be found using the formula

$$F_N[u] = \int_0^1 \langle N(\hat{u} + \lambda(u - \hat{u})), u - \hat{u} \rangle \, d\lambda + \text{const}, \tag{2.3}$$

where $\hat{u}$ is an arbitrary fixed element in $D(N)$.

Let us denote $N_j \equiv \sum\limits_{k=1}^{2n} C_{jk} u_t^k + E_j$, and $N(u) \equiv (N_1(u), N_2(u), \dots, N_{2n}(u))$.

Let us find the Gâteaux derivative of the operator N_j

$$(N'_u h)_j = \sum_{k=1}^{2n} \sum_{i=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} h^i u_t^k + \sum_{i=1}^{2n} C_{ji} h_t^i + \sum_{i=1}^{2n} \sum_{|\alpha|=0}^{s} \frac{\partial E_j}{\partial u_\alpha^i} h_\alpha^i.$$

Using condition (2.2), we get

$$\begin{aligned}\langle N_u' h, g\rangle &= \int_0^T \int_\Omega \sum_{j=1}^{2n} (N_u' h)_j . g^j dxdt \\ &= \int_0^T \int_\Omega \sum_{j=1}^{2n} \left[\sum_{k=1}^{2n} \sum_{i=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} h^i u_t^k + \sum_{i=1}^{2n} C_{ji} h_t^i + \sum_{i=1}^{2n} \sum_{|\alpha|=0}^{s} \frac{\partial E_j}{\partial u_\alpha^i} h_\alpha^i \right] g^j dxdt\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\langle N_u' g, h\rangle &= \int_0^T \int_\Omega \sum_{i=1}^{2n} (N_u' g)_i . h^i dxdt \\ &= \int_0^T \int_\Omega \sum_{i=1}^{2n} \left[\sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial C_{ik}}{\partial u^j} g^j u_t^k + \sum_{j=1}^{2n} C_{ij} g_t^j + \sum_{j=1}^{2n} \sum_{|\beta|=0}^{s} \frac{\partial E_i}{\partial u_\beta^j} g_\beta^j \right] h^i dxdt.\end{aligned}$$

Integrating by parts, from (2.4) we find

$$\begin{aligned}\langle N_u' h, g\rangle = \int_0^T \int_\Omega \sum_{j=1}^{2n} &\left[\sum_{k=1}^{2n} \sum_{i=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} h^i g^j u_t^k - \sum_{i=1}^{2n} \frac{d}{dt} \left(C_{ji} g^j \right) h^i + \right. \\ &\left. + \sum_{i=1}^{2n} \sum_{|\alpha|=0}^{s} (-1)^{|\alpha|} D_\alpha \left(\frac{\partial E_j}{\partial u_\alpha^i} g^j \right) h^i \right] dxdt.\end{aligned}$$

It should be noted that

$$\frac{d}{dt}\left(C_{ji} g^j \right) = \frac{d}{dt}\left(C_{ji} \right) g^j + C_{ji} g_t^j = \sum_{k=1}^{2n} \frac{\partial C_{ji}}{\partial u^k} u_t^k g^j + \frac{\partial C_{ji}}{\partial t} g^j + C_{ji} g_t^j,$$

$$\sum_{|\alpha|=0}^{s} (-1)^{|\alpha|} D_\alpha \left(\frac{\partial E_j}{\partial u_\alpha^i} g^j \right) = \sum_{|\alpha|,|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left(\frac{\partial E_j}{\partial u_\alpha^i} \right) g_\beta^j,$$

where

$$\binom{\alpha}{\beta} = \begin{cases} \binom{\alpha_1}{\beta_1}\binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_m}{\beta_m}, \text{if } \forall i \in \{1, 2, ..., m\} : \alpha_i \geq \beta_i, \\ 0, \text{if } \exists i \in \{1, 2, ..., m\} : \alpha_i < \beta_i, \end{cases}$$

$$\binom{\alpha_i}{\beta_i} = \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}.$$

Taking into account this for the potentiality of operator N (1.1), we obtain

$$\langle N_u' h, g\rangle - \langle N_u' g, h\rangle = \int_0^T \int_\Omega \sum_{i=1}^{2n} \left\{ \left[\sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial C_{jk}}{\partial u^i} g^j u_t^k - \sum_{j=1}^{2n} \sum_{k=1}^{2n} \frac{\partial C_{ji}}{\partial u^k} u_t^k g^j - \right.\right.$$
$$\left. - \sum_{j=1}^{2n} \frac{\partial C_{ji}}{\partial t} g^j - \sum_{j=1}^{2n} C_{ji} g_t^j + \sum_{j=1}^{2n} \sum_{|\alpha|,|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left(\frac{\partial E_j}{\partial u_\alpha^i} \right) g_\beta^j \right] -$$
$$\left.- \left[\sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial C_{ik}}{\partial u^j} g^j u_t^k + \sum_{j=1}^{2n} C_{ij} g_t^j + \sum_{j=1}^{2n} \sum_{|\beta|=0}^{s} \frac{\partial E_i}{\partial u_\beta^j} g_\beta^j \right] \right\} h^i dxdt$$
$$= \int_0^T \int_\Omega \sum_{i=1}^{2n} \left\{ \sum_{j,k=1}^{2n} \left(\frac{\partial C_{jk}}{\partial u^i} - \frac{\partial C_{ji}}{\partial u^k} - \frac{\partial C_{ik}}{\partial u^j} \right) g^j u_t^k - \sum_{j=1}^{2n} (C_{ji} + C_{ij}) g_t^j - \sum_{j=1}^{2n} \frac{\partial C_{ji}}{\partial t} g^j + \right.$$
$$\left. + \sum_{j=1}^{2n} \sum_{|\alpha|,|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left(\frac{\partial E_j}{\partial u_\alpha^i} \right) g_\beta^j - \sum_{j=1}^{2n} \sum_{|\beta|=0}^{s} \frac{\partial E_i}{\partial u_\beta^j} g_\beta^j \right\} h^i dxdt = 0.$$

By virtue of arbitrariness of the functions h^i, we come to the conditions:

$$\begin{cases} C_{ij} + C_{ji} = 0, \\ \dfrac{\partial C_{ji}}{\partial u^k} + \dfrac{\partial C_{ik}}{\partial u^j} + \dfrac{\partial C_{kj}}{\partial u^i} = 0, \\ \dfrac{\partial C_{ji}}{\partial t} = \displaystyle\sum_{|\alpha|=0}^{s} (-1)^{|\alpha|} D_\alpha \left(\frac{\partial E_j}{\partial u_\alpha^i} \right) - \frac{\partial E_i}{\partial u^j}, \\ \displaystyle\sum_{|\alpha|=1}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left(\frac{\partial E_j}{\partial u_\alpha^i} \right) - \frac{\partial E_i}{\partial u_\beta^j} = 0 \end{cases} \tag{2.5}$$

for $\forall (x,t) \in Q_T, \forall u \in D(N)$, where $i,j,k = \overline{1,2n}$, and $|\beta| = \overline{1,s}$.

Theorem 2.1. *System (1.1) is potential on $D(N)$ (1.2) with respect to bilinear form (2.1) if and only if conditions (2.5) are satisfied.*

3 Construction of the Hamiltonian action

If conditions (2.5) are satisfied, the desired functional F_N can be constructed using formula (2.3). Another approach can be taken to this problem. Let us look for the Hamiltonian action for (1.1) in the form

$$F_N = \int_0^T \int_\Omega \left(\sum_{i=1}^{2n} R_i u_t^i - B \right) dxdt, \tag{3.1}$$

where $R_i(x,t,u)$, $B(x,t,u_\alpha)$ are the unknown smooth functions.

The Gâteaux differential of functional (3.1) is given by

$$\delta F_N[u,h] = \int_0^T \int_\Omega \left[\sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_i}{\partial u^k} h^k u_t^i + \sum_{i=1}^{2n} R_i h_t^i - \sum_{i=1}^{2n} \sum_{|\gamma|=0}^{s} \frac{\partial B}{\partial u_\gamma^i} h_\gamma^i \right] dxdt.$$

Integrating by parts, we obtain

$$\delta F_N[u,h] = \int_0^T \int_\Omega \left[\sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_k}{\partial u^i} h^i u_t^k - \sum_{i=1}^{2n} \frac{dR_i}{dt} h^i - \right. \\ \left. - \sum_{i=1}^{2n} \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u_\gamma^i} \right) h^i \right] dx dt. \tag{3.2}$$

Since

$$\frac{dR_i}{dt} = \sum_{k=1}^{2n} \frac{\partial R_i}{\partial u^k} u_t^k + \frac{\partial R_i}{\partial t},$$

we can write (3.2) as

$$\delta F_N[u,h] = \int_0^T \int_\Omega \sum_{i=1}^{2n} \left[\sum_{k=1}^{2n} \left(\frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k} \right) u_t^k - \frac{\partial R_i}{\partial t} - \right. \\ \left. - \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u_\gamma^i} \right) \right] h^i dx dt.$$

From the definition of potentiality, we have

$$\int_0^T \int_\Omega \sum_{i=1}^{2n} \left[\sum_{k=1}^{2n} \left(\frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k} \right) u_t^k - \frac{\partial R_i}{\partial t} - \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u_\gamma^i} \right) \right] h^i dx dt = \\ = \int_0^T \int_\Omega \sum_{i=1}^{2n} \left[\sum_{k=1}^{2n} C_{ik} u_t^k + E_i \right] h^i dx dt.$$

Since the elements h^i are arbitrary, we obtain

$$\sum_{k=1}^{2n} \left(\frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k} \right) u_t^k - \frac{\partial R_i}{\partial t} - \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u_\gamma^i} \right) = \sum_{k=1}^{2n} C_{ik} u_t^k + E_i, \tag{3.3}$$

where $i = \overline{1,2n}$.

Comparing the left- and right-hand sides of (3.3), we find

$$\begin{cases} \dfrac{\partial R_k}{\partial u^i} - \dfrac{\partial R_i}{\partial u^k} = C_{ik}, \\ -\dfrac{\partial R_i}{\partial t} - \displaystyle\sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u_\gamma^i} \right) = E_i, \end{cases} \tag{3.4}$$

where $i,k = \overline{1,2n}$. For the first group of equations in system (3.4), we obtain the following solution [4]:

$$R_i = -\int_0^1 \sum_{k=1}^{2n} \lambda C_{ik} (x, t, \widehat{u} + \lambda(u - \widehat{u})) (u^k - \widehat{u}^k) d\lambda, \ i = \overline{1,2n}.$$

Let $\mathfrak{B}[t,u] = \int\limits_{\Omega} B(x,t,u_\alpha)dx$; $\dfrac{\delta \mathfrak{B}}{\delta u^i}$ be the functional derivative of $\mathfrak{B}$ with respect to u^i, $i = \overline{1,2n}$.

Let us rewrite the second group of equations in system (3.4) in the following form

$$\sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B}{\partial u^i_\gamma}\right) = -\frac{\partial R_i}{\partial t} - E_i.$$

or

$$\frac{\delta \mathfrak{B}}{\delta u^i} = -\frac{\partial R_i}{\partial t} - E_i, i = \overline{1,2n}.$$

Using formula (2.3), we obtain

$$\mathfrak{B}[t,u] = -\int\limits_{\Omega}\int\limits_{0}^{1}\sum_{i=1}^{2n}\left[\frac{\partial R_i}{\partial t}(x,t,\widehat{u}+\lambda(u-\widehat{u})) + E_i(x,t,\widehat{u}_\alpha+\lambda(u_\alpha-\widehat{u}_\alpha))\right]\cdot$$
$$\cdot(u^i-\widehat{u}^i)d\lambda dx + \text{const}.$$

Thus, we arrive to the following Birkhoff's equations for infinite-dimensional systems:

$$N_i \equiv \sum_{k=1}^{2n}\left(\frac{\partial R_k}{\partial u^i} - \frac{\partial R_i}{\partial u^k}\right)u^k_t - \frac{\partial R_i}{\partial t} - \frac{\delta \mathfrak{B}}{\delta u^i} = 0, i = \overline{1,2n}. \tag{3.5}$$

Theorem 3.1. *The extremals of functional (3.1) are solutions to the system of equations (3.5).*

4 Discretization

Let us divide the interval $[0,T]$ into l equal parts with nodes $t_j = j\tau$, $j = \overline{0,l}$, where $\tau = l^{-1}T$. Let us introduce the narrowing operators [7]

$$\overline{T}_r u(x,t) = \overline{u}_r = \big(u^1(x,t_1), u^2(x,t_1), \ldots, u^{2n}(x,t_1), u^1(x,t_2), u^2(x,t_2), \ldots,$$
$$u^{2n}(x,t_2), \ldots, u^1(x,t_{l-1}), u^2(x,t_{l-1}), \ldots, u^{2n}(x,t_{l-1})\big),$$

where $r = 2n(l-1)$. Such vectors form a linear space, which we will denote as $\overline{U}_r$. For convenience, let us denote $\widetilde{u}_j = u(x,t_j)$, $\widetilde{u}^i_j = u^i(x,t_j)$, $i = \overline{1,2n}$, $j = \overline{0,l}$.

Denote by $\overline{N}$ the operator of the discrete analogue of problem (3.5), (1.2), obtained on the basis of functional (3.1).

Let us define

$$D\left(\overline{N}\right) = \Big\{(\widetilde{u}_0, \overline{u}_r, \widetilde{u}_l) : \overline{u}_r \in \overline{U}_r, \widetilde{u}^i_0 = \varphi^i_0(x), \widetilde{u}^i_l = \varphi^i_1(x), \widetilde{u}^i_j \in C^{2s}\left(\overline{\Omega}\right),$$
$$\left.\frac{\partial^\nu \widetilde{u}^i_j}{\partial n^\nu_x}\right|_{\partial\Omega} = \psi^i_\nu(x,t_j), i = \overline{1,2n}, |\nu| = \overline{0,s-1}, j = \overline{0,l}\Big\},$$
$$D\left(\overline{N}'_u\right) = \Big\{(\widetilde{h}_0, \overline{h}_r, \widetilde{h}_l) : \overline{h}_r \in \overline{U}_r, \widetilde{h}^i_0 = 0, \widetilde{h}^i_l = 0, \widetilde{h}^i_j \in C^{2s}\left(\overline{\Omega}\right),$$
$$\left.\frac{\partial^\nu \widetilde{u}^i_j}{\partial n^\nu_x}\right|_{\partial\Omega} = 0, i = \overline{1,2n}, |\nu| = \overline{0,s-1}, j = \overline{0,l}\Big\},$$

where $\overline{N}'_u$ is the Gâteaux derivative of the operator $\overline{N}$.

Next, we approximate the integrals as follows:

$$\int_{t_j}^{t_{j+1}} \int_{\Omega} \left(\sum_{i=1}^{2n} R_i u_t^i - B \right) dx dt \approx \frac{T}{l} \int_{\Omega} \left(\sum_{i=1}^{2n} R_{i,j} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - B_j \right) dx,$$

where $R_{i,j} = R_i \left(x, t_j, \widetilde{u}_j\right)$, $B_j = \left(x, t_j, D_\gamma \widetilde{u}_j\right)$.

We replace functional (3.1) by the discrete Hamiltonian action:

$$\overline{F}\left(\overline{u}_r\right) = \frac{T}{l} \sum_{j=0}^{l-1} \int_{\Omega} \left(\sum_{i=1}^{2n} R_{i,j} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - B_j \right) dx.$$

Then, we have

$$\delta\overline{F}\left[\overline{u}_r, \overline{h}_r\right] = \frac{T}{l} \sum_{j=0}^{l-1} \int_{\Omega} \left(\sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_{i,j}}{\partial \widetilde{u}_j^k} \widetilde{h}_j^k \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} + \sum_{k=1}^{2n} R_{k,j} \frac{\widetilde{h}_{j+1}^k - \widetilde{h}_j^k}{\tau} - \right.$$
$$\left. - \sum_{k=1}^{2n} \sum_{|\gamma|=0}^{s} \frac{\partial B_j}{\partial D_\gamma \left(\widetilde{u}_j^k\right)} D_\gamma \widetilde{h}_j^k \right) dx. \tag{4.1}$$

Since

$$\widetilde{h}_0^i = 0, \widetilde{h}_l^i = 0, \left.\frac{\partial^\nu \left(\widetilde{h}_j^i\right)}{\partial n_x^\nu}\right|_{\partial\Omega} = 0, i = \overline{1, 2n}, j = \overline{0, l}, |\nu| = \overline{0, s-1},$$

integrating by parts in (4.1), we get

$$\delta\overline{F}\left[\overline{u}_r, \overline{h}_r\right] = \frac{T}{l} \sum_{j=1}^{l-1} \int_{\Omega} \left[\sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial R_{i,j}}{\partial \widetilde{u}_j^k} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - \sum_{k=1}^{2n} \frac{R_{k,j} - R_{k,j-1}}{\tau} - \right.$$
$$\left. - \sum_{k=1}^{2n} \sum_{|\gamma|=0}^{s} (-1)^{|\gamma|} D_\gamma \left(\frac{\partial B_j}{\partial D_\gamma \left(\widetilde{u}_j^k\right)} \right) \right] \widetilde{h}_j^k dx.$$

From the equality $\delta F\left[\overline{u}_r, \overline{h}_r\right] = 0$, $\forall \overline{u}_r \in D\left(\overline{N}\right)$, $\forall \overline{h}_r \in D\left(\overline{N}_u'\right)$ we obtain the system of equations for discrete-time motion as follows:

$$\overline{N}_{k,j} \equiv \sum_{i=1}^{2n} \frac{\partial R_{i,j}}{\partial \widetilde{u}_j^k} \frac{\widetilde{u}_{j+1}^i - \widetilde{u}_j^i}{\tau} - \frac{R_{k,j} - R_{k,j-1}}{\tau} - \frac{\delta \mathfrak{B}_j}{\delta \widetilde{u}_j^k} = 0,$$
$$j = \overline{1, l-1}, k = \overline{1, 2n}, \tag{4.2}$$

where $\mathfrak{B}_j = \mathfrak{B}[t_j, \widetilde{u}_j]$.

Theorem 4.1. *Equations (4.2) are discrete-time analogues of (3.5).*

5 Example

Let us consider the wave equation with axial symmetry [3]:

$$w_{tt} = a^2 \left(w_{\rho\rho} + \frac{1}{\rho} w_\rho \right), t \in [0, T], \rho \in [\rho_1, \rho_2], \tag{5.1}$$

with the following boundary conditions:

$$w\big|_{t=0} = \varphi_1(\rho), w\big|_{t=T} = \varphi_2(\rho),$$
$$w\big|_{\rho=\rho_1} = \phi_1(t), w\big|_{\rho=\rho_2} = \phi_2(t),$$

where $w(t,\rho)$ is the unknown function, ρ is the radial coordinate, and a is a constant coefficient.

Let us denote

$$\begin{cases} w = u^1, \\ w_t = u^2. \end{cases}$$

We can represent equation (5.1) as a system of equations:

$$N(u) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_t^1 \\ u_t^2 \end{pmatrix} + \begin{pmatrix} -u^2 \\ -a^2 \left(u_{\rho\rho}^1 + \frac{1}{\rho} u_\rho^1 \right) \end{pmatrix} = 0. \tag{5.2}$$

According to Theorem 2.1, operator (5.2) is not potential. Using conditions (2.5), we can find the matrix variational multiplier as follows:

$$M = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}.$$

Then, the system

$$\widehat{N} = MN = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix} \begin{pmatrix} u_t^1 \\ u_t^2 \end{pmatrix} + \begin{pmatrix} -a^2 \rho u_{\rho\rho}^1 - a^2 u_\rho^1 \\ \rho u^2 \end{pmatrix} = 0 \tag{5.3}$$

admits a representation in the form of Birkhoff's equations and we find

$$R_1 = -\frac{1}{2}\rho u^2, R_2 = \frac{1}{2}\rho u^1, B = -\frac{1}{2}a^2 \rho \left(u_\rho^1\right)^2 - \frac{1}{2}\rho \left(u^2\right)^2.$$

Therefore, according to formula (3.1), the Hamiltonian action for (5.3) takes the following form:

$$F_{\widehat{N}}[u] = \int_0^1 \int_\Omega \frac{1}{2} \left[-\rho u^2 u_t^1 + \rho u^1 u_t^2 + a^2 \rho \left(u_\rho^1\right)^2 + \rho \left(u^2\right)^2 \right] dx dt.$$

By converting it to its discrete version, we can easily obtain the discrete version of system of equations (5.3) as follows:

$$\overline{N}_{1,j} \equiv \frac{1}{2}\rho \frac{\widetilde{u}_{j+1}^2 - \widetilde{u}_j^2}{\tau} + \frac{1}{2}\rho \frac{\widetilde{u}_j^2 - \widetilde{u}_{j-1}^2}{\tau} - a^2 \rho \frac{\partial^2}{\partial \rho^2} \widetilde{u}_j^1 - a^2 \frac{\partial}{\partial \rho^1} \widetilde{u}_j^1 = 0, j = \overline{1, l-1},$$
$$\overline{N}_{2,j} \equiv -\frac{1}{2}\rho \frac{\widetilde{u}_{j+1}^1 - \widetilde{u}_j^1}{\tau} - \frac{1}{2}\rho \frac{\widetilde{u}_j^1 - \widetilde{u}_{j-1}^1}{\tau} + \rho \widetilde{u}_j^2 = 0, j = \overline{1, l-1}.$$

6 Conclusion

Necessary and sufficient conditions for the potentiality of the system of equations of the form $C(x,t,u)\,u_t + E(x,t,u_\alpha) = 0$ with respect to a given bilinear form have been obtained. An algorithm for constructing the corresponding Hamiltonian action and transforming this system into the form of Birkhoff's equations for infinite-dimensional systems is presented. Based on the derived Hamiltonian action, a discrete analogue of this system of equations has been obtained. An illustrative example has been considered.

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