EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 17, Number 2 (2026), 10 – 17

CONSTRUCTION OF A DISCRETE D-OPTIMAL DESIGN FOR A LINEAR REGRESSION MODEL WITH HAAR BASIS FUNCTIONS

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Communicated by E.L. Presman

Key words: regression model, optimal design, optimality criterion, information matrix, Haar functions.

AMS Mathematics Subject Classification: 62F10, 62K05, 49K10.

Abstract. Parametrically linear regression models are widely used in practice to describe various types of dependencies. The goal of such experiments is to estimate the unknown parameters of the model and to verify the optimality of the chosen design points according to certain criteria.

DOI: https://doi.org/10.32523/2077-9879-2026-17-2-10-17

1 Introduction

The aim of this paper is to construct a D-optimal experimental design if the basis functions are the Haar functions. It is known that many Monte Carlo algorithms and experimental design methods are based on selecting a certain probability distribution ρ of the design points defined on a measurable space X [1, 4]. In this work, it is proved that a given discrete design is D-optimal if the design points are distributed almost uniformly across the intervals of constancy of the Haar functions.

Let us consider a parametrically linear regression model, where the measurement results $y(x_j)$ at points $x_j \in X$ are represented as follows [2, 4]:

$$y_j = y\left(x_j\right) = \sum_{i=1}^{n} \theta_i \varphi_i(x_j) + \varepsilon(x_j), \quad j = 1, 2, \dots, N, \tag{1.1}$$

where $\varphi\left(x\right) = \left\{\varphi_1\left(x\right), \varphi_2\left(x\right), \dots, \varphi_N\left(x\right)\right\}^T$– is the basis function vector on X, in our case the Haar functions, θ_i– are the unknown parameters, $\varepsilon\left(x_j\right)$– are random errors, X – is the design space. Standard assumptions are made regarding the errors [2]:

$$E\varepsilon\left(x\right) = 0, E\varepsilon^2\left(x\right) = \sigma^2\left(x\right) = const < \infty,\ E\varepsilon\left(x_i\right)\varepsilon\left(x_j\right) = 0, \quad i \neq j,$$

where E is the mathematical expectation.

The regression model (1.1) can also be written in the form $Y = F\theta + \varepsilon$, where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \quad F = \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \dots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \dots & \varphi_n(x_2) \\ \vdots & \vdots & \dots & \vdots \\ \varphi_1(x_N) & \varphi_2(x_N) & \dots & \varphi_n(x_N) \end{pmatrix} = \begin{pmatrix} \varphi^T(x_1) \\ \varphi^T(x_2) \\ \vdots \\ \varphi^T(x_N) \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}.$$

It is required to estimate the unknown parameters θ and their variances, as well as to verify the D-optimality of the obtained design, i.e., to find the optimal density

$$\rho^* = \underset{\rho \in P}{argmin}(det[D(\rho)]) \tag{1.2}$$

The Haar function system on the set X=[0,1] with a given (probability) measure μ is defined as follows [3].

We divide the set X with measure $\mu(X) = 1$ into 2^m disjoint subsets $d\,(m,s)\,(1 \le s \le 2^{m-1}, m = 1, 2, \ldots)$ each of the same measure μ. The subsets $d\,(m,s)(1 \le s \le 2^{m-1})$ are defined by the equality:

$$d\,(m,s) = \left[\tfrac{s-1}{2^{m-1}}\ ,\ \tfrac{s}{2^{m-1}}\right),$$

where s ranges from 1 to 2^{m-1}, and $m = 1, 2, \ldots$ (of course, for $s = 2^{m-1}$ we consider $d(m,\ s)$ to be closed on the right as well). It is easy to see that for each m:

$$d(m,1)\bigcup d(m,2)\bigcup ... \bigcup d(m,2^{m-1}) = [0,1].$$

The Haar function system $\chi_{ms}(x)$ is conveniently defined in groups: the group with number m contains 2^{m-1} functions $\chi_{ms}\,(x)$,$s = 1, 2, \ldots, 2^{m-1}$, defined by the following equalities:

$$\chi_{ms}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & \text{at} \quad x \in d(m+1, 2s-1), \\ -2^{\frac{m-1}{2}}, & \text{at} \quad x \in d(m+1, 2s), \\ 0, & \text{at} \quad x \notin d(m,s). \end{cases} \tag{1.3}$$

Let k_j $(j = 1, 2, ..., n)$ be the number of points belonging to the subset $d(m,j)$ of the set X=[0,1] and $\sum\limits_{j=1}^{n} k_j = N$.

It is known [2], that if the matrix of the system of the normal equations F^TF is non-degenerate, then the least squares estimate has the form:

$$\hat{\theta} = \left(F^TF\right)^{-1} F^TY \tag{1.4}$$

and the variance has the form:

$$D\hat{\theta} = \sigma^2 \left(F^TF\right)^{-1}, \tag{1.5}$$

where

$$F^TF = \begin{pmatrix} (\phi_1,\phi_1) & (\phi_1,\phi_2) & \cdots & (\phi_1,\phi_n) \\ (\phi_2,\phi_1) & (\phi_2,\phi_2) & \cdots & (\phi_2,\phi_n) \\ \cdots & \cdots & \cdots & \cdots \\ (\phi_n,\phi_1) & (\phi_n,\phi_2) & \cdots & (\phi_n,\phi_n) \end{pmatrix},$$

$$(\phi,\psi) = \sum_{j=1}^{N} \phi(x_j)\psi(x_j), \quad x_j \in X.$$

The matrix $M = F^TF$ is called the information matrix, and the matrix $D\hat{\theta} = \sigma^2 M^{-1}$–is called the variance matrix for model (1.1).

2 Cases with two and four basis functions

To simplify the analysis, let us first consider the cases of $m = 1,\ 2$, i.e., the interval $[0,1]$ is divided into 1 and 2 subsets respectively, and the regression model contains 2 and 4 unknown parameters θ.

In the case of $m = 1$, the Haar functions take the following form:

$$\phi_1\,(x) = 1, \phi_2\,(x) = \begin{cases} +1, & for \quad x \in [0, \frac{1}{2}) \\ -1, & for \quad x \in [\frac{1}{2}, 1] \end{cases} \tag{2.1}$$

Let k_1 and k_2 be the number of points selected from the subsets $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$. respectively, with $k_1 + k_2 = N$. The information matrix takes the following form:

$$F^T F = \begin{pmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + k_2 \end{pmatrix}.$$

The determinant of this matrix is equal to $k_1 k_2$. Therefore, for the matrix to be non-degenerate, it is necessary and sufficient that the conditions k_1 and $k_2 > 0$ are satisfied.

$$\left(F^T F\right)^{-1} = \frac{1}{4} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} - \frac{1}{k_2} \\ \frac{1}{k_1} - \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \end{pmatrix}, \quad F^T Y = \begin{pmatrix} \sum_{i=1}^{k_1} y_i & + & \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \sum_{i=1}^{k_1} y_i & - & \sum_{i=k_1+1}^{k_1+k_2} y_i \end{pmatrix}.$$

Therefore, the estimate of the unknown parameters $\theta = (\theta_1, \theta_2)$ is given by:

$$\hat{\theta} = \left(F^T F\right)^{-1} F^T Y = \begin{pmatrix} \frac{1}{2k_1} \sum_{i=1}^{k_1} y_i & + & \frac{1}{2k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \frac{1}{2k_1} \sum_{i=1}^{k_1} y_i & - & \frac{1}{2k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \end{pmatrix},$$

and their variance has the following form:

$$D\hat{\theta} = \frac{\sigma^2}{4} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} - \frac{1}{k_2} \\ \frac{1}{k_1} - \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \end{pmatrix}.$$

In the case $m = 2$ we have four Haar functions: $\phi_1(x)$, $\phi_2(x)$ from (2.1) and

$$\phi_3(x) = \begin{cases} +\sqrt{2}, & for \quad x \in \left[0, \frac{1}{4}\right), \\ -\sqrt{2}, & for \quad x \in \left[\frac{1}{4}, \frac{1}{2}\right), \\ 0, & for \quad x \in \left[\frac{1}{2}, 1\right], \end{cases} \quad \phi_4(x) = \begin{cases} +\sqrt{2}, & for \quad x \in \left[\frac{1}{2}, \frac{3}{4}\right), \\ -\sqrt{2}, & for \quad x \in \left[\frac{3}{4}, 1\right), \\ 0, & for \quad x \in \left[0, \frac{1}{2}\right]. \end{cases} \tag{2.2}$$

Let k_1, k_2, k_3 и k_4 be the number of points selected from the subsets $\left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{4}\right)$ and $\left[\frac{3}{4}, 1\right]$, respectively, with $k_1 + k_2 + k_3 + k_4 = N$. The matrix $F^T F$ has the following form:

$$F^T F = \begin{pmatrix} k_1 + k_2 + k_3 + k_4 & k_1 + k_2 - k_3 - k_4 & \sqrt{2}(k_1 - k_2) & \sqrt{2}(k_3 - k_4) \\ k_1 + k_2 - k_3 - k_4 & k_1 + k_2 + k_3 + k_4 & \sqrt{2}(k_1 - k_2) & -\sqrt{2}(k_3 - k_4) \\ \sqrt{2}(k_1 - k_2) & \sqrt{2}(k_1 - k_2) & 2\left(k_1 + k_2\right) & 0 \\ \sqrt{2}(k_3 - k_4) & -\sqrt{2}(k_3 - k_4) & 0 & 2\left(k_3 + k_4\right) \end{pmatrix}.$$

The determinant of this matrix is $\det\left(F^T F\right) = 256 k_1 k_2 k_3 k_4$, where $k_i > 0, i = 1, 2, 3, 4$. The invers matrix to the matrix $\left(F^T F\right)^{-1}$ and the vector $F^T Y$ have the following form:

$$\left(F^T F\right)^{-1} = \frac{1}{16} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} & \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} - \frac{1}{k_4} & \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & \sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) \\ \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} - \frac{1}{k_4} & \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} & \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & -\sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) \\ \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & \sqrt{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & 2\left(\frac{1}{k_1} + \frac{1}{k_2}\right) & 0 \\ \sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) & -\sqrt{2}\left(\frac{1}{k_3} - \frac{1}{k_4}\right) & 0 & 2\left(\frac{1}{k_3} + \frac{1}{k_4}\right) \end{pmatrix},$$

$$F^T Y = \begin{pmatrix} \sum_{i=1}^{k_1+k_2+k_3+k_4} y_i \\ \sum_{i=1}^{k_1+k_2} y_i - \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3+k_4} y_i \\ \sqrt{2} \sum_{i=1}^{k_1} y_i - \sqrt{2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \sqrt{2} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \sqrt{2} \sum_{i=k_1+k_2+k_3+1}^{k_1+k_2+k_3+k_4} y_i \end{pmatrix}.$$

Then, the least squares estimates of the unknown parameters $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ determined by the formula $\hat{\theta} = \left(F^T F\right)^{-1} F^T Y$ are given by:

$$\begin{pmatrix} \frac{1}{4k_1} \sum_{i=1}^{k_1} y_i + \frac{1}{4k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i + \frac{1}{4k_3} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i + \frac{1}{4k_4} \sum_{i=k_1+k_2+k_3+1}^{N} y_i \\ \frac{1}{4k_1} \sum_{i=1}^{k_1} y_i + \frac{1}{4k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i - \frac{1}{4k_3} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \frac{1}{4k_4} \sum_{i=k_1+k_2+k_3+1}^{N} y_i \\ \frac{\sqrt{2}}{4k_1} \sum_{i=1}^{k_1} y_i - \frac{\sqrt{2}}{4k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \frac{\sqrt{2}}{4k_3} \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \frac{\sqrt{2}}{4k_4} \sum_{i=k_1+k_2+k_3+1}^{N} y_i \end{pmatrix},$$

and the variance of these estimates by: $D\hat{\theta} = \sigma^2 \left(F^T F\right)^{-1}$.

3 General case

We obtain the same result using another approach, which allows us to generalize the findings to the case of an arbitrary n. To do this, we introduce a system of characteristic functions $f_1(x)\, f_2(x), f_3(x), f_4(x)$ corresponding to the segments $d(2,1) = \left[0, \frac{1}{4}\right)$, $d(2,2) = \left[\frac{1}{4}, \frac{1}{2}\right)$, $d(2,3) = \left[\frac{1}{2}, \frac{3}{4}\right)$ and $d(2,4) = \left[\frac{3}{4}, 1\right]$ defined as follows:

$$f_i(x) = \begin{cases} 1, & if \quad x \in d(2,i), \\ 0, & if \quad x \notin d(2,i). \end{cases} \tag{3.1}$$

It is then evident that the Haar functions can be expressed in terms of the characteristic functions$f_i(x), i = 1, 2, 3, 4$, in the following form:

$$\phi_1(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x),$$

$$\phi_2(x) = f_1(x) + f_2(x) - f_3(x) - f_4(x),$$

$$\phi_3(x) = \sqrt{2}\left(f_1(x) - f_2(x)\right),$$

$$\phi_4(x) = \sqrt{2}\left(f_3(x) - f_4(x)\right)$$

or

$$\phi(x) = Lf(x), \tag{3.2}$$

where

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}, \tag{3.3}$$

$$\phi(x) = \left(\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)\right)^T, \ f(x) = \left(f_1(x), f_2(x), f_3(x), f_4(x)\right)^T.$$

Substituting (3.2) into (1.1), we obtain

$$Y = F\theta + \epsilon = \theta^T LG + \epsilon, \tag{3.4}$$

where

$$G = \begin{pmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_N) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_N) \\ f_3(x_1) & f_3(x_2) & \dots & f_3(x_N) \\ f_4(x_1) & f_4(x_2) & \dots & f_4(x_N) \end{pmatrix}. \tag{3.5}$$

From this we obtain:

$$\theta^T L = \theta_{(1)}^T,\ \theta^T = \theta_{(1)}^T L^{-1},\ \theta = \left(L^{-1}\right)^T \theta_{(1)}. \tag{3.6}$$

If now $\hat{\theta}_{(1)}$ is the least squares estimate of the unknown parameters under an arbitrary design ξ with the dispersion matrix $D\hat{\theta}_{(1)}$, then, according to Theorem 1.4 [2], we have that the estimate $\hat{\theta} = \left(L^{-1}\right)^T \theta_{(1)}$ is also a least squares estimate, and its variance is calculated using the formula:

$$D\hat{\theta} = \left(L^{-1}\right)^T D\hat{\theta}_{(1)} L^{-1}, \tag{3.7}$$

Hence,

$$\det D\hat{\theta} = (\det L)^{-2} \det D\hat{\theta}_{(1)}, \tag{3.8}$$

which coincides with the previously obtained result, because $\det L = -16$.

The latter method for obtaining least squares estimates for the system of Haar functions allows us to generalize the estimation process to the general case. Given that Haar functions are linear combinations of step (characteristic) functions of the type (3.1), the more general case where the number of basis functions exceeds four can be studied using the previously obtained results for step functions [4].

In the general case of arbitrary m, we have $n = 2^m$ Haar functions defined on the interval [0,1], which is divided into 2^m subintervals $d(m, s)\, s = 1, 2, ..., 2^m$ of equal length.

Let us consider the system of characteristic functions $f_s(x)$ defined by the equalities:

$$f_s(x) = \begin{cases} 1, & if \quad x \in \left[\frac{s-1}{2^m}, \frac{s}{2^m}\right] \\ 0, & if \quad x \notin \left[\frac{s-1}{2^m}, \frac{s}{2^m}\right] \end{cases}, \; s = 1, 2, \ldots, 2^m.$$

Lemma 3.1. *Let m be a natural number. Then the Haar basis functions $\{\phi_s(x)\}\,(s = 1, 2, \ldots, 2^m)$ can be expressed in terms of the characteristic functions $\{f_s(x)\}\,(s = 1, 2, \ldots, 2^m)$ using formula (3.2), where the elements of the matrix $L_m = (l_{m,i,j})_{i,j=1}^n$ are defined by the following recursive formula:*

$$l_{m,i,j} = \begin{cases} l_{m-1,i,\left[\frac{j+1}{2}\right]}, \quad if \quad i \le 2^{m-1} \\ 2^{\frac{m-1}{2}}, \quad if \quad i > 2^{m-1}, \quad j = i - 2^{m-1} \\ -2^{\frac{m-1}{2}}, \quad if \quad i > 2^{m-1}, \quad j = i + 1 - 2^{m-1} \\ 0, \quad if \quad i > 2^{m-1}, \quad j \ne i - 2^{m-1}, \quad j \ne i + 1 - 2^{m-1} \end{cases}. \tag{3.9}$$

Proof. We proceed by the method of mathematical induction. For $m = 1$, the system of Haar functions $\phi_1(x), \phi_2(x)$ is defined in equation (2.1), and the matrix

$$L_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Assume that the conditions of the theorem hold for $m = k$. We will prove formula (3.9) for the case $m = k + 1$. The subintervals $d(k + 1, s)s = 1, 2, ..., 2^{k+1}$ are obtained from the subintervals $d(k, s)s = 1, 2, ..., 2^k$ by dividing each interval in half. The Haar function system consists of 2^k Haar functions $\{\phi_s(x)\}\,(s = 1, 2, \ldots, 2^k)$ for the case $m = k$, and 2^k functions $\chi_{k+1,s}(x), s = 1, 2, \ldots, 2^k$ from equation (1.3). Then, the first 2^k rows of the matrix L_{k+1} consist of the elements of the matrix L_k. The next 2^k rows are defined by the formulas in equation (1.3), in which two elements are non-zero and the rest are zero, as reflected in the last three expressions of formula (3.9). The first 2^k rows of the matrix L_k have 2^k columns, but since the subintervals $d(k + 1, s)s = 1, 2, ..., 2^{k+1}$ are halves of the subintervals $d(k, s)s = 1, 2, ..., 2^k$, each element must be repeated twice. □

Lemma 3.2. *For the matrix $L_m$the following equality holds $L_m^T L_m = 2^m E$, where E is the identity matrix.*

Proof. The statement of the lemma means that all rows of the matrix L_m are orthogonal, and the sum of the squares of the elements in each row is equal to 2^m. The second statement is obvious. Let us prove the orthogonality of the rows, i.e., that the scalar product of any two distinct rows is zero.

The first row is orthogonal to all other rows, since the remaining rows contain an even number of non-zero elements, half of which have opposite signs.

The last 2^{m-1} rows are mutually orthogonal, as each row contains exactly two non-zero elements located in different columns.

The first 2^{m-1} rows are also mutually orthogonal, since they are obtained by duplicating the elements of the matrix L_{m-1}, whose rows are orthogonal by assumption.

The scalar product of any row from the first half with any row from the second half consists of two terms with opposite signs and therefore equals zero. $\square$

Substituting (3.2) into (1.1), we obtain:

$$Y = \theta^T F + \epsilon = \theta^T LG + \epsilon = \theta^T_{(1)} G + \epsilon, \tag{3.10}$$

where

$$G = \begin{pmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_N) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_N) \\ \dots & \dots & \dots & \dots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_N) \end{pmatrix}. \tag{3.11}$$

$\theta = (\theta_{1,}\theta_{2,}..., \theta_n)^T$are the unknown parameters in the case of Haar functions, and $\theta_1 = \left(\theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_n^{(1)}\right)^T$ in the case of characteristic functions. From this follow equalities (3.6).

If now $\hat{\theta}_{(1)}$ is the least squares estimate of the unknown parameters under an arbitrary design ξ with the dispersion matrix $D\hat{\theta}_{(1)}$, then, according to Theorem 1.4 [2], the estimate $\hat{\theta} = (L^{-1})^T \theta_{(1)}$ is also a least squares estimate, and its variance is calculated using formula (3.7), while the determinant is given by formula (3.8).

Since the determinant of the information matrix $M(\hat{\theta}_{(1)})$ in the case of step functions is easily computed and equals [1]

$$\det M\left(\hat{\theta}_{(1)}\right) = n^n k_1 k_2 \dots k_n, \tag{3.12}$$

then the determinant reaches its maximum value when $k_1 = k_2 = ... = k_n$.

Thus, if the number of measurement points N is divisible by the number of unknown parameters $n = 2^m$, then the following theorem holds.

Theorem 3.1. *Let $X = [0, 1], n = 2^m, \{\phi_i(x)\}\,(i = 1, 2, ..., n)$ be the system of Haar functions. Consider the linear regression model (1.1) $Y = F\theta + \epsilon$. If the number of measurement points N is divisible by n, i.e., $N = ln$, then the determinant in formula (3.8) reaches its minimum value when $k_1 = k_2 = ... = k_n = l$*

Theorem 3.2. *Let $X = [0, 1], n = 2^m, \{\phi_i(x)\}\,(i = 1, 2, ..., n)$ be the system of Haar functions. Consider the linear regression model (1.1) $Y = F\theta + \epsilon$. Now suppose the number of measurement points N is not divisible by the number of partitions n, i.e., $N = ln + r$, where $0 < r < n$.*

Then, in the optimal design, all k_i differ from each other by no more than one, i.e.,

$$k_1 = k_2 = ... = k_r = l + 1, \quad k_{r+1} = k_{r+2} = ... = k_n = l.$$

The number of such designs is equal to C_n^r.

Proof. Indeed, to proceed, we order the values $k_1, k_2, \dots, k_n$ and obtain a variational series

$$k_1 \le k_2 \le \dots \le k_n,$$

where

$$k_1 k_2 \dots k_n = \frac{1}{n^n} \det M(\xi).$$

Suppose that $k_n - k_1 > 1$. We construct a new design in which all k_i, except for k_1 and k_n, remain unchanged, while k_n is replaced by $k_n - 1$ and k_1 is replaced by $k_1 + 1$. For this new design, the determinant equals $(k_1 + 1)k_2 \dots k_n - 1(k_n - 1) > k_1, k_2, \dots,\ k_n$, i.e., the original design is not optimal, and therefore our assumption is incorrect. Thus, in the optimal design, all k_i differ from each other by no more than one. The number of such designs is equal to C_n^r. One such design is:

$$\xi = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ \frac{l+1}{N} & \frac{l+1}{N} & \cdots & \frac{l+1}{N} & \frac{l}{N} & \cdots & \frac{l}{N} \end{pmatrix}.$$

□

Thus, a D-optimal design has been constructed for the system of Haar functions in a linear regression model with parameters appearing linearly.

Discussion

Thus, we have obtained an explicit form of the discrete optimal design under the D-optimality criterion for a linear-in-parameters regression model in the case of the system of Haar basis functions. Furthermore, we derived an explicit form of the transformation matrix L_m converting step functions into Haar functions, which satisfies the conditions of Theorem 1.4 in [2]. The resulting construction extends naturally to functions defined on an arbitrary finite interval $X = [a, b]$. One may also replace the Haar system with other basis functions possessing similar structural properties. In addition, it is possible to construct optimal designs for other optimality criteria (G, MV) and to verify the assertion of the Kiefer–Wolfowitz equivalence theorem (Theorem 2.3 in [2]).

Acknowledgments

The authors express their gratitude for the financial support provided by the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant number BR20280990).

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Received: 26.10.2023

Revised: 13.01.2026