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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THIRD-KIND LINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper, there are studied third-kind linear Volterra integral equations with smooth data and the operator of multiplying by a smooth function that degenerates at the initial point of the integration interval. A theorem of existence, uniqueness, and continuity of a solution is proved. Conditions for smoothness and the degree of smoothness of a solution are obtained. Additionally, the existence and uniqueness of the solution in the L^p space are proved.

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1 Introduction

According to the theory of integral equations, the solution of first-kind linear Volterra integral equations exists and is unique in the space of continuous functions if the input data is smooth and the kernel $K(x,t)$ does not vanish on the diagonal $t = x$. In this case, it is possible to precisely determine the value of the desired function at the initial point of the segment, which is requried for the numerical solution of the equation. Naturally, the question arises whether similar results can be obtained for third-kind linear Volterra integral equations in the case of smooth known functions.

G.C. Evans, in [4, 5], thoroughly studied the solvability conditions for Volterra integral equations

$$a(x)u(x) + \int_0^x K(x,t)u(t)dt = f(x), \tag{1.1}$$

with the right-hand side $f(x) = a(x)g(x)$ and a singular kernel.

Since then, only a few works have been published addressing the existence of solutions for third-kind Volterra integral equations [1, 6, 8, 15, 19]. These works have investigated only special classes of third-kind equations with continuous data [1, 19] and weakly singular kernels [1, 6, 8, 15].

T. Sato [19] constructed the solution of nonlinear third-kind Volterra integral equations in the form of a power series when $a(x) = x$. T.F. Fényes [6] demonstrated the existence of a locally integrable solution to equation (1.1) with a convolution-type kernel and $a(x) = x + c, c < 0$. A.M. Nakhushev [15], using the apparatus of fractional differentiation, investigated integral equation (1.1) with a weakly singular kernel for $a(x) = x^\beta, 0 < \beta < 1$. P. Grandits [8] obtained conditions under which equation (1.1) with the special kernel $K(x,t) = 1 + \Gamma(x,t)$ has a unique continuous solution. S.S. Allaei, Z.-W. Yang, and H. Brunner [1], using the properties of cordial Volterra integral operators, proved the existence of a unique continuous solution to equation (1.1) with a continuous (or weakly singular) kernel for $a(x) = x^\beta, \beta > 0$. A multiparameter family of solutions to equation (1.1) in Banach spaces of a special type was obtained by N.A. Magnitskii [14].

G.C. Evans's method [4] has been further developed in the theory of regularization of third-kind Volterra integral equations. In the works of A. Asanov [2], M.I. Imanaliev and A. Asanov [10], and T.T. Karakeev [12], issues of regularizing the solution of equation (1.1) in the space of continuous functions, when $a(x)$ is monotonic (in the scalar case), have been investigated. T.D. Omurov [16] and S.B. Tagaeva [20] demonstrated the applicability of the regularization method in the space of summable functions for (1.1) with a non-decreasing function $a(x)$. The regularization of equation (1.1) with a weakly singular kernel is addressed in the work of S.V. Pereverzev and S.A. Prössdorf [18]. S. Iskandarov [11] studied the conditions for the uniqueness of a solution of equation (1.1) on the half-line.

In this work, we will prove the existence and uniqueness of a solution to integral equations (1.1) both in the class of continuous functions and in the space $L_p(0,b)$. We will define the smoothness order of a solution.

2 Resolving equation

G.C. Evans [4] proved the existence of a unique bounded solution to a second-kind Volterra integral equation with a non-integrable order kernel. The study of such equations was continued by L.I. Panov [17], who demonstrated the unique solvability of the equation in a special Banach space of continuous functions. In this section, we will consider a special class of second-kind Volterra integral equations, which can be categorized as the type of equations studied in [4, 17]. The results of this section will be used in subsequent sections.

Let $G(x), a(x), f(x)$ be known functions on the segment $[0,b]$ and a function $Q(x,t)$ be defined in the domain $D := \{(x,t) : 0 \le t \le x \le b\}$. Assume that the following condition holds:

(A) $\quad a(0) = 0, \int\limits_0^x \frac{dt}{a(t)} = +\infty, a(x) > 0, \forall x \in (0,b]$.

Consider the integral equation

$$v(x) = \int\limits_0^x exp\left(-\int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(t)}{a(t)}\left\{\int\limits_0^t Q(t,s)v(s)ds + f(t)\right\}dt. \tag{2.1}$$

By $C^{n,0}(D)$ we denote the space of continuous functions $z(x,t)$ on D that have continuous derivatives $\frac{\partial^n z}{\partial x^n}$, and functions $\frac{\partial d^i z}{\partial dx^i}(x,x), i = 0,1,...,n-1$, are differentiable on $[0,b]$ up to order $n-i-1$ inclusively. $C^{n,1}(D)$ is the space of continuous functions $w(x,t)$ on D that have continuous derivatives $\frac{\partial w}{\partial t}, \frac{\partial^n w}{\partial x^n}$ and $\frac{\partial w}{\partial t} \in C^{n,0}(D)$.

Theorem 2.1. *Let functions $a(x), f(x)$, and $G(x)$ be continuous on $[0,b]$, a function $Q(x,t)$ be continuous on the domain D, and the function $a(x)$ satisfy condition (A), and $G(x) > 0$. Then, equation* (2.1) *has a unique solution $v(x) \in C[0,b]$ with $v(0) = f(0)$. If $a(x), f(x) \in C^n[0,b], G(x) \in C^{n-1}[0,b], Q(x,t) \in C^{n,0}(D)$, and the following condition holds*

$$G_m(x) = G(x) + ma'(x) \ge d_1 > 0, m = 0,1,...,n, \tag{2.2}$$

then the solution $v(x) \in C^n[0,b]$.

Proof. Let $Q_1 = max\{|Q(x,t)|, (x,t) \in D\}, N_0 = max\{|f(x)|, x \in [0,b]\}$. From (2.1), using Dirichlet's rule for the repeated integral, we obtain

$$v(x) = \int\limits_0^x L(x,t)v(t)dt + F(x),$$

where

$$L(x,t)=\int\limits_t^x exp\biggl(-\int\limits_s^x \frac{G(\tau)d\tau}{a(\tau)}\biggr)\frac{G(s)}{a(s)}Q(s,t)ds,$$

$$F(x)=\int\limits_0^x exp\biggl(-\int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)}\biggr)\frac{G(t)}{a(t)}f(t)dt.$$

Since, according to (A) and (2.2),

$$0\le exp\biggl(-\int\limits_0^x \frac{G(\tau)d\tau}{a(\tau)}\biggr)\le exp\biggl(-d_1\int\limits_0^x \frac{d\tau}{a(\tau)}\biggr)=0,$$

we have

$$exp\biggl(-\int\limits_0^x \frac{G(\tau)d\tau}{a(\tau)}\biggr)=0.$$

Thus, for any $x\in(0,b]$

$$\int\limits_0^x \frac{G(\tau)d\tau}{a(\tau)}=+\infty.$$

Let $\xi=\int\limits_x^b \frac{G(\tau)d\tau}{a(\tau)}, \eta=\int\limits_t^b \frac{G(\tau)d\tau}{a(\tau)}$. At $t=x$, we have $\eta=\xi$. If t=0, then $\eta=+\infty$. Then [4, Section 6]

$$\int\limits_0^x exp\biggl(\int\limits_x^b \frac{G(\tau)d\tau}{a(\tau)}-\int\limits_t^b \frac{G(\tau)d\tau}{a(\tau)}\biggr)\frac{G(t)}{a(t)}dt=\int\limits_\xi^\infty exp(\xi-\eta)d\eta=1$$

and

$$\int\limits_t^x exp\biggl(-\int\limits_s^x \frac{G(\tau)d\tau}{a(\tau)}\biggr)\frac{G(s)}{a(s)}ds=\int\limits_\xi^\eta exp(\xi-\sigma)d\sigma=1-exp\biggl(-\int\limits_t^x \frac{G(\tau)d\tau}{a(\tau)}\biggr),$$

where $\sigma=\eta$ at $s=t$. Due to this, the following estimates hold

$$|F(x)|\le N_0, \quad |L(x,t)|\le Q_1,$$

and

$$|v(x)|\le Q_1\int\limits_0^x |v(t)|dt+N_0. \tag{2.3}$$

If a function $g(x)$ is integrable on $[0,b]$, $g(x)\ge 0$, and a function $f(x)$ is continuous on $[0,b]$, then the theory of definite integrals allows the application of the mean value theorem in a generalized form [9, p. 324]: for some $x_1\in[0,b]$

$$\int\limits_a^b f(t)g(t)dt=f(x_1)\int\limits_a^b g(t)dt.$$

Due to this formula and condition (A) we have that for some $\bar{x} \in [0, x_0]$

$$F(x_0) = f(\bar{x}) \int_0^{x_0} exp\left(-\int_t^{x_0} \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(t)}{a(t)}dt = f(\bar{x}).$$

where x_0 is an arbitrary point from $[0, b]$. From this, when $x_0 \to 0$ it follows that $F(0) = f(0)$. So the kernel $L(x,t)$ degenerates on the diagonal $t = x : L(x,x) = 0$.

For the function $F(x)$ and the kernel $L(x,t)$, it is easily established by standard methods that the increments $\Delta F \to 0$ as $\Delta x \to 0$ and $\Delta L \to 0$ as $\Delta x \to 0, \Delta t \to 0$.

From the above and estimate (2.3), it follows that the function $F(x)$ and the kernel $L(x,t)$ are continuous, respectively, in the regions $[0,b]$ and D. According to the theory of Volterra integral equations of the second kind [3, p. 5], there exists a unique solution $v(x)$ of equation (2.1) in C[0,b] and $v(0) = f(0)$.

Let $a(x), f(x) \in C^n[0,b], G(x) \in C^{n-1}[0,b], Q(x,t) \in C^{n,0}(D)$, and condition (2.2) be satisfied. From (A) and (2.2) it follows that

$$exp\left(-\int_0^x \frac{G_m(\tau)d\tau}{a(\tau)}\right) = 0, m = 0, 1, ..., n. \tag{2.4}$$

By integrating by parts, due to (2.4), we get

$$\int_0^x exp\left(-\int_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(t)}{a(t)}f(t)dt = f(x) - \int_0^x exp\left(-\int_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)f'(t)dt$$

and

$$\int_0^x exp\left(-\int_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)\frac{G(t)}{a(t)}\int_0^t Q(t,s)v(s)dsdt$$

$$= \int_0^x Q(x,s)v(s)ds - \int_0^x exp\left(-\int_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)c(t)dt,$$

where $c(x) = Q(x,x)v(x) + \int_0^x Q_x(x,s)v(s)ds$.

We will demonstrate the existence of a continuous derivative for the solution of equation (2.1). Since

$$exp\left(-\int_t^x \frac{a'(\tau)d\tau}{a(\tau)}\right) = \frac{a(t)}{a(x)},$$

from condition (2.4) it follows that the function

$$w(x) = \int_0^x exp\left(-\int_t^x \frac{G(\tau)d\tau}{a(\tau)}\right)c_0(t)dt,$$

is continuously differentiable and

$$w'(x) = c_0(x) - G(x)\int_0^x exp\left(-\int_t^x \frac{G_1(\tau)d\tau}{a(\tau)}\right)\frac{c_0(t)}{a(t)}dt.$$

where $c_0(x) = c(x) + f'(x)$.

Thus, the solution of equation (2.1) has a continuous derivative

$$v'(x) = c_0(x) - w'(x) = G(x) \int_0^x exp\left(- \int_t^x \frac{G_1(\tau)d\tau}{a(\tau)} \right) \frac{c_0(t)}{a(t)} dt. \tag{2.5}$$

Let x_0 be an arbitrary point from $[0, b]$. For $v'(x_0)$, using the generalized mean value theorem, we obtain

$$v'(x_0) = G(x_0) \int_0^{x_0} exp\left(- \int_t^{x_0} \frac{G_1(\tau)d\tau}{a(\tau)} \right) \frac{G_1(t)}{a(t)} c_1(t)dt = G(x_0) \frac{c_0(\zeta)}{G_1(\zeta)},$$

for some $0 \le \zeta \le x_0$, where $c_1(x) = \frac{c_0(x)}{G_1(x)}$. From this, when $x_0 \to 0$

$$v'(0) = G(0) \frac{c_0(0)}{G_1(0)} = G(0) \frac{Q(0,0)f(0) + f'(0)}{G_1(0)}. \tag{2.6}$$

In a similar manner, for any continuous function $c_0(x)$, one can derive the following representation:

$$w(x_0) = \frac{c_0(\zeta)}{G_1(\zeta)} a(\zeta), 0 \le \zeta \le x_0.$$

Therefore, $w(0) = 0$, since $a(0) = 0$. This fact will be used repeatedly in the following computations.

We introduce the notations

$$c_1(x) = \frac{c_0(x)}{G_1(x)}, \quad c_m(x) = \frac{c'_{m-1}(x)}{G_m(x)}, m = 2, 3, ..., n,$$

$$W_m(x,t) = exp\left(- \int_t^x \frac{G_m(\tau)d\tau}{a(\tau)} \right), m = 0, 1, ..., n,$$

$$A_1(x) = \int_0^x \frac{c_0(t)}{a(t)} W_1(x,t)dt, \quad A_m(x) = \int_0^x \frac{c'_{m-1}(t)}{a(t)} W_m(x,t)dt, m = 2, 3, ..., n. \tag{2.7}$$

We write (2.5) in the form

$$v'(x) = G(x)A_1(x). \tag{2.8}$$

The function

$$A_1(x) = \int_0^x c_1(t) W_1(x,t) \frac{G_1(t)}{a(t)} dt = c_1(x) - \int_0^x W_1(x,t) c'_1(t)dt$$

is continuous on $[0, b]$ and $A_1(0) = c_1(0)$. According to condition (2.2), the following estimate holds

$$|A_1(x)| \le d_1^{-1} \|c_0(x)\|_\infty \int_0^x W_1(x,t) \frac{G_1(t)}{a(t)} dt \le d_1^{-1} \|c_0(x)\|_\infty .$$

The function $A_1(x)$ is continuously differentiable and

$$A'_1(x) = \frac{G_1(x)}{a(x)} \int_0^x W_1(x,t) c'_1(t)dt = G_1(x) \int_0^x W_2(x,t) \frac{c'_1(t)}{a(t)} dt = G_1(x) A_2(x).$$

Since

$$A_2(x) = \int_0^x c_2(t)W_2(x,t)\frac{G_2(t)}{a(t)}dt = c_2(x) - \int_0^x W_2(x,t)c_2'(t)dt,$$

we get

$$A_1'(0) = G_1(0)c_2(0), \quad |A_1'(x)| \le d_1^{-1}\|G_1(x)\|_\infty \|c_1'(x)\|_\infty .$$

Then, there exists a continuous second derivative on $[0,b]$ and

$$v''(x) = G'(x)A_1(x) + G(x)A_1'(x). \tag{2.9}$$

Note that formula (2.8), for determining $v'(x)$, contains the first derivatives of $f'(x), a'(x)$ and $Q_x(x,t)$, while in formula (2.9) for $v''(x)$, the second derivatives $f''(x)$, $a''(x)$, $Q_{xx}(x,t)$ and the first derivatives of the functions $Q(x,x), v(x), G(x)$, also the derivative $Q_x(x,x)$ are present.

By differentiating $n-1$ times from (2.8), we get

$$v^{(n)}(x) = (G(x)A_1(x))^{(n-1)}.$$

We will show that $A_1(x)$ has continuous derivatives of order $n-1$. Above, it was shown that the function $A_1(x)$ has a first derivative. Let us differentiate $A_2(x)$.

$$A_2'(x) = \frac{G_2(x)}{a(x)}\int_0^x W_2(x,t)c_2'(t)dt = G_2(x)\int_0^x W_3(x,t)\frac{c_2'(t)}{a(t)}dt = G_2(x)A_3(x)$$

and

$$A_2'(0) = G_2(0)c_3(0), \quad |A_2'(x)| \le d_1^{-1}\|G_2(x)\|_\infty \|c_2'(x)\|_\infty .$$

Next, the second derivative exists:

$$A_1''(x) = (G_1(x)A_2(x)))' = G_1'(x)A_2(x) + G_1(x)A_2'(x).$$

The third-order derivative $A_1^{(3)}(x)$ is given by the formulas:

$$A_1^{(3)}(x) = (G_1(x)A_2(x)))'' = G_1''(x)A_2(x) + 2G_1'(x)A_2'(x) + G_1(x)A_2''(x),$$

$$A_2''(x) = (G_2(x)A_3(x)))' = G_2'(x)A_3(x) + G_2(x)A_3'(x).$$

Here, we need to show the differentiability of

$$A_3(x) = \int_0^x W_3(x,t)\frac{c_2'(t)}{a(t)}dt.$$

All other functions on the right-hand sides of these equations are defined and continuous on $[0,b]$. By virtue of (2.2), we have

$$A_3'(x) = G_3(x)\int_0^x W_4(x,t)\frac{c_3'(t)}{a(t)}dt = G_3(x)A_4(x)$$

and

$$A_3'(0) = G_3(0)c_4(0), \quad |A_3'(x)| \le d_1^{-1}\|G_3(x)\|_\infty \|c_3'(x)\|_\infty .$$

Using the method of induction, it is easy to establish that $A_1^{(n-1)}(x)$ is determined by the following system of formulas:

$$A_1^{(n-1)}(x) = (G_1(x)A_2(x))^{(n-2)}, A_2^{(n-2)}(x) = (G_2(x)A_3(x))^{(n-3)},$$

$$..., A_{n-1}^{(1)}(x) = G_{n-1}(x)A_n(x).$$

At each step m, we need to find $A_m'(x)$, for which the following formulas hold:

$$A_m'(x) = G_m(x)A_{m+1}(x), \quad A_m'(0) = G_m(0)c_{m+1}(0), m = 1, 2, ..., n-1$$

and

$$|A_m'(x)| \leq d_1^{-1} \|G_m(x)\|_\infty \|c_m'(x)\|_\infty, m = 1, 2, ..., n-1.$$

The function $c_{n-1}'(x)$ in the equality

$$A_n(x) = \int_0^x \frac{c_{n-1}'(t)}{a(t)} W_n(x,t)dt,$$

contains $a^{(n)}(x), f^{(n)}(x), v^{(n-1)}(x), Q_x^{(n)}(x,t)$ and lower-order derivatives, and

$$A_n(0) = c_n(0), \quad |A_n(x)| \leq d_1^{-1} \left\|c_{n-1}'(x)\right\|_\infty.$$

Thus, the function $A_1(x)$ is continuously differentiable $n-1$ times. Then, by Leibniz's rule it follows that the function $v(x)$ is continuously differentiable up to order n inclusively. Therefore, the solution $v(x)$ of equation (2.1) belongs to $C^n[0,b]$. □

Definition 1. We will call equation (2.1) the resolving equation for equation (1.1) if $G(x) = K(x,x)$ and $Q(x,t) = \frac{\partial}{\partial t}\left(\frac{K(x,t)}{G(t)}\right)$.

3 Theorem of existence and uniqueness of regular solutions

Theorem 2.1 serves as the basis for establishing the conditions of existence, uniqueness, and continuity of the solution to the linear Volterra integral equation of the third kind (1.1), where $a(x), K(x,t)$ and $f(x)$ are known functions, and $u(x)$ is the desired function.

V. Volterra [21, pp. 97-98] demonstrated the possibility of solving Volterra integral equations of the first kind in two ways: by differentiating the equation and by integrating by parts with the introduction of a new desired function through an integral substitution. In both cases, we obtain a Volterra integral equation of the second kind. The first method has found wide application, including in the construction of numerical solutions. The second method involves finding the derivative of the new desired function and is rarely used. We are particularly interested in the second method, the generalization of which proves to be useful for investigating the existence, uniqueness, and continuity of the solution to Volterra integral equations of the third kind. Moreover, it is possible to identify the conditions that determine the order of smoothness of the solution.

For equation (1.1), condition (2.2) takes the form

$$G_m(x) = K(x,x) + ma'(x) \geq d_1 > 0, m = 0, 1, ..., n. \tag{3.1}$$

Theorem 3.1. *Let* $a(x), f(x) \in C^1[0,b], K(x,t) \in C^{1,1}(D), f(0) = 0$ *and let conditions (A), and* (3.1) *(for m=0,1) hold. Then equation* (1.1) *has a unique solution* $u(x) \in C[0,b]$*, which at x=0 takes the value*

$$u(0) = \frac{f'(0)}{K(0,0) + a'(0)} \tag{3.2}$$

and the following estimate holds:

$$\|u(x)\|_\infty \le N_1 \|f(x)\|_{C^1},$$

where N_1 is a positive constant, $\|\cdot\|_{C^1}$ is the norm in the space $C^1[0,b]$. Furthermore, if $a(x), f(x) \in C^n[0,b], K(x,t) \in C^{n,1}(D)$ and condition (3.1) holds, then $u(x) \in C^{n-1}[0,b]$.

Proof. Let us rewrite equation (1.1) as

$$a(x)u(x) + \int_0^x G(t)u(t)dt = \int_0^x Q(x,t)\int_0^t G(s)u(s)dsdt + f(x), \tag{3.3}$$

where $G(x) = K(x,x), Q(x,t) = \frac{\partial}{\partial t}\left(\frac{K(x,t)}{G(t)}\right)$. We introduce a new unknown function by substituting

$$\int_0^x G(t)u(t)dt = v(x). \tag{3.4}$$

Then, from (1.1), we obtain the integro-differential equation

$$a(x)v'(x) + G(x)v(x) = \int_0^x G(x)Q(x,t)v(t)dt + G(x)f(x), \tag{3.5}$$

with the initial condition $v(0) = 0$. We reduce this zero initial value problem to the integral equation of the followig form [5, Section 11]:

$$v(x) = exp\left(\int_x^b \frac{G(\tau)}{a(\tau)}d\tau\right)\int_0^x exp\left(-\int_t^b \frac{G(\tau)}{a(\tau)}d\tau\right)\frac{G(t)}{a(t)}\left[\int_0^t Q(t,s)v(s)ds + f(t)\right]dt,$$

which can be represented in form (2.1), where $G(x)$ and $Q(x,t)$ are defined according to (3.3). Any solution of integro-differential equation (3.5) with the initial condition $v(0) = 0$ is a solution of integral equation (2.1) and vice versa, because at $x = 0$ the solution of equation (2.1) takes the value $v(0) = f(0)$ and, by the assumptions of the theorem, $f(0) = 0$. Consequently, solving the zero Cauchy problem for (3.5) is equivalent to solving resolving integral equation (2.1). From the conditions imposed on the function $K(x,t)$, it follows that $Q(x,t) \in C^{1,0}(D)$. By virtue of Theorem 2.1, resolving equation (2.1) has a unique solution $v(x) \in C^1[0,b]$. Then, from (3.4), we find the unique solution of equation (1.1)

$$u(x) = \frac{v'(x)}{G(x)}, \tag{3.6}$$

which belongs to the space $C[0,b]$. At $x = 0$, from (3.6) and (2.6), we obtain (3.2). The estimate of this theorem directly follows from (3.6), (2.3), and (2.5).

Suppose $a(x), f(x) \in C^n[0,b], K(x,t) \in C^{n,1}(D)$ and condition (3.1) is satisfied. Then, according to Theorem 2.1, the solution of resolving equation (2.1) $v(x) \in C^n[0,b]$, and from (3.6) it follows that the solution of equation (1.1) $u(x) \in C^{n-1}[0,b]$. □

Formula (3.2) is well consistent with the theory of first-kind Volterra integral equations. If $a(x) \equiv 0$, then from (1.1) we obtain a linear first-kind Volterra integral equation. When the kernel and the right-hand side of the first-kind equation are smooth and $K(x,x) > 0$, it has a unique continuous solution $u(x)$ and

$$u(0) = \frac{f'(0)}{K(0,0)},$$

which we also obtain from (3.2) when $a(x) \equiv 0$.

Remark 1. Under the assumptions of Teorem 3.1 estimate (2.3), it follows that the solution of equation (1.1) can be constructed by the method of successive approximations according to the rule:

$$u_{n+1}(x) = \int_0^x \frac{W_1(x,t)}{a(t)} \left\{ Q(t,t)v_{n+1}(t) + \int_0^t Q_x(t,s)v_{n+1}(s)ds + f'(t) \right\} dt,$$

$$v_{n+1}(x) = \int_0^x W_0(x,t)\frac{G(t)}{a(t)} \left\{ \int_0^t Q(t,s)v_n(s)ds + f(t) \right\} dt, n = 0,1,....$$

Example 1. Let $K(x,t) = x - t + 1/2, 0 \le t \le x \le 1$, $f(x) = 3x/2 + 7x^2/4 + x^3/6$, $a(x) = x$, $0 \le x \le 1$. Then, $G(x) = 1/2, Q(x,t) = -2$, $F(x) = x/2 + 7x^2/20 + x^3/42$.
If we choose the initial approximation $v_0(x) = F(x)$ and apply the method of successive approximations to (2.1), then

$$v_n(x) = x/2 + x^2/4 + (-1)^n(\beta_n/105 + x\gamma_n/42)x^{n+2}, n = 1,2,...,$$

where $\beta_1 = 1, \beta_n = 2^{n-1}\prod_{m=4}^{n+2} \frac{1}{m(2m+1)}$ $(n = 2,3,...)$, $\gamma_n = \frac{2\beta_n}{(n+3)(2n+7)}$ $(n = 1,2,3,...)$.

Then, $v(x) = \lim_{n\to\infty} v_n(x) = x/2 + x^2/4$ and the solution of integral equation (1.1), according to (3.6), is $u(x) = 1 + x$ and $u(0) = \frac{f'(0)}{K(0,0)+a'(0)} = \frac{3/2}{1/2+1} = 1$.

4 Existence and uniqueness of a solution in $L^p(0,b)$

The smoothness properties of the function $a(x)$, the kernel $K(x,t)$, and the right-hand side $f(x)$ of equation (1.1) allowed us to prove the existence and uniqueness of a continuous solution to equation (1.1). If we discard some of these conditions, equation (1.1) may not have a solution in $C[0,b]$.

Let a function $f(x)$ be continuous on $[0,b]$ and $f'(x) \in L^p(0,b), p \ge 1$. Then, the fulfillment of conditions (A) and (3.1) (for m=0) does not imply that equation (1.1) has a solution in the space of continuous functions. In this case, it is reasonable to investigate the solvability of equation (1.1) in the space $L^p(0,b)$. In this case, by a solution of equation (1.1), we mean a function delonging to $L^p(0,b)$ which, when substituted into (1.1), transforms the equation into an identity in the integral sense

Let the function $a(x)$ be continuously differentiable on $[0,b]$ and satisfy condition (A), and let $K(x,t) \in C^{1,1}(D)$. We require the fulfillment of the condition

$$k_m G_0(x) + ma'(x) \ge d_{m+1}, m = 0,1, k_1 = min(k_0, \frac{q}{2}), k_0 = 1, \quad (4.1)$$

where d_1, d_2 are positive constants, $1/p + 1/q = 1, p \ge 1$, and $G_0(x) = K(x,x)$.

If $f(0) = 0, f(x) \in C^\gamma[0,b], 0 < \gamma \le 1$, then equation (2.1), where $G(x)$ and $Q(x,t)$ are defined according to (3.3), is the resolving equation for equation (1.1). By virtue of the continuity $f(x)$ and $Q(x,t) = \partial/\partial t(K(x,t)/G(t))$, respectively on $[0,b]$ and in D, according to Theorem 2.1, resolving equation (2.1) has a unique continuous solution $v(x)$. From (3.4) and (2.1) we find the solution of equation (1.1)

$$u(x) = F_0(x) + \int_0^x W_1(x,t)\frac{Q(t,t)}{a(t)}v(t)dt + \int_0^x v(t)\int_t^x W_1(x,s)\frac{Q_x(s,t)}{a(s)}dsdt, \quad (4.2)$$

where

$$F_0(x) = \frac{f(x)}{a(x)} - \frac{1}{a(x)}\int_0^x W_0(x,t)\frac{G_0(t)}{a(t)}f(t)dt. \quad (4.3)$$

Lemma 4.1. *Let a function $f(x) \in C[0,b]$ and $f'(x) \in L^p(0,b)$, $p \geq 1$. If $f(0) = 0$ and conditions (A) and (4.1) are satisfied, then the following estimate holds:*

$$\|F_0(x)\|_{L^p} \leq \Big(d_2\Big)^{-\frac{1}{q}} \left(\frac{pd_1}{2}\right)^{-\frac{1}{p}} ||f'(x)||_{L^p}. \tag{4.4}$$

Proof. Integrating by parts, from (4.3) we get:

$$\|F_0(x)\|_{L^p}^p = \int_0^b \left| \frac{f(x)}{a(x)} - \frac{1}{a(x)} \int_0^x exp\left(- \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) \frac{G_0(t)}{a(t)} f(t)dt \right|^p dx$$

$$= \int_0^b \left| \frac{1}{a(x)} \int_0^x exp\left(- \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) f'(t)dt \right|^p dx.$$

According to Hölder's inequality, we have:

$$\left(\int_0^x exp\left(- \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) |f'(t)|dt \right)^p \leq \left(\int_0^x exp\left(-\frac{q}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) dt \right)^{\frac{p}{q}}$$

$$\times \int_0^x exp\left(-\frac{p}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) |f'(t)|^p dt.$$

Then

$$\|F_0(x)\|_{L^p}^p = \int_0^b \left| \frac{1}{a(x)} \int_0^x exp\left(- \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) f'(t)dt \right|^p dx$$

$$\leq \int_0^b \left(\frac{1}{a(x)} \right)^p \left(\int_0^x exp\left(-\frac{q}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) dt \right)^{\frac{p}{q}} \int_0^x exp\left(-\frac{p}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) |f'(t)|^p dtdx.$$

Since

$$\left(\frac{1}{a(x)} \right)^p \left(\int_0^x exp\left(-\frac{q}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) dt \right)^{\frac{p}{q}} = \frac{1}{a(x)} \left(\int_0^x exp\left(-\frac{q}{2} \int_t^x \frac{G_{01}(\tau)}{a(\tau)} d\tau \right) \frac{dt}{a(t)} \right)^{\frac{p}{q}},$$

where $G_{01}(x) = \frac{q}{2} G_0(x) + a'(x)$, then

$$\|F_0(x)\|_{L^p}^p \leq (d_2)^{-\frac{p}{q}} \int_0^b \frac{1}{a(x)} \int_0^x exp\left(-\frac{p}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) |f'(t)|^p dtdx.$$

Using Dirichlet's rule, we get the estimate

$$\|F_0(x)\|_{L^p}^p \leq (d_2)^{-\frac{p}{q}} \int_0^b |f'(t)|^p \int_t^b exp\left(-\frac{p}{2} \int_t^x \frac{G_0(\tau)d\tau}{a(\tau)} \right) \frac{1}{a(x)} dxdt$$

$$\leq (d_2)^{-\frac{p}{q}} \frac{2}{pd_1} \int_0^b |f'(t)|^p dt.$$

From this, estimate (4.4) follows. □

Since

$$\left|\int_0^x W_1(x,t)\frac{Q(t,t)}{a(t)}v(t)dt\right| \le Q_1 d_2^{-1}||v(x)||_\infty$$

and

$$\left|\int_0^x v(t)\int_t^x W_1(x,s)\frac{Q_x(s,t)}{a(s)}dsdt\right| \le Q_2 b d_2^{-1}||v(x)||_\infty,$$

where $Q_{i+1} = max\{|Q_x^{(i)}(x,t)|, (x,t) \in D\}, i = 0,1$, by virtue of Minkowski's inequality and estimate (4.4), from (4.2) we arrive at the following estimate:

$$\|u(x)\|_{L^p} \le \left(d_2\right)^{-\frac{1}{q}}\left(\frac{pd_1}{2}\right)^{-\frac{1}{p}}\|f'(x)\|_{L^p} + (Q_1 + Q_2 b)b^{\frac{1}{p}}d_2^{-1}\|v(x)\|_\infty.$$

The uniqueness of a solution to resolving equation (2.1) and this estimate guarantee the uniqueness of the solution to equation (1.1) in $L^p(0,b)$.

Theorem 4.1. *Let $a(x) \in C^1[0,b], K(x,t) \in C^{1,1}(D)$, and a function f(x) be continuous on $[0,b]$, with $f(0) = 0$ and $f'(x) \in L^p(0,b)$. Suppose that conditions (A) and (4.1) are satisfied. Then, there exists a unique solution to equation (1.1) in $L^p(0,b)$, and the following estimate holds:*

$$\|u(x)\|_{L^p} \le \left(d_2\right)^{-\frac{1}{q}}\left(\frac{pd_1}{2}\right)^{-\frac{1}{p}}\|f'(x)\|_{L^p} + M_2\|v(x)\|_\infty,$$

where $M_2 = (Q_1 + Q_2 b)b^{\frac{1}{p}}d_2^{-1}$.

Example 2. Let the functions $a(x)$ and $K(x,t)$ be the same as in Example 1, and the function $f(x)$ be given by

$$f(x) = (7/4 + 9x/10)\sqrt[3]{x^2}, 0 \le x \le 1.$$

Then,

$$F(x) = \left(\frac{3}{4} + \frac{27}{130}x\right)\sqrt[3]{x^2},$$

and $G_0(x) = \frac{1}{2}, \quad Q(x,t) = -2$. If we choose the initial approximation $v_0(x) = F(x)$ and apply the method of successive approximations to (2.1), then we get:

$$v_1(x) = \left(\frac{3}{4} - \frac{243}{65}\beta_1 x^2\right)\sqrt[3]{x^2}, ...,$$

$$v_n(x) = \left(\frac{3}{4} + (-1)^n\frac{3^{2n+3}}{65}2^{n-1}\beta_n x^{n+1}\right)\sqrt[3]{x^2}, ...,$$

where $\beta_n = \prod_{m=1}^n \frac{1}{(3m+5)(6m+13)}, (n = 1,2,3,...)$. Hence, $v(x) = \lim_{n\to\infty} v_n(x) = \frac{3}{4}\sqrt[3]{x^2}$. Then, the solution of integral equation (1.1), according to (3.6), is $u(x) = \frac{1}{\sqrt[3]{x}}$.

5 Conclusions

In regularization theory, as well as in the numerical solution of Volterra integral equations of the third kind, one of the main difficulties lies in the fact that the value $u(0)$ is unknown. Therefore, an approximate value u_δ is often used instead of $u(0)$ [13], which is not always easy to determine.

Theorem 3.1 for equation (1.1) with smooth data eliminates this problem since the exact value of $u(0)$ is known and is determined by formula (3.2). Moreover, Theorem 3.1 can serve as a theoretical basis for constructing a regularizing operator and numerically solving equation (1.1) not only for a monotonic function $a(x)$ [7], but also for a broader class of functions $a(x)$.

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