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ON COINCIDENCE POINTS OF MAPPINGS ON COMPACT DOMAINS

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Abstract. In the paper, we study coincidence points of two mappings defined on a compact metric space. We assume that the first mapping satisfies the covering condition with a constant $\alpha > 0$ and the second mapping satisfies the strict Lipschitz inequality with the same constant α. We prove that under certain continuity assumptions these two mappings have a coincidence point. An analogous result on the existence of a coincidence point and a generalized coincidence point of set-valued mappings is obtained.

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1 Introduction

Given metric spaces (X, ρ_X) and (Y, ρ_Y), denote by $B_X(x_0, r)$ the closed ball centered at point $x_0 \in X$ with radius $r \geq 0$ in the space X. An analogous notation we will use for closed balls in the space Y. Let $\psi, \varphi : X \to Y$ be given mappings.

A point $\xi \in X$ is said to be a coincidence point of the mappings ψ and φ if $\psi(\xi) = \varphi(\xi)$. In [1], there was developed the coincidence point theory. It was shown, that under natural continuity and completeness assumptions, if the mapping ψ satisfies a certain covering assumption and the mapping φ is Lipschitz with a sufficiently small Lipschitz constant, then there exists a coincidence point of the mappings ψ and φ. Analogous results were derived for set-valued mappings. Let us recall the concepts in use.

Given a real number $\alpha > 0$, the mapping ψ is said to be α-covering if the following inclusion takes place

$$B_Y(\psi(x_0), \alpha r) \subset \psi(B_X(x_0, r)) \quad \forall\, x_0 \in X, \quad \forall\, r \geq 0.$$

Given a real $\beta > 0$, the mapping φ is said to be β-Lipschitz if the following inequality takes place

$$\rho_Y(\varphi(x_1), \varphi(x_2)) \leq \beta \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X.$$

The mapping ψ is said to be closed if its graph

$$\operatorname{gph} \psi := \{(x, \psi(x)) :\, x \in X\}$$

is a closed subset of the metric space $X \times Y$ endowed with the metric defined by formula

$$\rho((x_1, y_1), (x_2, y_2)) := \rho_X(x_1, x_2) + \rho_Y(y_1, y_2), \quad (x_1, y_1),\, (x_2, y_2) \in X \times Y.$$

Note that there could be defined other equivalent metrics in the spaces $X \times Y$. However, the one defined above is more convenient in the subsequent.

Recall now the coincidence point theorem from [1]. *Assume that the metric space* (X, ρ_X) *is complete, the mapping* $\psi : X \to Y$ *is* α*-covering and closed, the mapping* $\varphi : X \to Y$ *is* β*-Lipschitzian and* $\alpha > \beta > 0$. *Then for every point* $x_0 \in X$ *there exists a point* $\xi = \xi(x_0) \in X$ *such that*

$$\psi(\xi) = \varphi(\xi) \quad \text{and} \quad \rho_X(x_0, \xi) \leq \frac{\rho_Y(\psi(x_0), \varphi(x_0))}{\alpha - \beta}.$$

In the particular case, in which the space (Y, ρ_Y) coincides with the complete space (X, ρ_X) and ψ is the identity mapping, this coincidence point theorem implies the fixed point theorem by S. Banach and R. Caccioppoli (see, for example, [5, Chapter 1, §1]).

In this paper, we consider the coincidence point problem for the specific case $\alpha = \beta$.

If $\alpha = \beta$ and all assumptions of the cited coincidence point theorem are preserved except $\alpha > \beta$, then the mappings ψ and φ could have no coincidence point even under the additional assumption that the domain X is compact. For example, assume that X is a circle in the Euclidian plane $\mathbb{R}^2$ with the induced metric, $Y = X$, $\psi : X \to X$ is the identity mapping, φ is a rotation mapping by a fixed angle different from $2\pi m$ and $m \in \{..., -1, 0, 1, ...\}$. Take $\alpha = \beta = 1$. Then, all the assumption of the coincidence point Theorem 1 in [1] hold except the assumption $\alpha > \beta$. At the same time, the mappings ψ and φ have no coincidence points, although the domain X is compact.

Assume now that the mapping φ satisfies the strict Lipschitz inequality with the constant α, i.e.

$$\rho_Y(\varphi(x_1), \varphi(x_2)) < \alpha \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X : \quad x_1 \neq x_2. \tag{1.1}$$

Then, the α-covering mapping ψ and the mapping φ can also have no coincidence point. For example, assume that $X = Y = [1, +\infty)$ and the mappings $\psi : X \to X$ and $\varphi : X \to X$ are defined by formulae

$$\psi(x) = x, \quad \varphi(x) = x + \frac{1}{x} + c, \quad x \in [1, +\infty).$$

Here, real $c \geq 0$ is given. Take $\alpha := 1$. Then the mapping ψ is α-covering and continuous, whereas the mapping φ satisfies strict Lipschitz inequality (1.1). However, the mappings ψ and φ have no coincidence points. Moreover,

$$|\psi(x) - \varphi(x)| > c \quad \forall\, x \in [1, +\infty). \tag{1.2}$$

Note that, both considered examples were given in monograph [8] and illustrated the absence of a fixed point of the corresponding mappings.

As is known, if the metric space X is compact, $Y = X$ and the mapping $\varphi : X \to X$ is a strict contraction (i.e. inequality (1.1) holds with $\alpha = 1$), then the mapping φ has the only coincidence point (see, for example, [5, §1.6 (A.7)] or [8, §1.6]). In other words, in the contraction mapping theorem, the contraction assumption can be weakened by replacing it with the assumption of strict contraction, and the assumption of completeness of the space X can be strengthened by replacing it with the assumption of compactness of this space. For more results on fixed points of mappings satisfying various contraction assumptions see, for example, [6, 7, 9]. For the results on application to optimization see, for example, [3], [4].

In connection with the above, the following natural question arises. Is Theorem 1 in [1] true if the assumption of completeness of X is replaced by the assumption of compactness of X, and the assumption that φ is β-Lipschitz with $\beta < \alpha$ is replaced by the assumption that strict Lipschitz inequality (1.1) holds with the same constant α?

Below, we will provide a positive answer to this question. In addition, we obtain sufficient conditions for the existence of coincidence points and generalized coincidence points of set-valued mappings.

2 Coincidence points of single-valued mappings

Let us now present sufficient conditions for the existence of coincidence points of two mappings defined on compact domain.

Theorem 2.1. *Let a metric space (X,ρ_X) be compact. Given $\alpha>0$, assume that a mapping $\psi:X\to Y$ is α-covering and closed, a mapping $\varphi:X\to Y$ satisfies strict Lipschitz inequality (1.1) with the constant α.*

Then, there exists a coincidence point $\xi\in X$ of the mappings ψ and φ, i.e. $\psi(\xi)=\varphi(\xi)$.

Proof. Denote

$$\mathcal{D}:=\{(x_1,x_2)\in X\times X:\,\psi(x_2)=\varphi(x_1)\}.$$

Let us prove that the set $\mathcal{D}$ is a closed subset of the space $X\times X$ endowed with the metric

$$\rho((x_1,x_2),(\bar{x}_1,\bar{x}_2)):=\rho_X(x_1,\bar{x}_1)+\rho_Y(x_2,\bar{x}_2),\quad (x_1,x_2),\,(\bar{x}_1,\bar{x}_2)\in X\times X.$$

Take an arbitrary sequence $\{(x_1^i,x_2^i)\}\subset\mathcal{D}$ convergent to a point $(\bar{x}_1,\bar{x}_2)\in X\times X$. Then, $\varphi(x_1^i)\to\varphi(\bar{x}_1)$ as $i\to+\infty$ since the mapping φ is Lipschitz and, therefore, continuous. Since $\{(x_1^i,x_2^i)\}\subset\mathcal{D}$, the definition of the set $\mathcal{D}$ implies that $\psi(x_2^i)=\varphi(x_1^i)$ for every i. Therefore, $\psi(x_2^i)\to\varphi(\bar{x}_1)$ as $i\to+\infty$ as well. The closedness of the mapping ψ and the relations $x_2^i\to\bar{x}_2$ and $\psi(x_2^i)\to\varphi(\bar{x}_1)$ as $i\to+\infty$ imply that the point $(\bar{x}_2,\varphi(\bar{x}_1))$ belongs to the graph of the mapping ψ. Therefore, the equality $\psi(\bar{x}_2)=\varphi(\bar{x}_1)$ takes place. Hence, $(\bar{x}_1,\bar{x}_2)\in\mathcal{D}$. So, it is shown that the set $\mathcal{D}$ is a closed subset of the space $X\times X$.

Let us consider the following constrained optimization problem:

$$\text{minimize }\ \rho_X(x_1,x_2)\quad\text{subject to the condition }\ (x_1,x_2)\in\mathcal{D}. \tag{2.1}$$

The set $\mathcal{D}$ is compact since this set is a closed subset of the compact space $X\times X$. Moreover, the function $(x_1,x_2)\mapsto\rho_X(x_1,x_2)$ is continuous on the entire space $X\times X$. Therefore, the Weierstrass theorem implies that there exists at least one point $(\xi_1,\xi_2)\in\mathcal{D}$ which is a solution to Problem (2.1), i.e. $\rho_X(\xi_1,\xi_2)\le\rho_X(x_1,x_2)$ for every $(x_1,x_2)\in\mathcal{D}$.

Let us prove that the minimal value $\rho_X(\xi_1,\xi_2)$ to Problem (2.1) equals zero. Assume the contrary, i.e. $\rho_X(\xi_1,\xi_2)>0$ or equivalently $\xi_1\ne\xi_2$. Since the mapping ψ is α-covering, there exists a point $\xi_3\in X$ such that $\psi(\xi_3)=\varphi(\xi_2)$ and the inequality $\rho_X(\xi_2,\xi_3)\le\alpha^{-1}\rho_Y(\psi(\xi_2),\varphi(\xi_2))$ takes place. Applying this inequality we obtain

$$\rho_X(\xi_2,\xi_3)\le\alpha^{-1}\rho_Y(\psi(\xi_2),\varphi(\xi_2))=\alpha^{-1}\rho_Y(\varphi(\xi_1),\varphi(\xi_2))<\rho_X(\xi_1,\xi_2).$$

Here, the equality follows from the inclusion $(\xi_1,\xi_2)\in\mathcal{D}$ and the strict inequality follows from (1.1) since $\xi_1\ne\xi_2$.

So, we have $(\xi_2,\xi_3)\in\mathcal{D}$, since $\psi(\xi_3)=\varphi(\xi_2)$. At the same time, it is shown that $\rho_X(\xi_2,\xi_3)<\rho_X(\xi_1,\xi_2)$. Therefore, the point (ξ_1,ξ_2) is not a solution to Problem (2.1). The contradiction obtained proves that $\rho_X(\xi_1,\xi_2)=0$.

Denote $\xi:=\xi_1$. Since $\rho_X(\xi_1,\xi_2)=0$, we have $\xi_2=\xi$. Thus, the inclusion $(\xi_1,\xi_2)\in\mathcal{D}$ implies that $\psi(\xi)=\psi(\xi_1)=\varphi(\xi_1)=\varphi(\xi)$. □

Let us compare the obtained coincidence point theorem with Theorem 7.2 in [2]. In Theorem 7.2 it is assumed that the domain X as well as the target space Y are Banach spaces, whereas in Theorem 2.1 these spaces are metric ones and X is compact. In [2], it is assumed that the α-covering mapping ψ is smooth, whereas in Theorem 2.1 ψ is closed. In [2], the mapping φ satisfies a stronger assumption than (1.1). So, these two coincidence point theorems do not follow from each other. These two theorems also differs from Theorem 1 in [1]. The key difference is that they can be applied if the mapping φ is not β-Lipschitz with $\beta<\alpha$. However, Theorem 2.1 here as well as the result in [2] are not applicable if the metric space X is neither compact nor Banach. At the same time Theorem 1 in [1] can be applied to certain mappings in this case.

3 Coincidence points of set-valued mappings

Let us pass to the study of the coincidence points of set-valued mappings.

For a nonempty set $M \subset Y$ and a real number $r \geq 0$ denote $B_Y(M,r) := \bigcup\limits_{y \in M} B_Y(y,r)$. Denote by h_Y the Hausdorff distance between subsets of the space Y, i.e.

$$h_Y(M,N) := \inf\{r > 0 : \; B_Y(M,r) \supset N, \;\; B_Y(N,r) \supset M\}$$

for nonempty closed subsets $M, N \subset Y$. Here, if the set in the right-hand side of the equality is empty, then we assume that $h_Y(M,N) := +\infty$. Below we will also use the following distance function between sets

$$\mathrm{dist}_Y(M,N) := \inf\{\rho_Y(y_1,y_2) : \; y_1 \in M, \;\; y_2 \in N\}.$$

Here $M, N \subset Y$ are arbitrary nonempty closed subsets.

Let $\Psi, \Phi : X \rightrightarrows Y$ be given set-valued mappings, i.e. the mappings that correspond to each point $x \in X$ closed nonempty subsets of the space Y. A point $\xi \in X$ is said to be a coincidence point of the mappings Ψ and Φ if $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$. A point $\xi \in X$ is said to be a generalized coincidence point of the mappings Ψ and Φ if $\mathrm{dist}(\Psi(\xi), \Phi(\xi)) = 0$.

Given a number $\alpha > 0$, recall that the set-valued mapping Ψ is said to be α-covering if

$$B_Y(\Psi(x_0), \alpha r) \subset \Psi(B_X(x_0, r)) \quad \forall\, x_0 \in X, \quad \forall\, r \geq 0.$$

The set-valued mapping Ψ is said to be upper semicontinuous if for every point $x \in X$ and every sequence $\{(x^i, y^i)\} \subset \mathrm{gph}\, \Psi$ the relation $\mathrm{dist}(y^i, \Psi(x)) \to 0$ as $i \to \infty$ takes place. Here,

$$\mathrm{gph}\, \Psi := \{(x,y) \in X \times Y : \; x \in X, \;\; y \in \Psi(x)\}.$$

Let us formulate now sufficient conditions for the existence of a generalized coincidence point of two set-valued mappings.

Theorem 3.1. *Let a metric space (X, ρ_X) be compact. Given $\alpha > 0$, assume that a set-valued mapping $\Psi : X \rightrightarrows Y$ is α-covering and upper semicontinuous, and a set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies the strict Lipschitz inequality with the constant α, i.e.*

$$h_Y(\Phi(x_1), \Phi(x_2)) < \alpha \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X : \quad x_1 \neq x_2. \tag{3.1}$$

Then, there exists a generalized coincidence point $\xi \in X$ of the set-valued mappings Ψ and Φ, i.e. $\mathrm{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$.

Before moving on to the proof of the main results of this section, we present and discuss auxiliary constructions. First, consider a constrained optimization problem (2.1) with the set $\mathcal{D}$ defined in a new way by the formula

$$\mathcal{D} := \{(x_1, x_2) \in X \times X : \; \Psi(x_2) \cap \Phi(x_1) \neq \emptyset\}. \tag{3.2}$$

If this problem has a solution and the minimal value for this problem equals zero, then obviously there exists a generalized coincidence point $\xi \in X$ of the set-valued mappings Ψ and Φ. In the particular case when the mappings Ψ and Φ are single-valued, the set $\mathcal{D}$ coincides with the set $\mathcal{D}$ in the proof of Theorem 2.1. In this case, the set $\mathcal{D}$ is compact. This fact was shown in the proof of Theorem 2.1. However, if the mapping Φ is set-valued, then the set $\mathcal{D}$ is not necessarily compact. Consider the corresponding example.

Example 1. Put $X := [0,1]$, $Y := \{0\} \cup [1, +\infty)$,

$$\Psi(x) := 1/x, \quad x \in (0,1], \quad \Psi(0) := 0,$$

$$\Phi(x) := [1, +\infty), \quad x \in [0,1].$$

Then, the space X is compact, the mapping Ψ is closed and 1-covering. The mapping Φ is constant, so it satisfies strict Lipschitz inequality (3.1) with the constant $\alpha = 1$.

The points $(1/i, 1/i)$ belong to the set $\mathcal{D}$ for every i, since $\Psi(1/i) = i \in [1, +\infty) = \Phi(1/i)$. At the same time the limit $(0,0)$ of the sequence $(1/i, 1/i)$ does not belong to the set $\mathcal{D}$, since $\Psi(0) = 0 \notin [1, +\infty) = \Phi(0)$. Therefore, the set $\mathcal{D}$ is not compact.

This example shows that the reasonings from the proof of Theorem 2.1 are not valid when the assumptions of Theorem 3.1 hold. Namely, we cannot apply the Weierstrass theorem to Problem (2.1), since the set of admissible points $\mathcal{D}$ is not necessarily compact. Thus, we cannot prove the existence of a solution to Problem (2.1). In fact, Example 2 below shows that a solution may not exist. So, we need the following auxiliary assertions.

Lemma 3.1. *Let a metric space (X, ρ_X) be compact. Given $\alpha > 0$, assume that a set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies inequality (3.1).*

Then, for every real number $\mu > 0$ there exists a nonnegative real number $\beta = \beta(\mu) < \alpha$ such that

$$h_Y(\Phi(x_1), \Phi(x_2)) \le \beta \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in X : \quad \rho_X(x_1, x_2) \ge \mu.$$

Proof. Take an arbitrary real number $\mu > 0$. Denote

$$\mathcal{M} := \{(x_1, x_2) : \rho_X(x_1, x_2) \ge \mu\}.$$

It is obvious that $\mathcal{M}$ is a compact subset of the space $X \times X$.

Consider the function $f : \mathcal{M} \to \mathbb{R}$ defined by formula

$$f(x_1, x_2) := \frac{h_Y(\Phi(x_1), \Phi(x_2))}{\rho_X(x_1, x_2)}, \quad (x_1, x_2) \in \mathcal{M}.$$

Since $\rho(x_1, x_2) \ge \mu > 0$ for every $(x_1, x_2) \in \mathcal{M}$, inequality (3.1) implies that this function f is continuous. The Weierstrass theorem implies that the function f attains its maximal value β at some point $(\bar{x}_1, \bar{x}_2) \in \mathcal{M}$. Then $\bar{x}_1 \ne \bar{x}_2$. So, inequality (3.1) implies that $\beta < \alpha$. Moreover,

$$h_Y(\Phi(x_1), \Phi(x_2)) = f(x_1, x_2)\rho_X(x_1, x_2) \le \beta \rho_X(x_1, x_2) \quad \forall\, (x_1, x_2) \in \mathcal{M}.$$

Thus, the constructed value β is the desired one. □

Lemma 3.2. *Let a metric space (X, ρ_X) be compact. Given $\alpha > 0$, assume that a set-valued mapping $\Psi : X \rightrightarrows Y$ is α-covering and a set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies inequality (3.1).*

Then the infimum in problem (2.1) equals zero, i.e.

$$\inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\} = 0. \tag{3.3}$$

Here, the set $\mathcal{D}$ is defined by formula (3.2).

Proof. Denote $\mu := \inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\}$. It is obvious that $\mu \ge 0$. Let us prove that $\mu = 0$.

Assume the contrary: $\mu > 0$. For each $(x_1, x_2) \in \mathcal{D}$ we have $\rho_X(x_1, x_2) \ge \mu$. Then Lemma 3.1 implies that there exists a nonnegative real number $\beta < \alpha$ such that

$$h_Y(\Phi(x_1), \Phi(x_2)) \le \beta \rho_X(x_1, x_2) \quad \forall\, x_1, x_2 \in \mathcal{D}.$$

Thus, there exists a real number $\theta > 1$ such that $\beta\theta < \alpha$.

Since $\mu > 0$, the definition of μ implies that there exists a point $(x_1, x_2) \in \mathcal{D}$ such that $\rho_X(x_1, x_2) < \mu\alpha/(\beta\theta)$. Since $(x_1, x_2) \in \mathcal{D}$, there exists a point $y_1 \in Y$ such that $y_1 \in \Psi(x_2) \cap \Phi(x_1)$. Thus, the definition of the Hausdorff distance implies that the inequality $\operatorname{dist}(y_1, \Phi(x_2)) < \theta h_Y(\Phi(x_1), \Phi(x_2))$ holds. So, there exists a point $y_2 \in \Phi(x_2)$ such that $\rho_Y(y_1, y_2) \le \theta h_Y(\Phi(x_1), \Phi(x_2))$.

Since the mapping Ψ is α-covering, there exists a point $x_3 \in X$ which satisfies the relations

$$y_2 \in \Psi(x_3) \quad \text{and} \quad \rho_X(x_2, x_3) \le \frac{1}{\alpha}\rho_Y(y_1, y_2).$$

Note that the obtained inequalities imply that

$$\rho_X(x_2, x_3) \le \frac{1}{\alpha}\rho_Y(y_1, y_2) \le \frac{\theta}{\alpha} h_Y(\Phi(x_1), \Phi(x_2)) \le \frac{\beta\theta}{\alpha}\rho_X(x_1, x_2) < \mu.$$

Moreover, $(x_2, x_3) \in \mathcal{D}$, since $y_2 \in \Psi(x_3)$ and $y_2 \in \Phi(x_2)$ as is shown above. The obtained inequality $\rho_X(x_2, x_3) < \mu$ and the inclusion $(x_2, x_3) \in \mathcal{D}$ imply that the value μ is less than $\inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\}$. This contradicts the fact that μ is the infimum. The obtained contradiction proves equality (3.3). ☐

Lemma 3.2 shows that infimum of $\rho_X(x_1, x_2)$ in (2.1) equals zero. The compactness assumption is essential for this assertion. Indeed, let $X = Y = [1, +\infty)$, $\psi(x) = x$, $\varphi(x) = x + 1/x + c$, $x \in [1, +\infty)$ and $c > 0$ be a given real number. Then, by virtue of (1.2), the statement of Lemma 3.2 fails even for single-valued mappings.

Proof of Theorem 3.1. Lemma 3.2 implies that

$$\inf\{\rho_X(x_1, x_2) : (x_1,\, x_2) \in \mathcal{D}\} = 0.$$

Here, $\mathcal{D}$ is the set defined by formula (3.2). So, there exists a sequence $\{(x_1^j, x_2^j)\} \subset \mathcal{D}$ such that $\rho_X(x_1^j, x_2^j) \to 0$ as $j \to \infty$. This sequence has a convergent subsequence, since the space X is compact. Denote this subsequence by $\{(x_1^j, x_2^j)\}$ as well. Denote its limit by $(\xi_1, \xi_2) \in X \times X$. Since $\rho_X(x_1^j, x_2^j)$ tends zero as $j \to \infty$, we obtain that $\xi_1 = \xi_2$. Denote $\xi := \xi_1$. So, we have $x_1^j \to \xi$ and $x_2^j \to \xi$ as $j \to \infty$.

Since $\{(x_1^j, x_2^j)\} \subset \mathcal{D}$, we have $\Psi(x_2^j) \cap \Phi(x_1^j) \neq \emptyset$ for each j. Therefore, there exists a sequence $\{y_j\} \subset Y$ such that $y_j \in \Psi(x_2^j) \cap \Phi(x_1^j)$ for each j. Since the set-valued mapping Ψ is upper semicontinuous, $y_j \in \Psi(x_2^j)$ for all j and $x_2^j \to \xi$, then $\operatorname{dist}_Y(y_j, \Psi(\xi)) \to 0$ as $j \to \infty$. Since $y_j \in \Phi(x_1^j)$ for each j, inequality (3.1) implies that $\operatorname{dist}_Y(y_j, \Phi(\xi)) \to 0$ as $j \to \infty$.

Let us prove now that $\operatorname{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$. For each j take $\psi_j \in \Psi(\xi)$, $\varphi_j \in \Phi(\xi)$ and a sequence of positive real numbers δ_j such that

$$\rho_Y(\psi_j, y_j) \le \operatorname{dist}_Y(\Psi(\xi), y_j) + \delta_j, \quad \rho_Y(\varphi_j, y_j) \le \operatorname{dist}_Y(\Phi(\xi), y_j) + \delta_j, \quad \delta_j \to 0+ \quad \text{as} \quad j \to \infty.$$

Applying the triangle inequality, we have

$$\rho_Y(\psi_j, \varphi_j) \le \rho_Y(\psi_j, y_j) + \rho_Y(y_j, \varphi_j) \le \operatorname{dist}_Y(\Psi(\xi), y_j) + \operatorname{dist}_Y(\Phi(\xi), y_j) + 2\delta_j.$$

This inequality and the obtained relations imply that $\rho_Y(\psi_j, \varphi_j) \to 0$ as $j \to \infty$. Thus, the inequality $\operatorname{dist}_Y(\Psi(\xi), \Phi(\xi)) \le \rho_Y(\psi_j, \varphi_j)$ implies that $\operatorname{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$. ☐

Theorem 3.1 implies the following assertion on coincidence points.

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold. Namely,* $\alpha > 0$ *is given, a metric space* (X, ρ_X) *is compact, a set-valued mapping* $\Psi : X \rightrightarrows Y$ *is* α*-covering and upper semicontinuous, a set-valued mapping* $\Phi : X \rightrightarrows Y$ *satisfies inequality (3.1). Assume additionally that for each* $x \in X$ *at least one of two sets either* $\Psi(x)$ *or* $\Phi(x)$ *is compact.*

Then there exists a coincidence point $\xi \in X$ *of the set-valued mappings* Ψ *and* Φ*, i.e.* $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$.

Proof. Theorem 3.1 implies that there exists a point $\xi \in X$ such that $\mathrm{dist}_Y(\Psi(\xi), \Phi(\xi)) = 0$. Since at least one of two sets either $\Psi(x)$ or $\Phi(x)$ is compact, it follows from this equality that $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$. □

Remark 1. Theorem 2.1 does not follow from Corollary 3.1 of Theorem 3.1, since the closedness assumption in Theorem 2.1 is weaker than upper continuity assumption in Corollary 3.1.

In Theorem 3.1, the coincidence point may not exist. Let us demonstrate this fact by the following example which was presented in [1].

Example 2. Put $\Pi := \{(x, x^2/2) : x \in [0,1]\} \subset \mathbb{R}^2$. Consider the spaces $X = [0,1]$, $Y = ([0,1] \times [0,1]) \setminus \Pi$ with the metrics induced by the metric of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Put

$$\Psi(x) := \{(x,t) : t \in [0,1]\} \setminus \Pi, \quad \Phi(x) := \{(t, tx/2) : t \in [0,1]\} \setminus \Pi, \quad x \in [0,1].$$

Obviously, the metric space X is compact. Put $\alpha := 1$. The set-valued mapping $\Psi : X \rightrightarrows Y$ is α-covering and upper semicontinuous. Moreover, the set-valued mapping $\Phi : X \rightrightarrows Y$ satisfies strict Lipschitz inequality (3.1) with the constant $\alpha = 1$. So, all the assumptions of Theorem 3.1 are satisfied. Thus, Ψ and Φ has a generalized coincidence point. However, $\Psi(x) \cap \Phi(x) = \emptyset$ for each $x \in X$. For more details see [1, Example 1].

Finally, note that the upper semicontinuity assumption in Theorem 3.1 cannot be replaced by the weaker closedness assumption. For the corresponding example see [1, Example 2].

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