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PSEUDOFINITE UNAR THEORY WITH ARBITRARY NUMBER OF SEMICHAINS AND ANTICHAINS

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Abstract. The paper focuses on constructing a pseudofinite theory of a unar with an arbitrary number of antichains and semichains and studying its model-theoretic properties. The question of elementary equivalence of unars with different numbers of antichains and semichains remains open. The obtained theory is shown to be omega-stable, strengthening the known result stating that all complete theories of unars are superstable. Prime models of this theory turn out to be omega-categorical, implying their smooth approximability.

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1 Introduction

The study of pseudofinite structures is important because it provides a way to model infinite structures that are similar to finite structures. This allows the transfer of techniques and results from finite mathematics to the study of infinite structures. For example, in model theory, pseudofinite structures are used to study the behaviour of infinite structures under different model-theoretic interpretations.

The family of (pseudofinite) theories is studied in [32, 33, 34, 27, 16], their rank characteristics and topologies in [14, 15], pseudofinite acyclic graphs in [18], and equivalence relation in [10, 19].

In the paper [3] there are considered surjective quadratic Jordan algebras, which has connections with problems of decomposition of algebraic structures, as in [11], where there is studied the effective decomposition of Abelian groups. In both cases, the issues of decomposition and model construction are important. Paper [4] is devoted to countable models of small stable theories, which has connections with the topics of [5, 12], where there is studied the difficulty of recognizing decidable theories.

Problems concerning complete theories of unars were considered in works [9, 13, 18, 22, 23, 30, 28, 8]. Yu.E. Shishmarev [30] obtained a description of uncountably and countably categorical theories of unars, in [13] L. Marcus solved the problem of the number of non-isomorphic countable models in the complete theory of unars, A.A. Ivanov [9] obtained a criterion for elementary equivalence of unars.

This paper is a continuation of the study on approximations of unar theory by finite unar theories starting from [31, 17, 8]. The characterization of pseudofinite theories of unars depends essentially on the combinatorial structure of their connected components, particularly on the distribution of semichains and antichains. While the case in which these components occur in equal numbers is comparatively tractable, the general situation with arbitrary finite or infinite numbers of semichains and antichains is substantially more involved. The interaction between these two types of components significantly influences the existence of finite approximations and, consequently, the pseudofiniteness of the corresponding theories. Therefore, obtaining necessary and sufficient conditions for pseudofiniteness in the general case constitutes a natural and important extension of previous investigations and provides a complete characterization of pseudofinite unar theories.

In [29] an algebraic description of projective, weakly-, quasi- and pseudo-projective, injective, weakly-, quasi- and pseudo-injective unars is presented.

In [31], it is proved that a coproduct of finite acts over monoid is pseudofinite.

As usual, we will use the standard terminology. In particular, ω denotes the set of all non-negative integers.

Definition 1. A unar is a structure $\mathcal{U} = \langle U; f\rangle$ where $f : U \to U$ is a unary operation on a set U. For any $u \in U$, let $f^0(u) = u$, $f^{n+1}(u) = f(f^n(u))$ for all $n \in \omega$, $f^{-1}(u) = \{w \in U | f(w) = u\}$. A unar $\mathcal{U}$ is called a *cycle* of length $n \in \omega$ if there exists $u \in U$ such that $U = \{f^i(u)|0 \leq i < n\}$, $f^n(u) = u$, $f^i(u) \neq f^j(u)$ for all distinct $i, j \in \{0, \ldots, n-1\}$. A set $\{u_i|i \in \omega\} \subseteq U$ is called a *semichain* if $f(u_i) = u_{i+1}$ and $u_i \neq u_j$ for all distinct $i, j \in \omega$. A set $\{u_i|i \in \omega\} \subseteq U$ is called an infinite *antichain* if $f(u_{i+1}) = u_i$ and $u_i \neq u_j$ for all distinct $i, j \in \omega$. If the cardinality of the set $\{w \in U|f(w) = u\}$ is k, we say that u is a *k-branching point*, or *k-valence point*. In the language of the graph theory we say that u is a point of *degree k*.

In the first section basic facts are given about model-theoretic properties of unars with some explanations. In the second section there are given equivalent definitions of pseudofiniteness. The third section is devoted to approximations developed by S.V. Sudoplatov in [33]. In the fourth section, a pseudofinite theory of a unar with an arbitrary number of semichains and antichains is constructed. It is proved that this theory is omega-stable.

2 Preliminaries

Definition 2. Let $X \subseteq U$ and $u, v \in U$. We say that u, v are *connected* if there are $n, m \in \omega$ such that $f^n(u) = f^m(v)$. We say that X is *connected* if any two elements of X are connected. A maximal connected subset of U is called a *connected component* of U.

A connected component is defined as for graphs. Two vertices u, v are connected if there is a f-path connecting them: $u = w_0, w_1, \ldots, w_k = v$, where either $f(w_i) = w_{i+1}$ for each $0 \leq i \leq k-1$ or $f(w_i) = w_{i-1}$ for each $1 \leq i \leq k$. We can define the distance $\rho(u, v)$ as the minimum length of a path connecting u, v if these vertices lie in the same component; and if they lie in different components, then we assume $\rho(u, v) = \infty$.

The cycle (ring) of a component (graph) Γ is the set consisting of all $u \in \Gamma$ for which there exists $n > 0$ such that $f^n(u) = u$. A component Γ is called a component with a cycle if its cycle is not empty.

If a component does not contain a cycle of length $n > 0$, we call it a *tree*. If a unar is disconnected and each component does not contain a cycle of length $n > 0$, we call it a *forest*.

The *root of depth* n of an element u is the set $K_n(u) = \{w \in \mathcal{U}|$ there is $i \leq n$ such that $f^i(w) = u\}$. The *root* of u is

$$K(u) = \bigcup_{i\in\omega} K_n(u).$$

A connected subset of the root $K_n(u)$ that contains u is called a subroot of depth n of the element u.

Fact 1. *There is at most one cycle in one component.*

If u does not belong to a cycle, then $K(u)$ is a tree. Indeed, suppose for vertices $u', v' \in K(u)$ there are two paths $u' = x_0, x_1, \ldots, x_k = v'$ and $u' = x_0, y_1, \ldots, y_{m-1}, x_k = v'$. We can assume that the common vertices of these two paths are only u' and v'. One of these paths "comes" to u', let us say $f(x_1) = u'$. Then, we have $f(x_{i+1}) = f(x_i)$ for each $i \in \{0, 1, \ldots, k\}$, and the second path leads from u' to v'. The union of these paths gives a cycle containing u' and v', but not containing u, a contradiction with $u', v' \in K(u)$.

Similarly, it can be proved that if a vertex u belongs to a cycle C, then $(K(u) \setminus C) \cup \{u\}$ is a tree. For simplicity of further exposition, we will also denote this tree by $K(u)$ (when the vertex u belongs to a cycle).

Description of connected components of a unar.

(1) There is a cycle in a component. Then the component is a forest of trees planted on the vertices of this cycle.

(2) There is no cycle in a component. If there is an element that does not have an f-preimage, then the component is a forest of trees planted on the f-trajectory (semichain) of this element.

(3) There is no cycle in a component. All elements have an f-preimage and the component is a forest of trees planted on the f-trajectory ($\mathbb{Z}$-chain) of any element.

Definition 3. A theory T is said to be *limited* if there exists a natural number N such that the following formula is true in T:

$$(\forall u)[\bigvee_{n,m=1}^{N} (f^n(u) = f^{n+m}(u))].$$

Fact 2. [23, 30] *A theory T is ω-categorical if and only if*

i) T is limited,

ii) if $\mathcal{U} \models T$, then there are only finitely many non-isomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(u)$ in $\mathcal{U}$ or equivalently, $\mathcal{U}$ realizes finitely many 1-types.

In components of form (2) and (3) there are vertices u, v with the formula property $\rho(u, v) = n$ for any natural n, and there is a unique f-path without repetitions connecting these vertices u, v. Therefore, in a countably categorical unar, all components contain a cycle. The lengths of cycles are uniformly bounded, and all trees planted on the vertices of a cycle have finite height, and these heights are also uniformly bounded.

Let us prove the following: all trees of the form $K(u)$ can be assigned labels so that isomorphic trees and only they have the same labels. A label is a formula that tells about the property of the root of this tree. The total number of labels is finite, which is not surprising since up to equivalence the number of formulas (with one free variable) in a countably categorical structure is finite. A label of a tree of height h will have the superscript h.

We will assign labels by induction on the height of the trees. There is only one tree of height zero, it is assigned the label p_0^0, and the formula says that the root does not have a f-preimage.

Let us assume that all trees of height $\leq k$ are assigned labels. For each label p_j^i with $j \leq k$, there is a natural number m_j^i such that if the f-preimage of a vertex contains more than m_j^i vertices u for which $K(u)$ has the label p_j^i, then the f-preimage contains infinitely many such vertices (again, due to countable categoricity).

Let a tree of the form $K(u)$ have height $k+1$. Then we study the composition of the set $f^{-1}(u)$, each vertex v from this set defines a tree $K(v)$, and therefore some label p_j^i with $j \leq k$. We count how many vertices from $f^{-1}(u)$ define a tree with label p_j^i: if there are $n \leq m_j^i$ pieces, then we compose the formula $\psi^{-1}(x)$, which says that "$f^{-1}(x)$ contains n vertices with label p_j^i", and if there are $n > m_j^i$ pieces, then we take the formula $\psi^{-1}(x)$, which says that "$f^{-1}(x)$ contains $m_j^i + 1$ vertices with label p_j^i". The conjunction of all formulas $\psi^{-1}(x)$ will be a label for u.

The distribution of labels to vertices is complete.

It is easy to prove by induction on height that if the labels of the vertices u, v coincide in a countable model, then $K(u) \cong K(v)$.

Now consider two components. They are isomorphic if and only if their cycles have the same length and the labels of the vertices on the cycles are such that there is an isomorphism between the cycles that maps the vertices with the same labels. Since there are finitely many labels and the lengths of the cycles are bounded by some number, there will be finitely many non-isomorphic components in a countable model. A complete theory of a unar simply specifies how many components of a certain type of isomorphism there are in its model.

Note also that in a countably categorical unar the connectivity relation is formulaic and therefore each component is distinguished by a formula (with a parameter).

Fact 3. [30] *A limited theory T is uncountably categorical if and only if it satisfies the following conditions:*

i) $|f^{-1}(u)|$ is infinite for at most one u, and is otherwise bounded.

ii) If $|f^{-1}(u)|$ is finite for all u, then all except one type of connected component of $\mathcal{U}$ are finite.

iii) If there is $u \in \mathcal{U}$ such that $|f^{-1}(u)|$ is infinite, then all types of connected components of $\mathcal{U}$ are finite and all $K_n(u)$ for $f(w) = u$ are isomorphic except for finitely many u.

A set of N-neighborhood of $V \subseteq U$ is the set

$$\{u \in U : \text{there is } v \in V \text{ such that } \bigvee_{n,m}^{N} f^n(v) = f^m(u)\}.$$

Fact 4. [30] *An unlimited theory T is uncountably categorical if it satisfies the following conditions:*

i) $|f^{-1}(u)|$ is bounded.

ii) For each $n \in \omega$ there are only finitely many connected components whose cycle consists of n elements.

iii) There exist a finite set $V_0 \subseteq U$, a set $V \subseteq U$, $m \in \omega$, and a set $\{P_v : v \in V\}$ such that $U = V_0 \cup \bigcup_{v \in V} P_v$, P_v is a subroot of depth m for $v \in V$, and for $v, w \in V$, the subroots P_v and P_w are isomorphic and this isomorphism can be continued to an isomorphism of their $2m$-neighborhoods.

The following statement perfectly and succinctly summarizes the last two facts above.

Theorem 2.1. [22] *Let $\langle M, f\rangle$ be a unar. Then $Th(\langle M, f\rangle)$ is ω_1- categorical if and only if it is quasisimilar to the theory of infinite sets without any structure.*

In work of J. Ax [2] the concept of pseudofiniteness was first defined. The ground works obtained to date for pseudofinite structures directly depend on the results of J. Ax. The basic definitions of pseudofiniteness are the following:

Definition 4. [2] An infinite structure $\mathcal{M}$ of a fixed language L is *pseudofinite* if for all L-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called *pseudofinite*.

Many beautiful theorems in model theory of the 1950-60s were proved using the ultraproducts. Set theorists love ultraproducts since they give rise to elementary embeddings, a staple of large cardinal theory. J. Ax in [2] connects the notion of pseudofiniteness and the construction of ultraproducts.

Theorem 2.2. [2] *Fix a language L and an L-structure $\mathcal{M}$. Then the following are equivalent:*
1) an L-structure $\mathcal{M}$ is pseudofinite;
2) $\mathcal{M} \models T_f$, where T_f is the common theory of all finite L-structures;
3) $\mathcal{M}$ is elementarily equivalent to an ultraproduct of finite L-structures.

In classical logic, the following property is a straightforward corollary of pseudofiniteness.

Theorem 2.3. *Let $\mathcal{M}$ be a pseudofinite structure and $f : M^k \to M^k$ be a definable function. Then f is injective if and only if f is surjective.*

Pseudofinite fields are studied in [2], more detailed information can be found in [7]. Also a survey on pseudofinite groups is given by D. Macpherson in [20, 21]. The investigation of pseudofinite rings was started in [6, 1], more complicated pseudofinite rings are studied in the series of papers [24, 25, 26].

Definition 5. [33] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family $\mathcal{T}$*, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T.

Definition 6. [35] *A disjoint union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint systems $\mathcal{M}_n$ of pairwise disjoint predicate signatures $\Sigma_n, n \in \omega$, is a system of the signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} | n \in \omega\}$ with the support $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n that coincide with their interpretations in systems $\mathcal{M}_n$, $n \in \omega$.

A disjoint union of theories T_n, of pairwise disjoint predicate signatures Σ_n, respectively, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons Th(\bigsqcup_{n\in\omega} \mathcal{M}_n),$$

where $\mathcal{M}_n \models T_n, n \in \omega$.

Obviously, the $T_1 \sqcup T_2$ theory does not depend on choice of the disjunctive union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of models $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$.

3 An example of pseudofinite unar theory with arbitrary number of semichains and antichains

In this section, we give an example of a complete, pseudofinite graph theory T in a language $L = \{=, f^{(1)}\}$ with equality and a unary function $f^{(1)}$.

We immediately move on to listing the axioms of our theory. We present them in groups, sometimes providing corresponding semantic consequences. The groups of axioms are numbered in the order of their appearance in our presentation. We avoid formal writing of axioms in the language of first-order logic, limiting ourselves to their semantic description.

(1) The theory says that f is a unary function defined as in Definition 1. We will freely use the terms of unar theory (such as *k-branching vertice*, *distance between vertices*, *antichain*, *semichain* etc.) for the semantic description of the given axioms.

(2) For every natural number $i \geq 2$ there is an axiom φ_i, which says that in the models of the theory T there are no cycles of length i. This means that every connected component of a unar that

is a model of T will be acyclic. Moreover, for any two vertices from one connected component, the distance is defined as the length of the only chain connecting these vertices.

(3) There is one and only one vertex of valency ≥ 3, the remaining vertices have valency 1 or 0. Moreover, for each $i \geq 3$ there is an axiom δ_i, which says that if the valency of a vertex $\geq i$, then the degree of this vertex $\geq i+1$. As a consequence, in the model of T, there is exactly one vertex of infinite valency, all other vertices have degree 1 or 0. Note that a vertex of infinite valency is formulaically definable (as a vertex of valency ≥ 3), so we will call it a root vertex, and in the future we will use this name to describe other axioms.

(4) For every natural number $i \geq 0$ there is an axiom ψ_i, which says that there is a unique vertex of valency 0, the distance from which to the root vertex is i.

(5) There is one axiom of the theory that says that the chain connecting two vertices of valency 0 has length ≥ 3. For every natural number $i \geq 3$ there is an axiom ϵ_i that says that every chain of length i connecting two vertices of valency 1 passes through the root vertex.

We call the component containing the root vertex the *root component*. We represent the root component as acyclic unar with a root (of infinite valence) to which branches (i.e. maximal chains descending to the root vertex) descend. If the distance from a vertex of valence 0 to the root is finite, then the branch is a finite chain. In the case where the distance is infinite, then the model contains a semichain starting from the vertex of valence 0 and an antichain descending to the root.

Note that different branches do not have common vertices because the valences of non-root vertices are ≤ 1 and there are no finite cycles. Since each vertex of the root component is connected to the root by some finite chain, the vertices on all branches exhaust all non-root vertices of the root component.

The group of axioms (4) states that for each natural number $i \geq 1$ there is exactly one branch of length i. On the other hand, the model of our theory may contain an arbitrary number of infinite branches (antichains) in the root component or none at all: our axioms do not contain any restrictions on the number of infinite branches.

Now, we turn to the description of non-root components of the potential model of our theory T. In such components, all vertices have valence ≤ 1.

For convenience, we introduce the following definitions. The standard models $\langle \mathbb{N}; =, succ\rangle$ and $\langle \mathbb{Z}; =, succ\rangle$, where *succ* is the successor function, will be called the $\mathbb{N}$-model and $\mathbb{Z}$-model, respectively. A graph component will be called the $\mathbb{N}$-component and $\mathbb{Z}$-component if it is isomorphic to the $\mathbb{N}$-model and $\mathbb{Z}$-model, respectively.

Suppose that the non-root component of a model of the theory T contains $k \geq 2$ vertices of valence 0. We choose two vertices of valence 0 in the component; By the definition of connectivity, there is a finite chain connecting these two vertices, and obviously this chain does not contain the root vertex. We obtain a contradiction with the group of axioms (5).

Thus, the non-root component of a model of the theory T can either contain exactly one vertex of valence 0, then it will obviously be an $\mathbb{N}$-component, or not contain a vertex of valence 0 at all, then it will obviously be a $\mathbb{Z}$-component, since by the axiom group (2) there are no finite cycles in a unar.

Thus, we have practically given a description of all models of the theory T: it consists of a root component, where for each natural number $i \geq 1$ there are exactly one branch of length i and an arbitrary number of infinite branches, and an arbitrary number of $\mathbb{N}$-components and $\mathbb{Z}$-components.

Theorem 3.1. *Any model of theory T has an elementary extension with countably many $\mathbb{N}$-components, countably many $\mathbb{Z}$-components, and countably many infinite branches in the root component.*

Proof. First, we will supplement our language with the following notations and axioms in accordance with them.

1. Since the root vertex r has infinite valency, it means that an infinite number of branches descend to it. If we denote the branch number by $j \in \omega$, then the valency of the vertex will be $|f^{(-1)}(r)| = \omega$, where each previous vertex $a_{1j}(a_{0j} = r)$ which is in the corresponding branch j which in turn descends to r, is written in general form as a_{ij}, and the lengths of each branch are determined through the indices $i \in \omega, k \in Z$. Now, we introduce an extension of the theory T by adding the following axiom

$$f(a_{ij} = a_{kj} \Leftrightarrow j \wedge i - k = 1 \wedge (i > k))$$

2. The vertex b_{0j} has valency 0, and the vertex b_{ij} has valency equal to 1. The corresponding axiom will have the form, for $j \in \omega, i \in \omega, k \in Z, f(b_{ij}) = b_{kj} \Leftrightarrow j \wedge i - k = 1 \wedge (i > k)$.

3. The vertex c_{ij} has valency 1, and the axiom for $j \in \omega, i \in \omega, k \in Z, f(c_{ij}) = c_{kj} \Leftrightarrow j \wedge i - k = 1 \wedge (i > k)$.

It is clear that the chain $a_{ij}{}_{i \in \omega}$ is an infinite branch, the chain $b_{ij}{}_{i \in \omega}$ is the N-component, and the chain $c_{ij}{}_{i \in \mathbb{Z}}$ is the Z-component for each fixed $j \in \omega$ in any model of the extended theory. Let us assume that we are given some model $\langle U; f \rangle$ of the theory T. Let us further enrich our language by highlighting each element of U as a constant. As a result, our language is represented as $L \cup \{a_{ij}, b_{ij}, c_{kj}, f\}_{i,j \in \omega, k \in \mathbb{Z}, f \in U}$. It is enough for us that the extended theory is consistent with the elementary diagram $T_{\triangle} = Th(\langle U; f \rangle)$ of the model $\langle U; f \rangle$.

The latter is a consequence of the compactness theorem. Let T_0 be a finite set of sentences consisting of several sentences of the elementary diagram $T_{\triangle}$ and several axioms of the extension T given above. We can assume that these axioms (extensions) mention all the constants from the set $a_{ij}, b_{ij}, c_{ij}, 0 \leq i, j \leq n \in \omega$, and only them, and T_0 includes all axioms describing the mapping between these constants and their properties; we will show that these constants can be interpreted in the $\langle U; f \rangle$ so that all expansion axioms included in T_0 will be true.

As for the finite fragment $T_{\triangle}$, we can assume that they only mention elements from the final branches of the root component, since other elements do not interfere with us when interpreting the constants $a_{ij}, b_{ij}, c_{ij}, 0 \leq i, j \leq n$, in the model $\langle U; f \rangle$.

First, we select all finite branches of length $\leq m$ of the root component, covering all constants from $\{f : A \to A\}$ mentioned in sentences from T_0, and here we interpret each constant a_{ij}, b_{ij}, c_{kj} as an element U. Let $N = max\{m + 1, n + 1\}$. Next, we use n branches with lengths $N + 1, N + 2, \ldots, N + n$ to interpret the constants $a_{ij}, 0 \leq i, j \leq n$. After this, n branches with lengths $N+n+1, N+n+2, \ldots, N+2n$ are used to interpret the constants $b_{ij}, 0 \leq i, j \leq n$. Finally, branches with lengths $N + 2n + 1, N + 2n + 2, \ldots, N + 3n$ are used to interpret the constants $c_{ij}, 0 \leq i, j \leq n$. □

It is easy to see that an unar containing only finite branches in the root component is a prime model of the theory, and the model with countably many $\mathbb{N}$-components, countably many $\mathbb{Z}$-components and countably many infinite branches in the root component is a countably saturated model of the theory T.

Theorem 3.2. *The theory T is ω-stable.*

Proof. As we mentioned, let us consider a saturated model M of the theory T. In this model, each element belongs to one of three components:

Root component: Contains a single root of infinite valency and infinite branches.

$\mathbb{N}$-components: Chains isomorphic to $(\mathbb{N}, f)$.

$\mathbb{Z}$-components: Chains isomorphic to $(\mathbb{Z}, f)$. We will show that the number of element types in the entire model is at most countable.

Let us directly count the number of types. In the root component, we have:

A unique root, which is definable by a formula. Vertices on finite branches, which are distinguished by their distance from the root. These vertices are fully determined by their distance, meaning their types depend only on an integer. Vertices on infinite branches, whose position is determined by their relative distance from the root. Due to this strict structure, the number of distinct types here does not exceed ω.

In each $\mathbb{N}$-chain, the f acts as: $i \mapsto i+1$.

Elements differ only by their position in the chain, meaning the type of an element is determined by an integer (its distance from a reference point). Since a saturated model contains only countably many such elements, the number of types is at most ω.

In $\mathbb{Z}$-chains, the situation is similar: each element's position is determined by its distance from a fixed point. Thus, the number of types is again at most ω.

We have proved that T is ω-stable since, in every saturated expansion, the number of element types remains at most countable.

□

4 Concluding remarks

An elementary equivalence of pseudofinite unars with n semichains and m antichains, where $n \neq m$, remained an interesting question. In this paper, a pseudofinite theory of a unar with an arbitrary number of antichains and semichains is constructed. It is known that any complete theory of unars is superstable. It is proved that this theory is omega-stable. A prime model of this omega-stable theory is omega categorical, hence, smoothly approximable.

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