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A PROBLEM WITH GELLERSTEDT CONDITIONS ON DIFFERENT CHARACTERISTICS FOR A MIXED LOADED EQUATION OF THE SECOND KIND

B.I. Islomov, D.A. Nasirova

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Keywords: loaded equation of the second kind, problem with Gellerstedt conditions, representation of the general solutions, energy integral method, extremum principle, integral equation with a shift.

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Abstract. This work is devoted to a formulation and an investigation of a boundary value problem with Gellerstedt conditions on different characteristics for the loaded parabolic-hyperbolic type equation of the second kind.By using the extremum principle and the method of energy integrals, there are proved the uniqueness of solution of the formulated problem, and the existence of a solution to the problem - by the method integral equations.

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1 Introduction

The study of loaded differential equations is one of the actual directions in the theory of ordinary differential equations and partial differential equations.

The first works on loaded equations were devoted to loaded integral equations. These include the works of L. Lichtenstein [28], N.N. Nazarov [35], N.M. Gunter and A.Sh. Gabibzade [12]. In the work of A.M. Nakhushev [34] there is given the most general definition of a loaded equation and a detailed classification of various loaded equations: loaded differential, integral, integro-differential, functional equations, as well as their numerous applications.

At present, the range of problems under consideration for loaded equations of the first kind of hyperbolic-parabolic and elliptic-parabolic types, when the loaded part contains only the trace or derivative of the desired function, has expanded significantly. Note the works [3], [5], [6], [8 -10], [16], [17], [19], [22], [23], [39].The obtained results on fractional differential and integral operators (see [13], [27], [32]) can be useful in the study of local and non-local problems for mixed loaded equations of the first kind, when the loaded part contains integro-differential operators in the sense of Riemann-Liouville and Caputo [7], [18], [25], [26], [40]. This is due to the fact, that the loaded equations describe the problems of optimal control [21], regulation of the soil water layer and ground moisture [33], modeling of particle transfer processes [45], problems of heat and mass transfer at a finite rate, modeling of fluid filtration in porous media [43], the study of inverse problems [29].The monographs [21], [33] contain various applications of loaded equations as a method for studying mathematical problems of biology, mathematical physics, theory of mathematical modeling of non-local processes and phenomena, theory of elastic shells.

The theory of boundary value problem with nonlocal integral condition for loaded equations was studied numerically in research work [1]. Boundary value problems for nonlinear loaded difference equations with multipoint boundary conditions have been studied by many researchers. We note works [2], [4], [36].

Boundary-value problems for mixed type equations of the second kind, in which the line of degeneracy is the envelope of a family of characteristics and is itself also a characteristic, are usually called as the mixed-type equations of the second kind, in the literature.

In works [15], [24], [30], [37], [38], [42], [46], introducing a generalized solution of the class R_2, there were studied the analognes of the Tricomi problem for a model degenerate equation of parabolic-hyperbolic and elliptic-hyperbolic types of the second kind.

Notice, that the boundary value problems for loaded degenerate equations of mixed type of the second kind have not yet been studied (see [20]). This is due, first of all, to the lack of representations of the general solution, on the other hand such problems are reduced to little-studied integral equations with a shift.

Proceeding from this, in this paper general representations of the solution to a degenerate loaded equation of parabolic-hyperbolic type of the second kind are constructed. Using the general representation and the method of energy integrals, the uniqueness of the solution to the problem with the Gellerstedt conditions on different characteristics, which were not previously known, is proved. The existence of a solution to the problem is equivalently reduced to little-studied integral equations with a shift, and a new approach is found for proving the unique solvability of such an equation.

2 Formulation of Problem

We consider the equation

$$0=\begin{cases} u_{xx}-x^{p}u_{y}-\mu_1 u\left(x,0\right), & (x,y)\in D_1,\\ u_{xx}-\left(-y\right)^{m}u_{yy}+\mu_2 u\left(x,0\right), & (x,y)\in D_2,\end{cases} \tag{2.1}$$

where $m,\quad p,\quad \mu_0\quad \mu_1\quad \mu_2$ are arbitrary real constants such that

$$0<m<1,\ p>0,\ \mu_1>0,\ \mu_2<0. \tag{2.2}$$

Let D_1 be the connected domain, bounded by segments AB, AA_0, BB_0, A_0B_0 on the lines $y=0,\quad x=0,\quad x=1,\quad y=h$, respectively;

D_{21} be the characteristic triangle, bounded by the segment $A\left(0,0\right)E\left(x_0,0\right)$ of the x axis and by two characteristics $AC_1: x-\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=0$, $EC_1: x+\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=x_0$ of equation (2.1), going out from the points $A(0;0)$, $E(x_0;0)$ and intersecting at the point $C_1\left[\frac{x_0}{2};\ -\left(\frac{2-m}{4}x_0\right)^{\frac{2}{2-m}}\right]$;

D_{22} be the characteristic triangle, bounded by the segment $E(x_0;0)B(1;0)$ of the x axis and by two characteristics $EC_2: x-\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=x_0$, $BC_2: x+\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=1$ of equation (2.1), going out from the points $E\left(x_0;0\right)$ and $B\left(1;0\right)$ and intersecting at the point $C_2\left[\frac{1+x_0}{2};-\left(\frac{2-m}{4}\left(1-x_0\right)\right)^{\frac{2}{2-m}}\right]$;

D_{23} be the characteristic rectangle, bounded by the characteristics $C_1C: x-\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=0$, EC_1, EC_2 and $C_2C: x+\frac{2}{2-m}\left(-y\right)^{\frac{2-m}{2}}=1$ of equation (2.1), intersecting at the points E, C_1, C_2 and $C\left[\frac{1}{2};\ -\left(\frac{2-m}{4}\right)^{\frac{2}{2-m}}\right]$, where $x>0,\ y<0$, and $x_0\in[0,1]$.

We denote: $J=\{(x,y):\ \ 0<x<1,\ \ y=0\}$,

$$J_1=\left\{(x,y):\ \ 0<x<x_0,\ \ y=0\right\},\quad J_2=\left\{(x,y):\ \ x_0<x<1,\ \ y=0\right\},$$

$$D_2=D_{21}\cup D_{22}\cup D_{23}\cup EC_1\cup EC_2,\quad D=D_1\cup D_2\cup J,\qquad 2\beta=m/(m-2)$$

moreover, we assume thet

$$-1 < 2\beta < 0, \tag{2.3}$$

$$\mathbb{D}^{\sigma}_{ax} f(x) = \begin{cases} \frac{sign(x-a)}{\Gamma(-\sigma)} \int_a^x \frac{f(t)dt}{|x-t|^{1+\sigma}}, & \text{at} \quad \sigma < 0, \\ f(x), & \text{at} \quad \sigma = 0, \\ [sign(x-a)]^{n+1} \frac{d^{n+1}}{dx^{n+1}} \mathbb{D}^{\sigma-(n+1)}_{ax} f(x), & \text{at} \quad \sigma > 0, \end{cases} \tag{2.4}$$

is the fractional integro-differential operator of order σ [44, c.16], $\mathbb{D}^{\sigma}_{ax} \equiv D^{\sigma}_{ax}$ at $x > a$ and $\mathbb{D}^{\sigma}_{ax} \equiv D^{\sigma}_{xa}$ at $x < a$, $n = [\sigma]$ is the integer part of the number σ.

In the domain D for equation (2.1) we investigate a boundary value problem with Gellerstedt conditions on the different characteristics.

Problem AG_1. Find in the domain D a function $u(x, y)$, with the following properties:

1) $u(x, y) \in C(\overline{D}) \cap C^1(D)$, besides $u_y(x, 0)$ can tend to infinity of order less than -2β at $x \to x_0$, in addition at $x \to 0$ and $x \to 1$ $u(x, y)$ is bounded;

2) $u(x, y) \in C^{2,1}_{x,y}(D_1)$ and it is a regular solution of equation (2.1) in the domain D_1;

3) $u(x, y)$ is a generalized solution of equation (2.1) belonging to the class R_2 [24] in the domain $D_2 \backslash \{EC_1 \cup EC_2\}$;

4) $u(x, y)$ satisfies the boundary conditions

$$u(x, y)|_{AA_0} = \varphi_1(y), \qquad u(x, y)|_{BB_0} = \varphi_2(y), \;\; 0 \le y \le h, \tag{2.5}$$

$$u|_{EC_1} = \psi_1(x), \;\; \frac{x_0}{2} \le x \le x_0, \quad u|_{EC_2} = \psi_2(x), \;\; x_0 \le x \le \frac{x_0 + 1}{2}, \tag{2.6}$$

where $\varphi_j(y)$, $\psi_j(x)$ $(j = 1, 2)$ are given functions, satisfyig the following conditions

$$\varphi_1(0) = \varphi_2(0) = 0, \quad \psi_1(x_0) = \psi_2(x_0), \tag{2.7}$$

$$\varphi_1(y), \;\; \varphi_2(y) \in C[0, h] \cap C^1(0, h), \tag{2.8}$$

$$\psi_1(x) \in C^1\left[\frac{x_0}{2}, x_0\right] \cap C^2\left(\frac{x_0}{2}, x_0\right), \psi_2(x) \in C^1\left[x_0, \frac{x_0+1}{2}\right] \cap C^2\left(x_0, \frac{x_0+1}{2}\right). \tag{2.9}$$

3 Investigation of Problem AG_1 for equation (2.1)

If conditions 1) - 3) of AG_1 are satisfied, then any regular solution to equation (2.1) can be represent in the form [16], [41]:

$$u(x, y) = \upsilon(x, y) + \omega(x), \tag{3.1}$$

where

$$\upsilon(x, y) = \begin{cases} \upsilon_1(x, y) & (x, y) \in D_1, \\ \upsilon_{2k}(x, y) & (x, y) \in D_{2k}, \end{cases} \quad (k = \overline{1, 3}), \tag{3.2}$$

$$\omega(x) = \begin{cases} \omega_1(x), & (x, 0) \in \bar{J}, \\ \omega_{2j}(x), & (x, 0) \in \bar{J}_j, \end{cases} \quad (j = 1, 2), \tag{3.3}$$

here $\upsilon_1(x, y)$ and $\upsilon_{2j}(x, y)$ are regular solutions to the equations

$$0 = \begin{cases} L\upsilon_1 \equiv \upsilon_{1\,xx} - x^p \upsilon_{1y}, & (x, y) \in D_1, \\ L\upsilon_{2j} \equiv \upsilon_{2\,jxx} - (-y)^m \upsilon_{2j\,yy}, & (x, y) \in D_{2j}, \end{cases} \tag{3.4}$$

$\omega_1(x)$, $\omega_{2j}(x)$ $(j = 1, 2)$ are arbitrary twice continuously differentiable solutions to the equations

$$\omega_1''(x) - \mu_1 \omega_1(x) = \mu_1 \upsilon_1(x, 0), \quad (x, 0) \in J, \tag{3.5}$$

$$\omega''_{2j}\left(x\right)+\mu_2\omega_{2j}\left(x\right)=-\mu_2\upsilon_{2j}\left(x,0\right),\quad (x,0)\in J_j. \tag{3.6}$$

Remark 3.1. Taking into account, that the function $ax+b$ satisfies equation (3.3), the functions $\omega_1(x)$ and $\omega_{2i}(x)$ can be defined uniquely if they satisfy the conditions

$$\omega_1(0)=\omega_1\left(1\right)=0, \tag{3.7}$$

$$\omega_{21}(0)=\omega_{21}\left(x_0\right)=0, \tag{3.8}$$

$$\omega_{22}\left(x_0\right)=\omega_{22}(1)=0. \tag{3.9}$$

Solutions to problems (3.5), (3.7) and (3.6), (3.8)((3.9)) have the forms

$$\omega_1(x)=\frac{\sqrt{\mu_1}sh\left(x-1\right)\sqrt{\mu_1}}{sh\sqrt{\mu_1}}\int\limits_0^1 sht\sqrt{\mu_1}\tau_1\left(t\right)dt-$$

$$-\sqrt{\mu_1}\int\limits_0^1 sh\sqrt{\mu_1}(x-t)\tau_1\left(t\right)dt,\quad (x,0)\in\overline{J}, \tag{3.10}$$

$$\omega_{2j}(x)=(-1)^j\frac{\sqrt{-\mu_2}sh\sqrt{-\mu_2}(x_0-x)}{sh\sqrt{-\mu_2}(x_0-\theta_j)}\int_{\theta_j}^{x_0}\tau_{2j}(t)sh\ \sqrt{-\mu_2}(t-\theta_j)\ dt-$$

$$-(-1)^j\sqrt{-\mu_2}\int_{x_0}^{x}\tau_{2j}(t)sh\sqrt{-\mu_2}\left((-1)^j(x-t)\right)dt,\qquad (x,0)\in\bar{J}_j, \tag{3.11}$$

respectively, where $\theta_j=0$ at $j=1$, $\theta_j=1$ at $j=2$, $\tau_1(x)=\upsilon_1(x,0)$, $(x,0)\in\bar{J}$, $\tau_{2j}(x)=\upsilon_{2j}(x,0)$, $(x,0)\in\bar{J}_j$.

By virtue of representation (3.1) owing to (3.7), (3.8), (3.9), Problem AG_1 is reduced to Problem AG_1^* of finding a solution to equation (3.4) in the domain D satisfying the conditions

$$\upsilon_1(x,y)|_{AA_0}=\varphi_1\left(y\right),\upsilon_1(x,y)|_{BB_0}=\varphi_2(y),\quad 0\leq y\leq h, \tag{3.12}$$

$$\upsilon_{21}|_{EC_1}=\psi_1(x)-\omega_{21}(x),\qquad \frac{x_0}{2}\leq x\leq x_0, \tag{3.13}$$

$$\upsilon_{22}|_{EC_2}=\psi_2(x)-\omega_{22}(x),\qquad x_0\leq x\leq\frac{x_0+1}{2}, \tag{3.14}$$

where $\omega_{2j}\ (x)\ (j=1,2)$ are defined in (3.11).

3.1. Function relations

The generalized solution of the class R_2 [24] of the Cauchy problem with the initial conditions

$$\upsilon_{2j}(x,-0)=\tau_{2j}(x),\ (x,0)\in\bar{J}_j,\quad \upsilon_{2jy}\left(x,-0\right)=\nu_{2j}\left(x\right),\ (x,0)\in J_j \tag{3.15}$$

for equation (3.4) in the domains $\Delta_{2j}(j=1,2)$ is given by the formula

$$\upsilon_{21}\left(\xi,\eta\right)=\int_{\xi}^{x_0}\left(t-\xi\right)^{-\beta}\left(t-\eta\right)^{-\beta}T_1\left(t\right)dt+\int_{\eta}^{\xi}\left(\xi-t\right)^{-\beta}\left(t-\eta\right)^{-\beta}N_1\left(t\right)dt, \tag{3.16}$$

$$\upsilon_{22}\left(\xi,\eta\right)=\int_{x_0}^{\eta}\left(\xi-t\right)^{-\beta}\left(\eta-t\right)^{-\beta}T_2\left(t\right)dt+\int_{\eta}^{\xi}\left(\xi-t\right)^{-\beta}\left(t-\eta\right)^{-\beta}N_2\left(t\right)dt, \tag{3.17}$$

where $\Delta_{21}=\{(\xi,\eta):\ 0<\eta<\xi, 0<\xi<x_0\}$, $\Delta_{22}=\{(\xi,\eta):\ x_0<\eta<1, \eta<\xi<1\}$,

$$\xi=x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}},\quad \eta=x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}},\tag{3.18}$$

$$\tau_{2j}(x)=(-1)^j\int\limits_{x_0}^{x}\left[(-1)^j(x-t)\right]^{-2\beta}T_j(t)dt,\quad (x,0)\in J_j,\tag{3.19}$$

$$N_j\left(x\right)=T_j\left(x\right)/2\cos\pi\beta-\gamma_2\nu_{2j}\left(x\right),\ (j=1,2),\tag{3.20}$$

besides, the functions $T_j\left(x\right)$ and $\nu_{2j}\left(x\right)$ are continuous on J_j and integrable on $\bar{J}_j$.

Substituting $\xi=x_0$, $\eta=x$ and $\eta=x_0$, $\xi=x$ into (3.16) and (3.17) respectively, taking into account (2.4), (3.13), (3.14), (3.20), $D_{xx_0}^{1-\beta}\cdot D_{xx_0}^{\beta-1}f(x)=f(x)$, $D_{x_0x}^{1-\beta}\cdot D_{x_0x}^{\beta-1}f(x)=f(x)$ [27], [44] we get

$$T_j(x)=\gamma_3\nu_{2j}(x)+\frac{2\cos\pi\beta}{\Gamma(1-\beta)}\left[(-1)^j(x-x_0)\right]^{\beta}(-1)^jD_{x_0x}^{1-\beta}\Psi_j(x),\quad (x,0)\in J_j,\tag{3.21}$$

where $\gamma_3=2\gamma_2\cos\pi\beta$, $\Psi_j(x)=\psi_j(x)-\omega_{2j}(x), (j=1,2)$.

From (3.21) and (3.19), we find the following functional relation between $\tau_{2j}(x)$ and $\nu_{2j}(x)$, which follows from D_{2i} on the I_j:

$$\tau_{2j}(x)=\gamma_3(-1)^j\int\limits_{x_0}^{x}\left[(-1)^j(x-t)\right]^{-2\beta}\nu_{2j}(t)dt+\Phi_j(x),\quad (x,0)\in\bar{J}_j,\tag{3.22}$$

where

$$\Phi_j(x)=\frac{2\Gamma(1-2\beta)\cos\pi\beta}{\Gamma(1-\beta)}D_{xx_0}^{-(1-2\beta)}\left[(-1)^j(x-x_0)\right]^{\beta}D_{xx_0}^{1-\beta}\Psi_j(x),\ \ (j=1,2).\tag{3.23}$$

According to the conditions 1) - 2) of Problem AG_1, taking into account (3.1), (3.7), passing to the limit in equation (3.4) as $y\to+0$, taking into account (3.12) and

$$\upsilon_1\left(x,+0\right)=\tau_1\left(x\right),\quad (x,0)\in\bar{J},\quad \upsilon_{1y}\left(x,+0\right)=\nu_1\left(x\right),\quad (x,0)\in J\tag{3.24}$$

we get

$$\tau_1''\left(x\right)=x^p\nu_1\left(x\right),\tag{3.25}$$

$$\begin{array}{ll}\tau_1(0)=\varphi_1(0), & \tau_1(x_0)=\psi_1(x_0),\\ \tau_1(x_0)=\psi_2(x_0), & \tau_1\left(1\right)=\varphi_2\left(0\right).\end{array}\tag{3.26}$$

Solving equations (3.25) and (3.26) considering gluing condition (see conditions of Problem AG_1), we get the second functional relation between $\tau_{2j}(x)$ and $\nu_{2j}(x)$, which follows from D_1 on J_j:

$$\tau_{2j}(x)=(-1)^{j-1}\int\limits_{\theta_j}^{x_0}G_j\left(x,t\right)t^p\nu_{2j}(t)dt+f_j(x),\quad (x,0)\in\bar{J}_j,\tag{3.27}$$

where $\theta_j=0$ at $j=1$, $\theta_j=1$ при $j=2$,

$$G_1\ \left(x,t\right)=\begin{cases}\frac{t(x-x_0)}{x_0}, & 0\le t\le x,\\ \frac{x(t-x_0)}{x_0}, & x\le t\le x_0,\end{cases}\quad G_2\ \left(x,t\right)=\begin{cases}\frac{(x-1)(t-x_0)}{1-x_0}, & x_0\le t\le x,\\ \frac{(t-1)(x-x_0)}{1-x_0}, & x\le t\le 1,\end{cases}\tag{3.28}$$

$$f_1(x)=\varphi_1(0)+\frac{x}{x_0}\left[\psi_1(x_0)-\varphi_1(0)\right], f_2(x)=\varphi_2(0)+\frac{1-x}{1-x_0}\left[\psi_2(x_0)-\varphi_2(0)\right].\tag{3.29}$$

3.2. Uniqueness of a solution to Problem AG_1

To prove the uniqueness of a solution to Problem AG_1, at the first step we prove the uniqueness of a solution to Problem AG_1^* for equation (3.4).

The following lemma plays an important role in proving the uniqueness of a solution to Problem AG_1^* for equation (3.4).

Lemma 3.1. *If conditions* (2.2), (2.3), (2.7) *are satisfied,*

$$p+2\beta>1, \quad (-y)^{-m/2}\upsilon_{21}(E)=0, \quad (-y)^{-m/2}\upsilon_{22}(B)=0, \tag{3.30}$$

and

$$\varphi_1(y)\equiv\varphi_2(y)\equiv 0, \forall y\in[0,h\,], \psi_1(x)\equiv 0, \forall x\in\left[\frac{x_0}{2},x_0\right], \psi_2(x)\equiv 0, \forall x\in\left[x_0,\frac{x_0+1}{2}\right],$$

then

$$\tau_{2j}(x)\equiv 0, \quad \forall x\in \bar J_j \quad (j=1,2), \tag{3.31}$$

where $\tau_{2j}(x)$ $(j=1,2)$ *if defined in* (3.15).

Proof. We prove this lemma using the method of energy integrals. Let $\upsilon_{2j}(x,y)$ be a twice continuously differentiable solution of the homogeneous problem AG_1^* in the domain $\bar D_{2j}^\varepsilon$, here D_{21}^ε is a domain with boundaries $\partial D_{21}^\varepsilon=\overline{A_\varepsilon C}_{1\varepsilon}\cup\overline{C_{1\varepsilon}E}_\varepsilon\cup\bar J_{1\varepsilon}$, strictly lying in the domain D_{21} for $j=1$, and for $j=2$, D_{22}^ε is a domain with boundaries $\partial D_{22}^\varepsilon=\overline{E_\varepsilon C}_{2\varepsilon}\cup\overline{C_{2\varepsilon}B}_\varepsilon\cup\bar J_{2\varepsilon}$, strictly lying in the region D_{22}, ε is a sufficiently small positive number.

Let $j=1$, then, integrating the equality

$$0=x^p(-y)^{-m}\upsilon_{21}(\upsilon_{21\,xx}-(-y)^m\,\upsilon_{21\,yy})=\frac{\partial}{\partial x}(x^p(-y)^{-m}\upsilon_{21}\upsilon_{21x})-\frac{\partial}{\partial y}(x^p\upsilon_{21}\upsilon_{21\,y})-$$

$$-x^p\left[(-y)^{-m}\upsilon_{21x}^2-\upsilon_{21y}^2\right]-px^{p-1}(-y)^{-m}\upsilon_{21}\upsilon_{21\,x} \tag{3.32}$$

over the domain $\bar D_{21}^\varepsilon$ and applying Green‘s formula, we have

$$\int_{\overline{A_\varepsilon C}_{1\varepsilon}\cup\overline{C_{1\varepsilon}E}_\varepsilon\cup\bar J_{1\varepsilon}} x^p(-y)^{-m}\upsilon_2\upsilon_{2x}dy+x^p\upsilon_2\upsilon_{2\,y}dx=\iint_{D_{21}^\varepsilon} x^p\left[(-y)^{-m}\upsilon_{2x}^2-\upsilon_{2y}^2\right]dxdy+$$

$$+p\iint_{D_{21}^\varepsilon} x^{p-1}(-y)^{-m}\upsilon_2\upsilon_{2\,x}dxdy.$$

From here, passing to the limit at $\varepsilon\to 0$, taking into account conditions (2.7) and 1)-3) of Problem AG_1^*, we obtain

$$\int_0^{x_0} x^p\tau_{21}(x)\nu_{21}(x)dx=-\int_{\overline{AC_1}} x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21}+\int_{\overline{C_1E}} x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21}-$$

$$-\iint_{D_{21}} x^p\left[(-y)^{-m}\upsilon_{21x}^2-\upsilon_{21y}^2\right]dxdy-p\iint_{D_{21}} x^{p-1}(-y)^{-m}\upsilon_{21}\upsilon_{21x}dxdy, \tag{3.33}$$

where $\tau_{21}(x)$, $\nu_{21}(x)$ are defined in (3.15) (see [11, Chapter 5, pp. 96-97]).

To calculate the right-hand side of equality (3.32), we move on to the characteristic coordinates $\xi=x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}}$, $\eta=x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}$. Further, considering (3.13), (3.14) with $\psi_1(x)=0$, $\psi_2(x)=0$ and using in the domain Δ_{21} the canonical form of hyperbolic equation (3.4) in the form: $\upsilon_{21\xi\eta} = \frac{\beta}{\xi-\eta}\left(\upsilon_{21\xi} - \upsilon_{21\eta}\right)$ from the right-hand side of equality (3.33), taking into account (3.30), we find

$$-\int_{\overline{AC_1}} x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21} = -\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta} x_0^{p+2\beta}\left(\omega_{21}\left(\frac{x_0}{2}\right)\right)^2 +$$

$$+\frac{p+2\beta}{2}\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{2\beta}\int_0^{x_0}\frac{\upsilon_{21}^2(\xi,0)}{\xi^{1-p-2\beta}}d\xi, \tag{3.34}$$

$$\int_{\overline{C_1E}} x^p(-y)^{-\frac{m}{2}}\upsilon_{21}d\upsilon_{21} = -\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta} x_0^{p+2\beta}\left(\omega_{21}\left(\frac{x_0}{2}\right)\right)^2 -$$

$$-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta} p\int_0^{x_0}\frac{(x_0+\eta)^{p-1}}{(x_0-\eta)^{-2\beta}}\upsilon^{\,2}_{21}(x_0,\eta)d\eta+$$

$$+\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta}\beta\int_0^{x_0}\frac{(x_0+\eta)^{p}}{(x_0-\eta)^{1-2\beta}}\upsilon^2_{21}(x_0,\eta)d\eta, \tag{3.35}$$

$$-\iint_{D_{21}} x^p\left[(-y)^{-m}\upsilon^2_{21x} - \upsilon^2_{21y}\right]dxdy =$$

$$= \left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta} x_0^{p+2\beta}\left(\omega_{21}\left(\frac{1}{2}\right)\right)^2 - (\beta+p)\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta}\int_0^1 \xi^{p+2\beta-1}\upsilon^2_{21}(\xi,0)d\xi+$$

$$+\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta} p\int_0^{x_0}(x_0+\eta)^{p-1}(x_0-\eta)^{2\beta}\upsilon^2_{21}(x_0,\eta)d\eta-$$

$$-\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta}\beta\int_0^{x_0}(x_0+\eta)^{p}(x_0-\eta)^{2\beta-1}\upsilon^2_{21}(x_0,\eta)d\eta-$$

$$-\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta} p(p-1))\iint_{\Delta_{21}}(\xi+\eta)^{p-2}(\xi-\eta)^{2\beta}\upsilon^2_{21}(\xi,\eta)d\xi d\eta, \tag{3.36}$$

$$-p\iint_{D_2} x^{p-1}(-y)^{-m}\upsilon_2\upsilon_{2\,x}dxdy = \left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{2\beta} p\ \times$$

$$\times\left[\int_0^{x_0}\xi^{p+2\beta-1}\upsilon^2_{21}(\xi,0)d\xi - \int_0^{x_0}(x_0+\eta)^{p-1}(x_0-\eta)^{2\beta}\upsilon^2_{21}(x_0,\eta)d\eta\right]+$$

$$+\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{2\beta} p(p-1)\iint_{\Delta_{21}}(\xi+\eta)^{p-2}(\xi-\eta)^{2\beta}\upsilon^2_{21}(\xi,\eta)d\xi d\eta. \tag{3.37}$$

Substituting (3.34)-(3.37) in (3.32) owing to (2.2), (2.3) and $p+2\beta>1$, we get

$$\int_0^{x_0} x^p\tau_{21}(x)\nu_{21}(x)dx = 0. \tag{3.38}$$

Let $j=2$, then integrating identity (3.31) over the domain D_{22} in the same way, we obtain

$$\int_{x_0}^{1} x^p\tau_{22}(x)\nu_{22}(x)dx = 0, \tag{3.39}$$

where $\tau_{22}(x)$, $\nu_{22}(x)$ are defined in (3.15).

Substituting (3.19) in (3.38) and (3.39), taking into account the conditions of Problem AG_1 and Lemma 1, as well as the equalities $\tau_{21}(0)=\tau_{22}(1)=0,\quad \tau_{2j}(x_0)=0,\ (j=1,2)$, we find

$$\int_0^{x_0} x^p\tau_{21}(x)\nu_{21}(x)dx=\int_0^{x_0}\tau_{21}(x)\tau''_{21}(x)dx=-\int_0^{x_0}\tau'^2_{21}(x)dx\le 0, \tag{3.40}$$

$$\int_{x_0}^1 x^p\tau_{22}(x)\nu_{22}(x;\lambda)dx=-\int_{x_0}^1\tau'^2_{22}(x)dx\le 0. \tag{3.41}$$

Comparing (3.40) and (3.41), we have

$$\int\limits_0^{x_0} x^p\tau_{21}(x)\nu_{21}(x)dx=0 \quad \text{if} \quad \int\limits_0^{x_0}\tau'^2_{21}(x)dx=0$$

$$\left(\int\limits_{x_0}^1 x^p\tau_{22}(x)\nu_{22}(x)dx=0 \quad \text{if} \quad \int\limits_{x_0}^1\tau'^2_{22}(x)dx=0\right).$$

This implies the validity of equality (3.31). □

By virtue of (3.2), (3.31) and condition 1) of Problem AG_1, due to the equalities $\upsilon_1(x,+0)=\upsilon_{21}(x,-0),\quad (x,0)\in\bar J_1,\ \upsilon_1(x,+0)=\upsilon_{22}(x,-0),\ (x,0)\in\bar J_2$, we get

$$\tau_1(x)\equiv 0,\quad (x,0)\in\bar J. \tag{3.42}$$

Taking into account (3.3), (3.15), (3.24), (3.31), (3.42) , from (3.10) and (3.11), we get

$$\omega(x)\equiv 0,\qquad \forall\, x\in\bar J. \tag{3.43}$$

Theorem 3.1. *If the conditions of Lemma* 3.1 *and* (3.43) *are satisfied, then Problem* AG_1^* *in the domain* D *cannot have more than one solution.*

Proof. According to the maximum principle for parabolic equations [14], boundary value problem AG_1^* for equation (3.4) in domain $\bar D_1$ with homogeneous conditions (3.12) and $\upsilon_1(x,0)=0,\quad (x,0)\in\bar J$ and (3.43) does not have a non-zero solution, i.e. $\upsilon_1(x,y)\equiv 0$ to $\bar D_1$.

Due to the uniqueness of a solution of the Cauchy problem with homogeneous conditions (3.15) for equation (3.4) in the domain D_2, taking into account (3.43), we get $\upsilon_2(x,y)\equiv 0$ in $\bar D_2$.

Consequently, from (3.2) we have

$$\upsilon(x,y)\equiv 0,\qquad (x,y)\in\bar D\ . \tag{3.44}$$

From (3.44) the uniqueness of a solution of Problem AG_1^* for equation (3.4). □

Theorem 3.2. *If the conditions of Theorem* 3.1 *are satisfied, then Problem* AG_1 *in* D *cannot have more than one solution.*

Proof. By virtue (3.42), (3.43) from (3.1) it follows, that

$$u(x,y)\equiv 0,\qquad (x,y)\in\bar D\ . \tag{3.45}$$

This proves the uniqueness of a solution to Problem AG_1 for equation (2.1). □

3.3. Existence of a solution to Problem AG_1

The existence of a solution to Problem AG_1 is proved by the method integral equations. To prove the existence of a solution to Problem AG_1, first we prove the existence of a solution to Problem AG_1^* for equation (3.4).

Theorem 3.3. *If* $p+2\beta>1$, *and conditions* (2.2), (2.3), (2.8), (2.9) *hold, then a solution to Problem* AG_1^* *in* D *exists.*

Proof. Substituting (3.27) in (3.19), taking into account the properties of operator (2.4) and gluing conditions (see the conditions of Problem AG_1), we find the function $T_i(x)$:

$$T_j(x)=\frac{sin2\beta\pi}{2\beta\pi}(-1)^{j-1}\int_{\theta_j}^{x_0}t^p\nu_{2j}(t)dt\frac{d^2}{dx^2}(-1)^j\int_{x_0}^{x}G_j\left(z,t\right)((-1)^j(x-z))^{2\beta}dz+$$

$$+\frac{(-1)^j\mathbb{D}_{xx_0}^{1-2\beta}f_j(x)}{\Gamma(1-2\beta)},\quad (j=1,2),\tag{3.46}$$

where $\theta_j=0$ at $j=1$, $\theta_j=1$ at $j=2$, $G_j\left(z,t\right)$ and $f_j(x)$ are defined in (3.28) and (3.29) respectively.

Now eliminating $T_j(x)$ from (3.21) and (3.37) owing to (3.7) and the equality $D_{0x}^{1-2\beta}g(x)=D_{0x}^{-2\beta}g'(x)$ we get the integral equation for $\nu_{2j}(x)$:

$$\nu_{2j}(x)-\int_{\theta_j}^{x_0}P_j(x,t)\nu_{2j}(t)dt=F_j(x),\quad (x,0)\in J_j,\tag{3.47}$$

where $\theta_j=0$ at $j=1$, $\theta_j=1$ at $j=2$,

$$P_j(x,t)=\frac{(-1)^{j-1}t^p}{\gamma_3}\left\{\frac{2\cos\pi\beta}{\beta\,\Gamma(1-\beta)}\frac{\mu_2\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^j\times\right.$$

$$\times\int_{\theta_j}^{x_0}G_j\left(z,t\right)sh\,\sqrt{-\mu_2}(z-\theta_j)dz+\frac{2\cos\pi\beta}{\beta\,\Gamma(1-\beta)}\frac{\mu_2\sqrt{-\mu_2}\,\left[(-1)^{j-1}(x_0-x)\right]^{\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^{j-1}\times$$

$$\times\int_{\theta_j}^{x_0}G_j\left(z,t\right)sh\,\sqrt{-\mu_2}(z-\theta_j)\,dz\int_x^{x_0}\left[(-1)^{j-1}(s-x)\right]^{\beta}sh\sqrt{-\mu_2}(x_0-s)ds-$$

$$+\frac{2\mu_2\cos\pi\beta}{\beta\Gamma(1-\beta)}\left[(-1)^{j-1}(x_0-x)\right]^{\beta}(-1)^{j-1}\int_x^{x_0}G_j\left(z,t\right)\left[(-1)^{j-1}(z-x)\right]^{\beta}dz+$$

$$+\frac{2\mu_2\sqrt{-\mu_2}\cos\pi\beta\left[(-1)^{j-1}(x_0-x)\right]^{\beta}}{\Gamma(1-\beta)}(-1)^{j-1}\int_x^{x_0}\left[(-1)^{j-1}(s-x)\right]^{\beta}ds\times$$

$$\times\int_s^{x_0}G_j\left(z,t\right)sh\sqrt{-\mu_2}(z-s)dz+$$

$$\left.+\frac{sin2\beta\pi}{2\beta\pi}\frac{d}{dx}(-1)^{j-1}\int_x^{x_0}\frac{\partial G_j\left(z,t\right)}{\partial z}\left[(-1)^{j-1}(z-x)\right]^{2\beta}dz\right\},\tag{3.48}$$

$$F_j(x)=\frac{2\mu_2\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\frac{\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^j\int_{\theta_j}^{x_0}f_j(t)sh\,\sqrt{-\mu_2}(t-\theta_j)\,dt+$$

$$+\frac{2\mu_2\sqrt{-\mu_2}\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\frac{\left[(-1)^{j-1}(x_0-x)\right]^{\beta}}{sh\sqrt{-\mu_2}\,(x_0-\theta_j)}(-1)^{j-1}\int_{\theta_j}^{x_0}f_j(z)sh\,\sqrt{-\mu_2}(z-\theta_j)\,dz\times$$

$$\times\int_{x}^{x_0}\left[(-1)^{j-1}(t-x)\right]^{\beta} sh\sqrt{-\mu_2}(x_0-t)dt-$$

$$-\frac{2\mu_2\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\left[(-1)^{j-1}(x_0-x)\right]^{\beta}(-1)^{j-1}\int_{x}^{x_0}\left[(-1)^{j-1}(t-x)\right]^{\beta} f_j(t)dt-$$

$$-\frac{2\mu_2\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\left[(-1)^{j-1}(x_0-x)\right]^{\beta}(-1)^{j-1}\int_{x}^{x_0}\left[(-1)^{j-1}(t-x)\right]^{\beta} dt\times$$

$$\times\int_{t}^{x_0} f_j(z)sh\sqrt{-\mu_2}\,(z-t)dz+\frac{sin2\pi\beta}{2\pi\beta\gamma_3}\frac{\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}}{x_0-\theta_j}\psi_j(x_0)+$$

$$+\frac{2\cos\pi\beta}{\beta\gamma_3\,\Gamma(1-\beta)}\left[(-1)^{j-1}(x_0-x)\right]^{2\beta}\psi_j'(x_0)+$$

$$\left.+\left[(-1)^{j-1}(x_0-x)\right]^{\beta}(-1)^{j-1}\int_{x}^{x_0}\left[(-1)^{j-1}(t-x)\right]^{\beta}\psi_j''(t)dt\right]. \tag{3.49}$$

By virtue of (2.2), (2.3), (2.8) and (2.9), the properties of the operator of integro-differentiation, Beta-function, hypergeometric functions [44, Chapter 1, §1, 2 and 4, pp. 4-32] and the functions $G_j(x,t)$ (3.48) and (3.49) imply that the kernel and the right-hand side of equation (3.47) admit the following estimates

$$|P_1(x,t)|\le c_1(x_0-x)^{2\beta},\qquad |P_2(x,t)|\le c_2(x-x_0)^{2\beta}, \tag{3.50}$$

$$|F_1(x)|\le c_3(x_0-x)^{2\beta},\qquad |F_2(x)|\le c_4(x-x_0)^{2\beta},\quad c_i=const>0. \tag{3.51}$$

Based on (2.8), (2.9), taking into account (3.51), we conclude that $F_j(x)\in C^2(J_j)$, and the functions $F_j(x)$ $(j=1,2)$ can go to infinity with order of growth less than -2β for $x\to x_0$, and for $x\to 0$ and $x\to 1$ they are bounded.

By virtue of (2.2), (3.50) and (3.51) equation (3.47) is a Fredholm integral equation of the second kind. According to the theory of Fredholm integral equations [31] and from the uniqueness of a solution to Problem AG_1^* (see Theorems 3.1), we conclude that integral equation (3.47) is uniquely solvable in the class $C^2\left(J_j\right)$, and the solutions $\nu_{2j}(x)$ can have the order of singularity less than -2β for $x\to x_0$, and for $x\to 0$ and $x\to 1$ are bounded and have the form:

$$\nu_{2j}(x)=F_j(x)+\int_{\theta_j}^{x_0}P_j^*(x,t)\,F_j(t)dt,\quad (x,0)\in J_j, \tag{3.52}$$

where $P_j^*(x,t)$ is the resolvent kernel.

Substituting (3.52) into (3.22) and (3.27) to the equalities $\upsilon_1(x,+0)=\upsilon_{21}(x,-0)$, $(x,0)\in\bar{J}_1$, $\upsilon_1(x,+0)=\upsilon_{22}(x,-0)$, $\quad(x,0)\in\bar{J}_2$, we find

$$\tau_j(x)\in C(\bar{J})\cap{}^2(J),\ \ (j=1,2)\,. \tag{3.53}$$

Therefore, Problem AG_1^* is uniquely solvable due to its equivalence to the Fredholm integral equation of the second kind (3.47).

Thus, the solution to Problem AG_1^* can be reconstructed in the domain D_1 as a solution of the first boundary value problem for equation (3.4), and in the domains D_{2j} (D_{23}) $(j=1,2)$ as a solution to the Cauchy (Goursat) problem for equation (3.4). This completes the study of the existence of a solution of Problem AG_1^* for equation (3.4). □

We turn to the proof of the existence of a solution to Problem AG_1.

The following theorem is true.

Theorem 3.4. *If the conditions of Theorem* 3.3 *are satisfied, then a solution to Problem* AG_1 *in* D *exists.*

Proof. By virtue (3.27) (or (3.22)) taking (3.52) into account, from (3.10) and (3.11) we find $\omega_1(x)$ and $\omega_{2j}(x)$ $(j=1,2)$. Then, a solution to Problem AG_1 in the domain can be found as $u_1(x,y) = \upsilon_1(x,y)+\omega_1(x)$, where $\upsilon_1(x,y)$ is a solution of the first boundary value problem for equation (3.4). In the domains D_{2i} and D_{23} it has the form $u_2(x,y) = \upsilon_{2j}(x,y)+\omega_{2i}(x)$, $(j=\overline{1,3})$, $(i=1,2)$, where $\upsilon_{2i}(x,y)$ $(\upsilon_{23}(x,y))$ is a solution of the Cauchy problem for equation (3.4) in the domain D_{2i} (D_{23}).

Thus, in the domain D, a solution to Problem AG_1 exists.

This completes the study of Problem AG_1 for equation (2.1). □

Example illustrating the problem.

Let $m=\frac{1}{2}$, $p=1$, $\mu_1=1$ $\mu_2=-1$, $x_0=0$, $\beta=-\frac{1}{6}$, $\varphi_1(y)\equiv\varphi_2(y)\equiv 0$, $\psi_2(x)=\psi(x)=x$, then the problem posed is reduced to Problem T_1 :

$$0=\begin{cases} u_{xx}-xu_y-u\left(x,0\right), & x>0, \quad y>0, \\ u_{xx}-\sqrt{-y}u_{yy}-u\left(x,0\right), & x>0, \quad y<0, \end{cases} \tag{3.54}$$

$$u(x,y)|_{AA_0}=0, \quad u(x,y)|_{BB_0}=0, \quad 0\le y\le h,$$

$$u|_{AC}=x, \quad 0\le x\le \frac{1}{2}.$$

In this case the conditions of Theorems 3.1, 3.2 and 3.3 are satisfied. Then formulas (3.22) and (3.27) take the form

$$\tau_2(x)=\tilde{\gamma}_3\int\limits_0^x (x-t)^{-\frac{1}{3}}\nu_2(t)dt+\Phi_2(x), \quad x\in[0,1], \tag{3.55}$$

$$\tau_1(x)=\int\limits_0^1 G_1\left(x,t\right)t\nu_1(t)dt, \quad x\in[0,1], \tag{3.56}$$

where

$$\tilde{\gamma}_3=16\sqrt{3}\left(\frac{3}{8}\right)^{4/3}\Gamma\left(\frac{1}{3}\right)/\Gamma^2\left(\frac{1}{6}\right),$$

$$G_1\ \left(x,t\right)=\begin{cases} t(x-1), & 0\le t\le x, \\ x(t-1), & x\le t\le 1, \end{cases} \tag{3.57}$$

$$\Phi_2(x)=\frac{2\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}D_{0x}^{-\frac{4}{3}}x^{-\frac{1}{6}}D_{0x}^{\frac{7}{6}}\left[x-\int\limits_0^x \tau_2\left(t\right)sh\left(x-t\right)dt\right], \quad x\in[0,1].$$

From (3.55) and (3.56) taking into account condition 1) of Problem AG_1 and that $D_{0x}^{4/3}g(x)=D_{0x}^{1/3}g'(x)$, we get the following integral equation for $\nu_2(x)$:

$$\nu_2(x)+\int\limits_0^1 \tilde{P}_2(x,t)\nu_2(t)dt=F_2(x), \quad x\in(0,1), \tag{3.58}$$

where $\tilde{P}_2(x,t)$ and $F_2(x)$ are the known functions satisfying the estimates

$$\left|\tilde{P}_2(x,t)\right|\le c_1x^{-\frac{1}{3}}, \quad |F_2(x)|\le c_2x^{-\frac{1}{3}}, \quad c_1,c_2=const>0.$$

According to the theory of Fredholm integral equations and from the uniqueness of a solution to Problem T_1 (see Theorem 3.2), we conclude that integral equation (3.58) is uniquely solvable in the class $C^2\left(0,1\right)$, and $\nu_2(x)$ has a singularity of order less than $\frac{1}{3}$ and for $x \to 0$, and for $x \to 1$, is bounded.

In the same way as above, the solution of Problem T_1 is restored.

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