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DEFORMATION OF SPECTRUM AND LENGTH SPECTRUM ON SOME COMPACT NILMANIFOLDS UNDER THE RICCI FLOW

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Abstract. In this article we study the eigenvalue variations of Heisenberg and quaternion Lie groups under the Ricci flow and we investigate the deformation of some characteristics of compact nilmanifolds $\Gamma \setminus N$ under the Ricci flow, where N is a simply connected 2-step nilpotent Lie group with a left invariant metric and Γ is a discrete cocompact subgroup of N, in particular Heisenberg and quaternion Lie groups.

1 Introduction

2-step nilpotent Lie groups play an important role in mathematics, and Heisenberg type, Heisenberg-like type Lie groups have a special significance. Heisenberg-type groups introduced by Kaplan [17], are examples of Carnot groups. A Carnot group is simply a connected m-step nilpotent Lie group N whose Lie algebra $\mathcal{N}$ decomposes into the direct sum of vector subspaces $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_m$ satisfying the following relations:

$$[\mathcal{V}_1, \mathcal{V}_k] = \mathcal{V}_{k+1}\ , 1 \leq k < m,\ \ [\mathcal{V}_1, \mathcal{V}_m] = \{0\}.$$

Carnot groups have applications in complex analysis, semiclassical analysis of quantum mechanics, control theory, and probability theory of degenerate diffusion processes, see [20] for further details. The geometry of Carnot groups has been studied extensively by many mathematicians, for instance, [2], [9], [10], [17].

The spectrum of a Riemannian manifold (M, g), denoted by $spec(M, g)$, is the collection of eigenvalues with multiplicities of the associated Laplace-Beltrami operator acting on smooth functions. Studying the eigenvalues of geometric operators plays a powerful role in geometric analysis. A basic question in spectral geometry is determining what geometric information is contained in the spectrum of a Riemannian manifold. Despite considerable research in the area, only a few geometric properties are known to be spectrally determined; for example, dimension, volume, and total scalar curvature. Two Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ are said to be isospectral if $spec(M, g) = spec(\widetilde{M}, \widetilde{g})$. Examples of isospectral manifolds provide us with the only means for determining properties not determined by the spectrum (see [3], [4], [5], [18], [19]).

On the other hand, the length spectrum of a Riemannian manifold is the set of lengths of closed geodesics, counted with multiplicity. The multiplicity of a length is defined as the number of distinct free homotopy classes in which the length occurs. The length spectrum and isospectral have relationship with each other (see [11], [13], [14], [16]).

Let N be a simply connected Lie group and let Γ be a cocompact discrete subgroup of N. A Riemannian metric g is left invariant if left translations of N are isometries. The left invariant metric g projects to a Riemannian metric on $\Gamma \setminus N$, which we also denote by g. Eberlein in [9] and [10] studied the differential geometry of simply connected, 2-step nilpotent Lie groups N with a left invariant Riemannian metric $<,>$. The goal of this article is to discuss relationships between the Laplace spectrum, the length spectrum and the geodesic flow of compact Riemannian manifolds.

Let M be a Riemannian manifold with a Riemannian metric g_0. The un-normalized Ricci flow on M is defined by the equation:

$$\frac{d}{dt}g(t) = -2Ric(g(t)), \; g(0) = g_0, \tag{1.1}$$

where Ric is the Ricci tensor of $g(t)$. The volume of a manifold does not remain constant under this Ricci flow, but it is preserved under the equation

$$\frac{d}{dt}g(t) = -2Ric(g(t)) + \frac{2r}{n}g, \; g(0) = g_0, \tag{1.2}$$

where

$$r = \frac{\int_M R d\mu}{\int_M d\mu}$$

is the average of the scalar curvature. (1.2) is called the normalized Ricci flow. Exitence and uniqueness of the solution to the Ricci flow over a sufficiently short time have been proved by Hamilton in [6], [7], by DeTurk in [8] on a compact Riemannian manifold and by G. Xu in [24] on a noncompact Riemannian manifold.

In [1] and [21] the Ricci flow equation has been constructed and solved for some classes of Carnot groups, the first is related to the multidimensional space of quaternion numbers which is called quaternion nilpotent Lie group, the second one is the higher-dimensional classical Heisenberg nilpotent Lie group.

Let M be a Riemannian manifold, Ω a bounded domain with smooth boundary in M and $f : \Omega \longrightarrow \mathbb{R}$ be a smooth function on Ω or $f \in W^{1,p}(\Omega)$, the Sobolev space. The p-Laplacian of f for $1 < p < \infty$ is defined as

$$\begin{aligned} \triangle_p f &= div(|\nabla f|^{p-2}\nabla f) \\ &= |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}(Hessf)(\nabla f, \nabla f), \end{aligned} \tag{1.3}$$

where

$$(Hessf)(X, Y) = \nabla(\nabla f)(X, Y) = Y.(X.f) - (\nabla_Y X).f, \quad X, Y \in \mathcal{X}(M).$$

Note that, for $p = 2$, Δ_p is the usual Laplace-Beltrami operator. We say that λ is an eigenvalue of the p-Laplacian on Ω whenever $f \neq 0$ on $\bar{\Omega}$ and

$$\begin{cases} \Delta_p f + \lambda|f|^{p-2}f = 0 & \text{on } \Omega, \\ f = 0 & \text{on } \partial\Omega, \end{cases}$$

or equivalently

$$\int_\Omega |\nabla f|^{p-2} < \nabla f, \nabla\varphi > d\mu = \lambda \int_\Omega |f|^{p-2} f\varphi d\mu \qquad \forall\varphi \in W_0^{1,p}(\Omega), \tag{1.4}$$

where $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in Sobolev space $W^{1,p}(\Omega)$. In this case we say that f is an eigenfunction associated to λ. Taking $\varphi = f$ in (1.4) implies

$$\int_\Omega |\nabla f|^p d\mu = \lambda \int_\Omega |f|^p d\mu,$$

hence

$$\lambda = \frac{\int_\Omega |\nabla f|^p d\mu}{\int_\Omega |f|^p d\mu}.$$

Normalized eigenfunctions are defined as follows :

$$\int_\Omega |f|^p d\mu = 1. \tag{1.5}$$

The nonzero first eigenvalue of the p-Laplacian is

$$\lambda_{1,p} = \inf_{f\neq 0} \left\{ \frac{\int_\Omega |\nabla f|^p d\mu}{\int_\Omega |f|^p d\mu} : \ f \in W^{1,p}(\Omega), \ \int_\Omega |f|^{p-2} f d\mu = 0 \right\}.$$

Let $(M^n, g(t))$ be a solution of the Ricci flow on the smooth manifold (M^n, g_0) in the interval $[0;T)$, then

$$\lambda(t) = \int_\Omega |\nabla f(x)|^p d\mu_t \tag{1.6}$$

defines the evolution of an eigenvalue of the p-Laplacian under the variation of $g(t)$ where an eigenfunction associated to $\lambda(t)$ is normalized. Suppose that for any metric $g(t)$ on M^n

$$Spec_p(g) = \{0 = \lambda_0(g) \leq \lambda_1(g) \leq \lambda_2(g) \leq ... \leq \lambda_k(g) \leq ...\}$$

is the spectrum of $\Delta_p = {}^g\Delta_p$. In [23] it is shown that along the Ricci flow the $\lambda(t)$ is differentiable on $[0, T]$, therefore, in what follows we assume the existence and C^1-differentiability of the elements $\lambda(t)$ and $f(t)$ under a Ricci flow deformation $g(t)$ of a given initial metric.

2 Variation of eigenvalues

In this section, we will be using evolution formulas for $\lambda(t)$ under the Ricci flow of [22] and compute those in particular cases.

In [22] and [23] it has been shown that the variation formula for the eigenvalues of the p-Laplacian under the Ricci flow is as follows:

i) for the un-normalized Ricci flow

$$\frac{d\lambda}{dt} = \lambda \int_\Omega R|f|^p d\mu - \int_\Omega R|\nabla f|^p d\mu + p \int_\Omega Ric(\nabla f, \nabla f)|\nabla f|^{p-2} d\mu, \tag{2.1}$$

ii) for the normalized Ricci flow

$$\frac{d\lambda}{dt} = -\frac{pr\lambda}{n} + \lambda \int_\Omega R|f|^p d\mu - \int_\Omega R|\nabla f|^p d\mu + p \int_\Omega Ric(\nabla f, \nabla f)|\nabla f|^{p-2} d\mu. \tag{2.2}$$

2.1 Variation of eigenvalues on the Heisenberg Lie group

We now recall the construction and properties of the higher-dimensional, classical Heisenberg Lie group. Let H_n be a $(2n+1)$-dimensional Heisenberg Lie group. Let

$$\begin{aligned} x &= (x^1, ..., x^n), \\ y &= (x^{n+1}, ..., x^{2n}). \end{aligned}$$

If $q = (x, y, z) \in H_n \times \mathbb{R}$ and $q = (x', y', z') \in H_n \times \mathbb{R}$, then the group multiplication is

$$(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + x.y')$$

where $x.y'$ is the usual inner product of vectors $x \in \mathbb{R}^n$ and $y' \in \mathbb{R}^n$. With respect to this multiplication, we have the following frame of left invariant vector fields,

$$e_i = \partial_i, \;\; e_{n+i} = \partial_{n+i} + x^i \partial_{2n+1}, \;\; e_{2n+1} = \partial_{2n+1}, \;\text{for all } 1 \leq i \leq n,$$

and the only nontrivial Lie bracket relation is

$$[e_i, e_{n+i}] = e_{2n+1}, \quad \text{for all } 1 \leq i \leq n.$$

The dual coframe is

$$\theta^i = dx^i, \;\; \theta^{n+i} = dx^{n+i}, \;\; \theta^{2n+1} = dx^{2n+1} - \sum_{k=1}^{n} x^k dx^{n+k}, \;\; \text{for all } 1 \leq i \leq n.$$

In [21] it has been shown that the solution to the Ricci flow equation in the Heisenberg nilpotent Lie group with the initial diagonal left-invariant metric

$$g_i(0) g_{n+i}(0) = g_{2n+1}(0), \;\; \text{for all } i, \;\; 1 \leq i \leq n$$

is

$$\begin{cases} g_i(t) = g_i(0)\Big((n+2)t + 1\Big)^{\frac{-2r_i}{n+2}}, & \text{if } 1 \leq i \leq 2n+1; \\ (r_1, \ldots, r_{2n+1}) = -\frac{1}{2}(1, 1, \ldots, 1, -n) \end{cases} \tag{2.3}$$

where $g_i(t) = g_{ii}(t)$. The orthogonality of vector fields is invariant under this metric. For the metric $g(t)$ defined by (2.3), the Ricci tensor is diagonal and for $1 \leq i \leq 2n+1$ we have

$$R_i = -2\frac{\partial g_i}{\partial t} = r_i g_i(0)\Big((n+2)t + 1\Big)^{\frac{-2r_i}{n+2} - 1} = r_i g_i(t)\Big((n+2)t + 1\Big)^{-1},$$

where $R_i = R_{ii}$, therefore

$$R = g^i R_i = \frac{-n}{2((n+2)t + 1)}$$

is a constant. Let $\nabla_i = \nabla_{e_i}$, then

$$\begin{aligned} Ric(\nabla f, \nabla f) &= R_{ij} \nabla^i f \nabla^j f = R_i (\nabla^i f)^2 \\ &= \frac{r_i}{(n+2)t + 1} g_i(t)(\nabla^i f)^2 \\ &= \frac{-1}{2((n+2)t + 1)} g_i(t)(\nabla^i f)^2 + \frac{n+1}{2((n+2)t + 1)} g_{2n+1}(t)(\nabla^{2n+1} f)^2 \\ &= \frac{-1}{2((n+2)t + 1)} |\nabla f|^2 + \frac{n+1}{2((n+2)t + 1)} g^{2n+1}(t)(\nabla_{2n+1} f)^2. \end{aligned}$$

Using (2.1) for the un-normalized Ricci flow eigenvalue variation of the p-Laplacian we get

$$\begin{aligned}\frac{d\lambda}{dt} &= \frac{-p\lambda}{2((n+2)t+1)} + \frac{(n+1)p}{2((n+2)t+1)}\int_\Omega \frac{((n+2)t+1)^{\frac{n}{n+2}}}{g_{2n+1}(0)}(\partial_{2n+1}f)^2|\nabla f|^{p-2}d\mu \\ &= \frac{-p\lambda}{2((n+2)t+1)} + \frac{(n+1)p((n+2)t+1)^{\frac{-2}{n+2}}}{2g_{2n+1}(0)}\int_\Omega (\partial_{2n+1}f)^2|\nabla f|^{p-2}d\mu\end{aligned}$$

and using (2.2) for the normalized Ricci flow of the p-Laplacian, we obtain

$$\frac{d\lambda}{dt} = -\frac{pr\lambda}{2n+1} + \frac{-p\lambda}{2((n+2)t+1)} + \frac{(n+1)p((n+2)t+1)^{\frac{-2}{n+2}}}{2g_{2n+1}(0)}\int_\Omega (\partial_{2n+1}f)^2|\nabla f|^{p-2}d\mu.$$

Example 1. Let $(\mathcal{N}, <,>)$ be a two-step nilpotent metric Lie algebra of a Lie group $(N, <,>)$, which has the orthogonal decomposition $\mathcal{N} = \mathcal{V}\oplus\mathcal{Z}$, where $\mathcal{Z}$ is center and $\mathcal{V}$ is the orthogonal complement of $\mathcal{Z}$. Define the linear transformation $j : \mathcal{Z} \to SO(\mathcal{V})$ by $j(Z)X = (adX)^*Z$ for $Z \in \mathcal{Z}$ and $X \in \mathcal{V}$. Equivalently, for each $Z \in \mathcal{Z}$, $j(Z) : \mathcal{V} \to \mathcal{V}$ is the skew-symmetric linear transformation defined by

$$< (adX)^*Z, Y >=< Z, (adX)Y > \tag{2.4}$$

for all $X, Y \in \mathcal{V}$. Here $adX(Y) = [X, Y]$ for all $X, Y \in \mathcal{N}$, and $(adX)^*$ denotes the (metric) adjoint of adX. We say that $(N, <,>)$ is of Heisenberg type whenever for any $Z \in \mathcal{Z}$ we have $j(Z)^2 = -|Z|^2Id$. Note, that given the above definitions, H_n is two-step nilpotent. Set

$$\mathcal{V} = \{e_i, e_{n+i}|1 \le i \le n\}, \qquad \mathcal{Z} = \{e_{2n+1}\}$$

If we choose an inner product on H_n such that $\mathcal{V}\bigcup\mathcal{Z}$ is an orthonormal basis for H_n then the Heisenberg Lie group is of Heisenberg type. Let $<,>$ be a left-invariant metric on a Lie group N and ∇ its metric connection, then for $X, Y, Z, W \in \mathcal{N}$, we have:

$$\nabla_X Y = \frac{1}{2}\{(adX)Y - (adX)^*Y - (adY)^*X\}.$$

Now, in the Heisenberg Lie group $(H_n, g(t))$ for all $1 \le I \le 2n+1$, we have

$$\nabla_{e_I}e_I = \frac{1}{2}\{[e_I, e_I] - (ade_I)^*e_I - (ade_I)^*e_I\} = -(ade_I)^*e_I$$

hence

$$< \nabla_{e_I}e_I\, ,\, e_J >= - < -(ade_I)^*e_I\, ,\, e_J >= - < e_I,\, [e_I, e_J] >= 0$$

this implies that $\Gamma^J_{II} = 0$. If $g(t)$ is the metric in (2.3) then it is diagonal and for $1 \le I, J, K \le 2n+1$ we have

$$\begin{aligned}\Delta f &= g^{IJ}(e_Ie_Jf - \Gamma^K_{IJ}e_Kf) = g^I(e_Ie_If - \Gamma^K_{II}e_Kf) \\ &= g^I(e_I(e_If)) \\ &= \sum_{i=1}^n g^i\partial_i^2 f + \sum_{i=1}^n g^{i+n}\left(\partial^2_{i+n}f + 2x^i\partial_{i+n}\partial_{2n+1}f + (x^i)^2\partial^2_{2n+1}f\right) \\ &+ g^{2n+1}\partial^2_{2n+1}f.\end{aligned}$$

For $p = 2$ if $f(x) = exp(x^1)$ where $x = (x^1, ..., x^n, x^{n+1}, ..., x^{2n}, x^{2n+1})$, then $\Delta f = g^1\partial_1^2 f$. Therefore

$$\lambda(t) = g^1 = (g_1(0))^{-1}\Big((n+2)t+1\Big)^{\frac{-1}{n+2}}.$$

For $p \neq 2$, we compute

$$|\nabla f|^2 = g^{IJ}(e_I f)(e_J f) = \frac{1}{g_1(0)}\Big((n+2)t+1\Big)^{\frac{-1}{n+2}} f^2,$$

and

$$|\nabla f|^{p-2}\nabla f = (\frac{1}{g_1(0)})^{\frac{p-2}{2}}\Big((n+2)t+1\Big)^{\frac{-(p-2)}{2(n+2)}} f^{p-2}\nabla f,$$

therefore

$$\begin{aligned}
\Delta_p f &= div(|\nabla f|^{p-2}\nabla f)\\
&= \nabla_I(|\nabla f|^{p-2}\nabla f)^I\\
&= g^{IJ}\left\{\nabla_I(|\nabla f|^{p-2}\nabla f)_J\right\}\\
&= g^{IJ}\left\{e_I(|\nabla f|^{p-2}\nabla f)_J - \Gamma^K_{IJ}(|\nabla f|^{p-2}\nabla f)_K\right\}\\
&= g^1 e_1(|\nabla f|^{p-2}\nabla f)_1\\
&= \frac{1}{g_1(0)}\Big((n+2)t+1\Big)^{\frac{-1}{n+2}}\partial_1\Big((\frac{1}{g_1(0)})^{\frac{p-2}{2}}\Big((n+2)t+1\Big)^{\frac{-(p-2)}{2(n+2)}} f^{p-1}\Big)\\
&= (\frac{1}{g_1(0)})^{\frac{p}{2}}\Big((n+2)t+1\Big)^{\frac{-(p)}{2(n+2)}}(p-1)f^{p-1},
\end{aligned}$$

which implies that

$$\lambda(t) = (\frac{1}{g_1(0)})^{\frac{p}{2}}\Big((n+2)t+1\Big)^{\frac{-(p)}{2(n+2)}}(p-1).$$

2.2 Variation of eigenvalues on quaternion Lie group

We now recall the construction and properties of higher-dimensional classical quaternion Lie groups. Let $N = Q_n$ be a $(4n+3)-$dimensional quaternion Lie group. Let

$$\begin{aligned}
x &= \left(x^{11}, x^{21}, ..., x^{n1}, ..., x^{1n}, x^{2n}, ..., x^{4n}\right),\\
z &= \left(z_1, z_2, z_3\right).
\end{aligned}$$

If $q = (x, z) \in N$ and $q' = (x', z') \in N$ and

$$M_1 = \begin{bmatrix} A_1 & O & \cdots & O\\ O & A_1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ O & \cdots & O & A_1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} A_2 & O & \cdots & O\\ O & A_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ O & \cdots & O & A_2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} A_3 & O & \cdots & O\\ O & A_3 & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ O & \cdots & O & A_3 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

then the multiplication on N is defined by

$$\begin{aligned} L_q(q') &= L_{(x,z)}(x', z') = (x, z) \circ (x', z') \\ &= \left(x + x', z_1 + z_1' + \frac{1}{2}(M_1 x, x'), z_2 + z_2' + \frac{1}{2}(M_2 x, x'), z_3 + z_3' + \frac{1}{2}(M_3 x, x')\right), \end{aligned}$$

where $(M_k x, x')$ is the usual inner product of the vectors $M_k x \in \mathbb{R}^{4n}$, $x' \in \mathbb{R}^{4n}$. With respect to this multiplication, we have the following vector fields

$$\begin{aligned} e_{1l} &= \frac{\partial}{\partial x^{1l}} + \frac{1}{2}\left(x^{2l}\frac{\partial}{\partial z^1} - x^{4l}\frac{\partial}{\partial z^2} - x^{3l}\frac{\partial}{\partial z^3}\right) \\ e_{2l} &= \frac{\partial}{\partial x^{2l}} + \frac{1}{2}\left(-x^{1l}\frac{\partial}{\partial z^1} - x^{3l}\frac{\partial}{\partial z^2} + x^{4l}\frac{\partial}{\partial z^3}\right) \\ e_{3l} &= \frac{\partial}{\partial x^{3l}} + \frac{1}{2}\left(x^{4l}\frac{\partial}{\partial z^1} + x^{2l}\frac{\partial}{\partial z^2} + x^{1l}\frac{\partial}{\partial z^3}\right) \\ e_{4l} &= \frac{\partial}{\partial x^{4l}} + \frac{1}{2}\left(-x^{3l}\frac{\partial}{\partial z^1} + x^{1l}\frac{\partial}{\partial z^2} + x^{2l}\frac{\partial}{\partial z^3}\right) \\ e_{4n+m} &= \frac{\partial}{\partial z^m} \\ & \quad l = 1, 2, ..., n\,, \qquad m = 1, 2, 3 \end{aligned}$$

The Lie brackets of these vector fields are

$$\begin{aligned} &[e_{1l}, e_{2l}] = -e_{4n+1}, \qquad [e_{1l} e_{3l}] = Z_{4n+3}, \qquad [e_{1l}, e_{4l}] = e_{4n+2} \\ &[e_{2l}, e_{2l}] = e_{4n+2}, \qquad [e_{2l}, e_{4l}] = -Z_{4n+3}, \qquad [e_{3l}, e_{4l}] = -e_{4n+1}. \end{aligned} \tag{2.5}$$

Other brackets are equal to zero. The duals of the above vector fields are as follows:

$$dx^{kl}, \qquad \theta_r = dz^r - \frac{1}{2}(M_r x, dx), \qquad k = 1, 2, 3, 4, \qquad 1 \le l \le n, \qquad r = 1, 2, 3.$$

Given the above definitions, Q_n is two-step nilpotent. Set

$$\mathcal{V} = \{X_{1l}, X_{2l}, X_{3l}, X_{4l} | 1 \le l \le n\}, \qquad \mathcal{Z} = \{Z_1, Z_2, Z_3\}$$

If we choose an inner product on Q_n such that $\mathcal{V} \bigcup \mathcal{Z}$ is an orthonormal basis for Q_n, then the quaternion Lie group is of Heisenberg type. The solution to the Ricci flow equation in a quaternion nilpotent Lie group with the initial diagonal left-invariant metric

$$g_j(0) = g_1(0), g_{4n+1}(0) = g_{4n+2}(0) = g_{4n+3}(0), \text{for } 1 \le j \le 4n$$

is

$$\begin{cases} g_i(t) = g_1(0)\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{\frac{3}{2n+6}}, & \text{if } 1 \leq i \leq 4n; \\ g_k(t) = g_{4n+1}(0)\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{\frac{-n}{n+3}}, & \text{if } 4n+1 \leq k \leq 4n+3;. \end{cases} \tag{2.6}$$

The orthogonality of vector fields is invariant under this metric. For metric (2.6) the Ricci tensor on Q_n is diagonal and for $1 \leq I \leq 4n+3$, we have

$$R_i = -\frac{1}{2}\frac{\partial g_i}{\partial t} = \begin{cases} \frac{-6g_{4n+1}(0)}{g_1^2(0)} g_i(t)\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1}, & \text{if } 1 \leq I \leq 4n; \\ \frac{4ng_{4n+1}(0)}{g_1^2(0)} g_i(t)\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1}, & \text{if } 4n+1 \leq I \leq 4n+3; \end{cases}$$

hence

$$R = g^i R_i = \frac{-12ng_{4n+1}(0)}{g_1^2(0)} g_i(t)\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1},$$

is a constant. Also

$$\begin{aligned} Ric(\nabla f, \nabla f) &= R_i(\nabla^i f)^2 \\ &= \frac{-6g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \sum_{I=1}^{4n} g_i(t)(\nabla^i f)^2 \\ &\quad + \frac{4ng_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \sum_{I=4n+1}^{4n+3} g_i(t)(\nabla^i f)^2 \\ &= \frac{-6g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} |\nabla f|^2 \\ &\quad + \frac{(4n+6)g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \sum_{I=4n+1}^{4n+3} g_i(t)(\nabla^i f)^2. \end{aligned}$$

Using (2.1), now we compute the eigenvalues variation of the p-Laplacian on $(Q_n, g(t))$. For the un-normalized Ricci flow we have

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{-6p\lambda g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \\ &\quad + \frac{(4n+6)g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \sum_{i=4n+1}^{4n+3} \int_\Omega g^i(t)(\nabla^i f)^2 |\nabla f|^{p-2} d\mu, \end{aligned}$$

and using (2.2) for the normalized Ricci flow we can write

$$\begin{aligned} \frac{d\lambda}{dt} &= -\frac{pr\lambda}{4n+3} + \frac{-6p\lambda g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \\ &= + \frac{(4n+6)g_{4n+1}(0)}{g_1^2(0)}\left(\frac{2n+6}{g_1^2(0)} g_{4n+1}(0)t + 1\right)^{-1} \sum_{i=4n+1}^{4n+3} \int_\Omega g^i(t)(\nabla^i f)^2 |\nabla f|^{p-2} d\mu. \end{aligned}$$

Example 2. Using the notation of Example 1 and the brackets of (2.5), we compute

$$\nabla_{e_i}e_i = \frac{1}{2}\{[e_i, e_i] - (ade_i)^*e_i - (ade_i)^*e_i\} = -(ade_i)^*e_i,$$

hence

$$< \nabla_{e_i}e_i\, ,\, e_i >= - < -(ade_i)^*e_i\, ,\, e_i >= - < e_i,\, [e_i, e_i] >= 0,$$

which implies that $\Gamma^j_{ii} = 0$. Similarly to Example 1, if $p = 2$, then for $f(x) = exp(x^{11})$ where $x = (x^{11}, x^{21}, ..., x^{n1}, ..., x^{1n}, x^{2n}, ..., x^{4n}, z^1, z^2, z^3)$, we obtain

$$\Delta f = g^i e_i e_i f = g^1 e_1 e_1 f = g^1 f$$

and

$$\lambda(t) = g^1(t) = \frac{1}{g_1(0)}\left(\frac{2n+6}{g_1^2(0)}g_{4n+1}(0)t + 1\right)^{\frac{-3}{2n+6}}.$$

If $p \neq 2$, then we have

$$\Delta_p f = g^1\partial_1(|\nabla f|^{p-2}\nabla f)_1 = (g^1)^{\frac{p}{2}}(p-1)f^{p-1}$$

and

$$\lambda(t) = \left(\frac{1}{g_1(0)}\left(\frac{2n+6}{g_1^2(0)}g_{4n+1}(0)t + 1\right)^{\frac{-3}{2n+6}}\right)^{\frac{p}{2}}(p-1).$$

Definition 1. Two Riemannian manifolds are said to be isospectral if the associated Laplace-Beltrami operators have the same eigenvalue spectrum.

Definition 2. A continuous family N_t of Riemannian manifold is said to be an isospectral deformation of M_0 if the manifolds are pairwise isospectral.

Definition 3. The solution $g(t)$ of the Ricci flow with the initial condition $g(0) = g_0$ is called a Ricci soliton if there exist a smooth function $u(t)$ and a 1-parameter family of diffeomorphisms ψ_t of M^n such that

$$g(t) = u(t)\psi_t^*(g_0),\ \ u(0) = 1,\ \ \psi_0 = id_{M^n}.$$

Remark 1. Now, let (M, g) and (N, h) be two manifolds and

$$\varphi : (M, g) \to (N, h)$$

an isometry, then we have

$$^g\Delta \circ \varphi^* = \varphi^* \circ {}^h\Delta.$$

Hence, for a given diffeomorphism $\varphi : M^n \to M^n$ we have that

$$\varphi : (M^n, \varphi^* g) \to (M^n, g)$$

is an isometry, therefore we conclude that $(M^n, \varphi^* g)$,and (M^n, g) have the same spectrum

$$Spec(g) = Spec(\varphi^* g)$$

with eigenfunctions f_k, $\varphi^* f_k$ respectively. If $g(t)$ is a Ricci soliton on (M^n, g_0), then (M, g_0) and $(M, \varphi_t^* g)$ are isospectral and this implies that the family $(M^n, \psi_t^* g)$ is an isospectral deformation of (M, g_0).

3 Deformation of marked length spectrum

In the last section we investigate the eigenvalue variations of the p-Laplacian under the Ricci flow and show that the spectrum on a closed manifold is preserved under the Ricci soliton. In this section we also show that the spectrum and the marked length spectrum on a nilmanifold are preserved under the Ricci soliton.

Suppose that the Lie group N is H_n or Q_n and $g(t)$ is the solution of Ricci flow in (2.3), (2.6) respectively. If $g(t)$ is a solution of the Ricci flow on N with the additional condition $g_1(0)g_{n+1}(0) = 1$ on H_n and $g_{4n+1}(0) = g_1^2(0)$ on Q_n, then for $t = 0$ we have

$$j(Z)^2 = -|Z|^2 Id \quad \text{for all } Z \in \mathcal{Z},$$

that is the group N is of Heisenberg type. But if for the Heisenberg Lie group $(H_n, g(t))$

$$\eta_t = ((n+2)t+1)^{\frac{-2}{n+2}},$$

then for $(H_n, g(t))$ we have

$$j(Z)^2 = -\eta_t |Z|_t^2 Id \quad \text{for all } Z \in \mathcal{Z}.$$

Also if for the quaternion Lie group $(Q_n, g(t))$ we suppose that,

$$\zeta_t = ((2n+6)t+1)^{-1}$$

then in $(Q_n, g(t))$ we obtain

$$j(Z)^2 = -\zeta_t |Z|_t^2 Id \quad \text{for all } Z \in \mathcal{Z}.$$

Let $P_t = \eta_t$ or ζ_t. For H_n or Q_n we have $j(Z)^2 = -P_t|Z|_t^2 Id$.

Proposition 3.1. *Let* $|\mathcal{N}, <,>_t|$ *be the Lie algebra of* N *where* N *is* H_n *or* Q_n *. Then we have*

1. $< j(Z)X, j(Z^*)X >_t = P_t < Z, Z^* >_t < X, X >_t$ for all $Z, Z^* \in \mathcal{Z}$ *and* $X \in \mathcal{V}$;
2. $< j(Z)X, j(Z)Y >_t = P_t < Z, Z >_t < X, Y >_t$ for all $Z \in \mathcal{Z}$ and $X, Y \in \mathcal{V}$;
3. $|j(Z)X|_t = P_t^{\frac{1}{2}} |Z|_t |X|_t$ for all $Z \in \mathcal{Z}$ and $X \in \mathcal{V}$;
4. $j(Z) \circ j(Z^*) + j(Z^*) \circ j(Z) = -2P_t < Z, Z^* >_t Id$ for all $Z, Z^* \in \mathcal{Z}$;
5. $[X, j(Z)X] = P_t < X, X >_t Z$ for all $Z \in \mathcal{Z}$ and$X \in \mathcal{V}$.

Proof. 1. Take $u = j(Z)X$ and $v = j(Z^*)X$ in the relation

$$< u+v, u+v >_t - < u-v, u-v >_t = 4 < u, v >_t .$$

j is a linear mapping, therefore

$$\begin{aligned} &< j(Z+Z^*)X, j(Z+Z^*)X >_t - < j(Z-Z^*)X, j(Z-Z^*)X >_t \\ &\qquad = 4 < j(Z)X, j(Z^*)X >_t . \end{aligned}$$

j is skew symmetric, hence

$$\begin{aligned} 4 < j(Z)X, j(Z^*)X >_t &= < j(Z+Z^*)^2X, X >_t - < j(Z-Z^*)^2X, X >_t \\ &= P_t|Z+Z^*|_t^2|X|_t^2 - P_t|Z-Z^*|_t^2|X|_t^2 \\ &= 4P_t|X|_t^2 < Z, Z^* >_t \end{aligned}$$

2. j is a linear mapping, therefore

$$< j(Z)X, j(Z)Y >_t = - < j(Z)^2X, Y >_t = P_t|Z|_t^2 < X, Y >_t .$$

3. In (2), let $X = Y$, then $|j(Z)X|_t^2 = P_t|Z|_t^2|X|_t^2$.

4. For all $Z, Z^* \in \mathcal{Z}$, we have

$$\begin{aligned} -2P_t < Z, Z^* >_t Id &= -P_t(|Z + Z^*|_t^2 - |Z|_t^2 - |Z^*|_t^2)Id \\ &= (j(Z + Z^*)^2 - j(Z)^2 - j(Z^*)^2)Id \\ &= j(Z) \circ j(Z^*) + j(Z^*) \circ j(Z). \end{aligned}$$

5. For all $Z \in \mathcal{Z}$,

$$\begin{aligned} < [X, j(Z)X], Z^* > &= < (adX)(j(Z)X), Z^* > = < j(Z)X, j(Z^*)X > \\ &= P_t|X|_t^2 < Z, Z^* >_t . \end{aligned}$$

□

From [9] we have

1. $\nabla_X Y = \frac{1}{2}[X, Y]$ for all $X, Y \in \mathcal{V}$;
2. $\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X$ for all $Z \in \mathcal{Z}$ and $X \in \mathcal{V}$;
3. $\nabla_Z Z^* = 0$ for all $Z, Z^* \in \mathcal{Z}$.

Definition 4. Let $(M, g(t))$ be a complete Riemannian manifold with the tangent bundle TM. For all $v \in TM$ and $s \in \mathbb{R}$, define $G^{(s,t)}(v) = \sigma_v'(s, t)$ to be the velocity in time s of the unique geodesic $\sigma(s, t)$ with the initial speed v. The $G^{(s,t)}$ is called a geodesic flow.

Let $\sigma(s, t)$ be a curve in 2-step nilpotent Lie group with a left invariant metric $(N, g(t))$, where N is H_n or Q_n, such that $\sigma(0, t) = e$ and $\sigma'(0, t) = X_0(t) + Z_0(t)$, where $X_0(t) \in \mathcal{V}(t)$, $Z_0(t) \in \mathcal{Z}(t)$ and e is the identity in N. Using the exponential coordinates, write $\sigma(s, t) = exp(X(s, t) + Z(s, t))$, with

$$\begin{aligned} &X(s,t) \in \mathcal{V}(t), \;\; X'(0,t) = X_0(t), \;\; X(0,t) = 0 \\ &Z(s,t) \in \mathcal{Z}(t), \;\; Z'(0,t) = Z_0(t), \;\; Z(0,t) = 0. \end{aligned} \tag{3.1}$$

With the above notation from [17], we have the following statement.

Proposition 3.2. *A curve $\sigma(s, t)$ is a geodesic if and only if the following geodesic equations are satisfied:*

$$\begin{cases} X''(s,t) = j(Z_0(t))X'(s,t), \\ Z'(s,t) + \frac{1}{2}[X'(s,t), X(s,t)] = Z_0(t). \end{cases} \tag{3.2}$$

From [9] we have the following lemma and corollary.

Lemma 3.1. *With the above notion we have*

$$\sigma'(s,t) = dL_{\sigma(s,t)}\Big(e^{sj(Z_0(t))}X_0(t) + Z_0(t)\Big).$$

Corollary 3.1. *Let $\{G^{(s,t)}\}$ be a geodesic flow in TN. With the above notation, if $n \in N$ then*

$$G^{(s,t)}\Big(dL_n(X_0(t)+Y_0(t))\Big) = dL_{\sigma(s,t)}\Big(e^{sj(Z_0(t))}X_0(t)+Z_0(t)\Big).$$

Proposition 3.3. *With the above notation, if the curve $\sigma(s,t) = exp(X(s,t)+Z(s,t))$ in $(N, g(t))$ satisfies in (2.3) and (2.6) where N is H_n or Q_n, then*

$$\begin{cases} X(s,t) = (coss\theta - 1)J^{-1}X_0(t) + \frac{sins\theta}{\theta}X_0(t), \\ Z(s,t) = \left(s(1+\frac{|X_0(t)|_t^2}{2|Z_0(t)|_t^2}) + \frac{sins\theta}{\theta}\frac{|X_0(t)|_t^2}{2|Z_0(t)|_t^2}\right)Z_0(t), \end{cases} \tag{3.3}$$

where $J = j(Z_0(t))$, $\theta = \sqrt{P_t}|Z_0(t)|_t$.

Proof. We verify that the expressions for $X(s,t)$ and $Z(s,t)$ given above satisfy (3.2) together with the initial conditions. Taking $s = 0$ in (3.3), we have

$$X(0,t) = Z(0,t) = 0, \;\; X'(0,t) = X_0(t), \;\; Z'(0,t) = Z_0(t),$$

therefore $X(s,t)$ and $Z(s,t)$ satisfy the initial conditions. Also by differentiating the first equation of (3.3) we obtain

$$\frac{\partial X}{\partial s}(s,t) = -\theta(sins\theta)J^{-1}X_0(t) + coss\theta X_0(t),$$

which implies that $X(s,t)$ satisfies the first equation of (3.2). Furthermore

$$\begin{aligned} \frac{\partial^2 X}{\partial s^2}(s,t) &= -\theta^2(coss\theta)J^{-1}X_0(t) - \theta(sins\theta)X_0(t) \\ &= J(-\theta^2(coss\theta)J^{-2}X_0(t) - \theta(sins\theta)J^{-1}X_0(t)) = JX'(s,t). \end{aligned}$$

Similarity by differentiating the second equation of (3.3) we obtain

$$\frac{\partial Z}{\partial s}(s,t) = \left((1+(1-coss\theta)\frac{< X_0(t), X_0(t) >_t}{2 < Z_0(t), Z_0(t) >_t}\right)Z_0(t),$$

which implies that

$$\begin{aligned} \frac{\partial Z}{\partial s} + \frac{1}{2}[\frac{\partial X}{\partial s}, X] &= \left((1+(1-coss\theta)\frac{|X_0(t)|_t^2}{2|Z_0(t)|_t^2}\right)Z_0(t) \\ &\quad + \frac{1}{2}sin^2s\theta[X_0, J^{-1}X_0] + \frac{1}{2}(cos^2s\theta - coss\theta)[X_0, J^{-1}X_0] \\ &= \left((1+(1-coss\theta)\frac{|X_0(t)|_t^2}{2|Z_0(t)|_t^2}\right)Z_0(t) + \frac{1}{2}(1-coss\theta)[X_0, J^{-1}X_0]. \end{aligned}$$

On the other hand, using property 5 of Proposition 3.1, we have

$$[X_0, J^{-1}X_0] = [X_0, -\frac{1}{\theta^2}JX_0] = -\frac{1}{\theta^2}P_t|X_0|_t^2 Z_0 = -\frac{|X_0|_t^2}{|Z_0|_t^2}Z_0,$$

therefore

$$\frac{\partial Z}{\partial s} + \frac{1}{2}\left[\frac{\partial X}{\partial s}, X\right] = Z_0$$

which completes the proof. □

Definition 5. A nonidentity element $\varphi(t)$ of $(N, g(t))$ translates a unit speed geodesic $\sigma(s,t)$ in $(N, g(t))$ by an amount $\omega(t) > 0$ if $\varphi(t).\sigma(s,t) = \sigma(s+\omega(t), t)$ for all $s \in \mathbb{R}$. The amount $\omega(t)$ is called a period of $\varphi(t)$.

Remark 2. Let G be a 2-step nilpotent Lie group with corresponding Lie algebra $\mathcal{G}$, then for all $X, Y \in \mathcal{G}$ and also $\varphi = exp(X)$, $\psi = exp(Y)$ from [15] we have

1. $exp(X)exp(Y) = exp(X + Y + \frac{1}{2}[X, Y])$
2. $\varphi\psi\varphi^{-1} = exp(Y + [X, Y])$

Proposition 3.4. *Let $\varphi(t)$ be a family of nonidentity elements of a two-step nilpotent Lie group with the left invariant metric $(N, g(t))$ and let $d_{\varphi(t)} : (N, g(t)) \to \mathbb{R}$ be the distance function defined by $d_{\varphi(t)}(n) = d_t(n, \varphi(t).n)$. Then $d_{\varphi(t)}$ attains its minimum $\omega(t) > 0$ on $(N, g(t))$ and translates any unit speed geodesic $\sigma(s,t)$ of N by an amount $\omega(t) > 0$. The amount $\omega(t)$ is called the smallest period of $\varphi(t)$.*

Proof. Choose $V^*(t) \in \mathcal{V}$ and $Z^*(t) \in \mathcal{Z}$ so that $\varphi(t) = exp(V^*(t) + Z^*(t))$. If

$$Z_{V^*(t)} = \{exp[V^*(t), \xi(t)]; \xi(t) \in \mathcal{N}\},$$

then $Z_{V^*(t)}$ is a closed subgroup of the center of N. Therefore the set $\varphi(t).Z_{V^*(t)} = Z_{V^*(t)}.\varphi(t)$ is closed in N, and we may choose an element $\psi^*(t) \in \varphi(t).Z_{V^*(t)}$ such that

$$d_t(e, \psi^*(t)) \leq d_t(e, \psi(t)), \quad \text{for all } \psi(t) \in \varphi(t).Z_{V^*(t)}.$$

So $\omega(t) = d_t(e, \psi^*(t))$ is the minimum value of $d_{\varphi(t)}$ and the $d_{\varphi(t)}$ attains its minimum value at $exp(\xi^*(t))$, where $\xi^*(t) \in \mathcal{N}$ is any element such that

$$\begin{aligned} \psi^*(t) &= \varphi(t).exp[V^*(t), \xi^*(t)] \\ &= exp(V^* + Z^* + [V^*, \xi^*(t)]). \end{aligned}$$

If $\xi(t) \in \mathcal{N}$ is arbitrary, then

$$\begin{aligned} \psi(t) &= exp(-\xi(t)).\varphi(t).exp(\xi(t)) \\ &= exp(V^* + Z^* + [V^*, \xi(t)]) \\ &= \varphi(t).exp[V^*(t), \xi(t)] \in \varphi(t).Z_{V^*(t)}, \end{aligned}$$

hence

$$\begin{aligned} d_{\varphi(t)}exp(\xi(t)) &= d_t(e, exp(-\xi^*(t)).\varphi(t).exp(\xi^*(t))) = d_t(e, \psi^*(t)) \leq d_t(e, \psi(t)) \\ &= d_{\varphi(t)}exp(\xi(t)). \end{aligned}$$

So $d_{\varphi(t)}$ attains its minimum $\omega(t) = d(e, \psi^*(t))$ at $exp(\xi^*(t))$ and $\omega(t)$ is the smallest period of $\varphi(t)$ and $\varphi(t)$ translates any minimizing geodesic from $exp(\xi^*(t))$ to $\varphi(t).exp(\xi^*(t))$. □

Lemma 3.2. *Let N be a Heisenberg Lie group or quaternion Lie group with the left invariant metric $g(t)$. Let $(\mathcal{N}, <,>_t)$ be the associated metric Lie algebra. Let $\varphi(t) = exp(V^*(t) + Z^*(t))$ be a family of nonidentity elements of N.*

a) *if $\varphi(t)$ does not lie in the center of N, let $\xi(t) \in \mathcal{N}$ be such that $Z^*(t) = [\xi(t), V^*(t)]$. Let $\sigma(s,t) = exp(\xi(t))exp(\frac{s}{|V^*(t)|_t}(V^*(t)))$. The period of $\varphi(t)$ is precisely $|V^*(t)|_t$ i.e. $\varphi(t).\sigma(s,t) = \sigma(s + |V^*(t)|_t, t)$,*

b) *if $\varphi(t)$ is an element of the center of N, let $\xi(t) \in \mathcal{N}$ be arbitrary and $\sigma(s,t) = exp(\xi(t))exp(\frac{s}{|Z^*(t)|_t}(Z^*(t)))$, then $\varphi(t).\sigma(s,t) = \sigma(s + |Z^*(t)|_t, t)$. Also if $\omega(t)$ is another period of $\varphi(t)$ then $\omega(t) \leq |Z^*(t)|_t$.*

Proof. **a)** if $a = exp(\xi(t))$, we define

$$\varphi^*(t) = a^{-1}.\varphi(t).a = exp(V^*(t) + Z^*(t) + [V^*(t), \xi(t)]) = exp(V^*(t)).$$

The condition $\varphi(t).\sigma(s,t) = \sigma(s + \omega^*(t), t)$ for all $s \in \mathbb{R}$ is equivalent to the condition $\varphi^*(t).\gamma^*(s,t) = \gamma^*(s + \omega^*(t), t)$ where

$$\gamma^*(s,t) = a^{-1}\gamma(s,t) = exp(-\xi(t))exp(V^*(t) + Z^*(t)) = exp(\frac{s}{\omega^*}V^*(t)).$$

Note that $(\sigma^*)'(0,t) = \frac{V^*(t)}{\omega^*(t)}$ is the unit vector by the definition of $\omega^*(t)$. $\sigma^*(s,t)$ is a geodesic and $\sigma(s,t) = a.\sigma^*(s,t)$ is a unit speed geodesic. The uniqueness of the period follows by Proposition 4.5 of [9].
b) The proof is similar to the proof of a). □

Definition 6. Let N be a simply connected, nilpotent Lie group with a left invariant metric, and let $\Gamma \subseteq N$ be a discrete subgroup of N. The group Γ is said to be a lattice in N if the quotient manifold $\Gamma \setminus N$ obtained by letting Γ act on N by left translation is compact.

Proposition 3.5. *Let $(N, g(t))$ be $(H_n, g(t))$ or $(Q_n, g(t))$ and Γ be a discrete subgroup of N. Let $\varphi(t) \in \Gamma$ be a family of nonidentity elements of the center of N, such that $log\varphi(t) \in \mathcal{Z}$. Then $\varphi(t) = exp(V^*(t) + Z^*(t))$ has the following periods*

$$\left\{|Z^*(t)|_t, \sqrt{(4\pi k)(|Z^*(t)|_t - \pi k)}; \text{where } k \text{ is an integer and } \{1 \leq k \leq \frac{1}{2\pi}|Z^*(t)|_t\}\right\}.$$

Proof. Every unit speed geodesic of N is translated by some element $\varphi(t)$ of N (see [9]) and (3.3) proves the proposition. □

Definition 7. Let M be a compact Riemannian manifold. For each nontrivial free homotopy class C of closed curves in M we define $l(C)$ to be the collection of all lengths of smoothly closed geodesics that belong to C.

Definition 8. The length spectrum of a compact Riemannian manifold M is the collection of all ordered pairs (L, m), where L is the length of a closed geodesic in M and m is the multiplicity of L, i.e. m is the number of free homotopy classes C of closed curves in M that contain a closed geodesic of length L.

Lemma 3.3. *If $g(t)$ is a solution of Ricci flow in (2.3) and (2.6) then $(\Gamma \setminus H_n, g(t))$ and $(\Gamma \setminus H_n, g_0)$ have the same length spectrum, also $(\Gamma \setminus Q_n, g(t))$ and $(\Gamma \setminus Q_n, g_0)$ have the same length spectrum.*

Proof. Let $(N, g(t))$ be $(H_n, g(t))$ or $(Q_n, g(t))$. If $\varphi(t)$ belongs to a discrete group $\Gamma \subseteq N$, then the periods of $\varphi(t)$ are precisely the lengths of the closed geodesic in $\Gamma \setminus N$ that belong to the free homotopy class of closed curves in $\Gamma \setminus N$ determined by $\varphi(t)$. Therefore a free homotopy class of closed curves in $\Gamma \setminus N$ corresponds to the conjugate class of an element φ in Γ and the collection $l(C)$ is then precisely the set of periods of φ; note that the conjugatae elements of Γ have the same periods. An arbitrary nonidentity element $\varphi(t) = exp(V^*(t) + Z^*(t)) \in N$ that does not lie in the center of N, by Lemma 3.2 has a unique period $\omega(t) = |V^*(t)|_t$. Therefore in a Heisenberg

Lie group $(H_n, g(t))$, if $\{e_1, ..., e_{2n}, e_{2n+1}\}$ is a basis for $\mathcal{H}_n$ and if $V^*(t) = \Sigma_{i=1}^n a_i e_i + b_i e_{n+i}$ for some $a_i, b_i \in \mathbb{R}$, then we obtain

$$\begin{aligned} |V^*(t)|_t^2 &= \Sigma_{i=1}^n a_i^2 |e_i|_t^2 + b_i^2 |e_{n+i}|_t^2 \\ &= \Sigma_{i=1}^n a_i^2 ((n+2)t+1)^{\frac{-1}{n+2}} |e_i|_0^2 + b_i^2 ((n+2)t+1)^{\frac{-1}{n+2}} |e_{n+i}|_0^2 \\ &= ((n+2)t+1)^{\frac{-1}{n+2}} |V^*(t)|_0^2 \end{aligned}$$

and in a quaternion Lie group, if

$$V^*(t) = \Sigma_{i=1}^n a_i X_{1i} + b_i X_{2i} + c_i X_{3i} + d_i X_{4i} \text{ for some } a_i, b_i, c_i, d_i \in \mathbb{R},$$

then

$$\begin{aligned} |V^*(t)|_t^2 &= \Sigma_{i=1}^n a_i^2 |X_{1i}|_t^2 + b_i^2 |X_{2i}|_t^2 + c_i^2 |X_{3i}|_t^2 + d_i^2 |X_{4i}|_t^2 \\ &= ((2n+6)t+1)^{\frac{3}{2n+6}} \Sigma_{i=1}^n a_i^2 |X_{1i}|_0^2 + b_i^2 |X_{2i}|_0^2 + c_i^2 |X_{3i}|_0^2 + d_i^2 |X_{4i}|_0^2 \\ &= ((2n+6)t+1)^{\frac{3}{2n+6}} |V^*(t)|_0^2. \end{aligned}$$

Let

$$W^*(t) = ((2n+6)t+1)^{\frac{-3}{4n+12}} V^*(t)$$

and $\psi(t) = exp(W^*(t) + Z^*(t))$, then $|W^*(t)|_t = |V^*(t)|_0$, hence the period of $\psi(t)$ is $\omega(t)$. Also, for arbitrary nonidentity elements $\varphi(t) = exp(V^*(t) + Z^*(t)) \in N$ which are in the center of N, we have the following periods:

$$\left\{ |Z^*(t)|_t, \sqrt{(4\pi k)(|Z^*(t)|_t - \pi k)}; \text{where } k \text{ is an integer and } \{1 \leq k \leq \frac{1}{2\pi} |Z^*(t)|_t\} \right\}.$$

Therefore in a Heisenberg Lie group $(H_n, g(t)))$ we see that $Z^*(t) = ae_{2n+1}$ for some $a \in \mathbb{R}$ and

$$\begin{aligned} |Z^*(t)|_t^2 &= a^2 |e_{2n+1}|_t^2 = a^2 ((n+2)t+1)^{\frac{n}{n+2}} |e_{2n+1}|_0^2 \\ &= ((n+2)t+1)^{\frac{n}{n+2}} |Z^*(t)|_0^2. \end{aligned}$$

If in a quaternion Lie group

$$Z^*(t) = \Sigma_{i=1}^3 a_i Z_{4n+i} \text{ for some } a_i \in \mathbb{R},$$

then

$$\begin{aligned} |Z^*(t)|_t^2 &= \Sigma_{i=1}^3 a_i^2 |Z_{4n+i}|_t^2 \\ &= ((2n+6)t+1)^{\frac{-n}{n+3}} \Sigma_{i=1}^3 a_i^2 |Z_{4n+i}|_0^2 \\ &= ((2n+6)t+1)^{\frac{-n}{n+3}} |Z^*(t)|_0^2. \end{aligned}$$

So in any case the set of periods of $\varphi(t)$ is similar and this implies that the length spectrum on (H_n, g_0) or (Q_n, g_0) is preserved under the metric in (2.3) and (2.6). □

Definition 9. Two Riemannian manifolds M_1 and M_2 are said to have the same marked length spectrum if there exists an isomorphism (called a marking) $T : \pi_1(M_1) \to \pi_1(M_2)$ such that, for each $\gamma \in \pi_1(M_1)$, the collection of lengths (counting multiplicities) of closed geodesics in the free homotopy class $[\gamma]$ of M_1 coincides with the analogous collection in the free homotopy class $[T(\gamma)]$ of M_2, i.e. $l(T_*(C)) = l(C)$ for all nontrivial free homotopy classes of closed curves in M_1, where T_* denotes the induced map on free homotopy classes.

Definition 10. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to have C^k-conjugate geodesic flows if there is a C^k diffeomorphism $F : S(M_1, g_1) \to S(M_2, g_2)$ between their unit tangent bundles that intertwines their geodesic flows i.e., $F \circ G^s_{M_1} = G^s_{M_2} \circ F$ where $G^s_{M_1}$ and $G^s_{M_2}$ are geodesic flow of M_1 and M_2 respectively.

Remark 3. *C^k-conjugate geodesic flow relation between Riemannian manifolds is a transitive relation.*

Definition 11. A compact Riemannian manifold M is said to be C^k-geodesically rigid within a given class $\mathcal{M}$ of Riemannian manifolds if any Riemannian manifold M_1 in $\mathcal{M}$ whose geodesic flow is C^k-conjugate to that of M is isometric to M.

Definition 12. **(a)** Let Γ be a uniform discrete subgroup of a simply connected nilpotent Lie group N. An automorphism Φ of N is said to be Γ-almost inner if $\Phi(\gamma)$ is conjugate to γ for all $\gamma \in \Gamma$. An automorphism is said to be almost inner if $\Phi(x)$ is conjugate to x for all $x \in N$.

(b) A derivation ϕ of the Lie algebra $\mathcal{N}$ corresponding to a Lie group N is said to be Γ-almost inner (respectively almost inner) if $\phi(x) \in image(ad(x))$ for all $x \in log\Gamma$ (respectively for all $x \in \mathcal{N}$).

Theorem 3.1. *The spectrum and the marked length spectrum on a compact nilmanifold are preserved under the Ricci soliton.*

Proof. If $g(t) = \varphi_t^* g_0$ is the Ricci soliton on a compact nilmanifold $(\Gamma \setminus N, g_0)$, then

$$(\varphi_t)_* : S(\Gamma \setminus N, \varphi^* g_0) \to S(\Gamma \setminus N, g_0)$$

is a diffeomorphism intertwining their geodesic flows. Therefore $(\Gamma \setminus N, \varphi^* g_0)$ and $(\Gamma \setminus N, g_0)$ are C^k-conjugate geodesic flow whenever φ_t is C^{k+1}. On the other hand according to [12], if $(\Gamma \setminus N, g)$ and $(\Gamma^* \setminus N^*, g^*)$ are compact two-step nilmanifolds and if $F : S(\Gamma \setminus N, g) \to S(\Gamma^* \setminus N^*, g^*)$ is a homeomorphism intertwining their geodesic flows, then there exists a Γ-almost inner automorphism Φ of N such that $(\Gamma^* \setminus N^*, g^*)$ is isometric to $(\Phi(\Gamma) \setminus N, g)$, also $(\Gamma \setminus N, g)$ and $(\Phi(\Gamma) \setminus N, g)$ are isospectral. Therefore, there exists a Γ_t-almost inner automorphism Φ_t of N such that $(\Gamma \setminus N, g_0)$ is isometric to $(\Phi_t(\Gamma) \setminus N, \varphi_t^* g_0)$, and the manifolds $(\Gamma \setminus N, g_0)$ and $(\Gamma \setminus N, \varphi^* g_0)$ are isospectral. Moreover, according to [9], two compact two-step nilmanifolds $(\Gamma \setminus N, g)$ and $(\Gamma^* \setminus N^*, g^*)$ have the same marked length spectrum if and only if there exists a Γ-almost inner automorphism Φ of N such that $(\Gamma^* \setminus N^*, g^*)$ is isometric to $(\Phi(\Gamma) \setminus N, g)$. Hence, the manifolds $(\Gamma \setminus N, \varphi^* g_0)$ and $(\Gamma \setminus N, g_0)$ have the same marked length spectrum. □

Theorem 3.2. *The geodesical rigidity on a compact nilmanifold of Heisenberg type is invariant under the Ricci soliton.*

Proof. If $g(t) = \varphi_t^* g_0$ is the Ricci soliton on a compact nilmanifold $(\Gamma \setminus N, g_0)$, then

$$(\varphi_t)_* : S(\Gamma \setminus N, \varphi^* g_0) \to S(\Gamma \setminus N, g_0)$$

is a diffeomorphism intertwining their geodesic flows. $(\Gamma \setminus N, g_0)$ is of Heisenberg type. Let (M, g) is an arbitrary compact nilmanifold such that $(\Gamma \setminus N, \varphi^* g_0)$ and (M, g) have C^0-conjugate geodesic flow such that

$$F : S(\Gamma \setminus N, \varphi^* g_0) \to S(M, g),$$

then

$$F \circ ((\varphi_t)_*)^{-1} : S(\Gamma \setminus N, g_0) \to S(M, g)$$

is a C^0 diffeomorphism between $S(\Gamma \setminus N, g_0)$ and $S(M, g)$ intertwining their geodesic flows. On the other hand as in [12], compact nilmanifolds of Heisenberg type are C^0-geodesically rigid within the class of all compact nilmanifolds. Hence, the manifold $(\Gamma \setminus N, g_0)$ is isometric to (M, g). Moreover $(\Gamma \setminus N, \varphi^* g_0)$ is isometric to $(\Gamma \setminus N, g_0)$, hence $(\Gamma \setminus N, \varphi^* g_0)$ is isometric to (M, g). This implies that $(\Gamma \setminus N, \varphi^* g_0)$ is C^0-geodesically rigid within the class of all compact nilmanifolds. $\square$

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