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Aims and Scope

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- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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MUKHTARBAY OTELBAEV

(to the 75th birthday)

On October 3, 2017 was the 75th birthday of Mukhtarbay Otelbaev, Doctor of Physical and Mathematical Sciences (1978), Professor (1983), academician of the National Academy of Sciences of the Republic of Kazakhstan (2004), Honored Worker of the Republic of Kazakhstan (2012), laureate of the State Prize of the Republic of Kazakhstan in the field of science and technology (2007), Director of the Eurasian Mathematical Institute (since 2001), Professor of the Department вЂњFundamental MathematicsвЂќ of the L.N. Gumilyov Eurasian National University, the editor-in-chief of the Eurasian Mathematical Journal (together with V.I. Burenkov and V.A. Sadovnichy).

M. Otelbaev was born in the village of Karakemer of the Kurdai district, Zhambyl region. He graduated from the M.V. Lomonosov Moscow State University (1969) and then completed his postgraduate studies at the same university (1972). There he defended his doctor of sciences thesis (1978).

Professor Otelbaev's scientific interests are related to functional analysis, differential equations, computational mathematics, and theoretical physics.

He introduced the q-averaging, which is now called the Otelbaev function; using it he obtained a number of fundamental results. For embedding of the Sobolev weighted spaces and the resolvent of the Schrödinger operator, he established criterions for the compactness and finiteness of the type, as well as estimates of the eigenvalues of the Schrödinger and Dirac operators that are exact in order. He was the first to establish that there is no universal asymptotic formula for the distribution function of the Sturm-Liouville operator. He obtained effective conditions for the separation of the differential operators with nonsmooth and oscillating coefficients, he developed an abstract theory of extension and contraction of operators which are not necessarily linear in linear topological spaces. M. Otelbaev proposed a new numerical method for solving boundary value problems, and a method for approximate calculation of eigenvalues and eigenvectors of compact operators. He obtained the fundamental results in the theory of nonlinear evolution equations and in theoretical physics.

He has published more than 70 scientific papers in leading international journals entering the rating lists of Thomson Reuters and Scopus. Under his supervision 70 postgraduate students have defended their candidate of sciences theses, 9 of them became doctors of sciences. In 2006 and 2011 he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal congratulates Mukhtarbay Otelbaev on the occasion of his 75th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Award for the Eurasian Mathematical Journal

Dear readers, authors, reviewers and members of the Editorial Board of the Eurasian Mathematical Journal,

we are happy to inform you that in November 2017 the Eurasian Mathematical Journal was awarded the title "Leader of Science 2017" by the National Center of State Scientific-Technical Expertise of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan in the nomination "Leader of Kazakhstan Scientific Publications" for the high level of publication activities and high level of citations in Web of Science Core Collection in 2014-2016.

Recall that the Eurasian Mathematical Journal was founded by the L.N. Gumilyov Eurasian National University in 2010 in co-operation with the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia and the University of Padua (see [1]).

The journal pulishes carefully selected original research papers in all areas of mathematics, survey papers, and short communications. It publishes 4 issues in a year. The language of the paper must be English only. Papers accepted for publication are edited from the point of view of English.

More than 280 papers were published written by mathematicians from more than 40 countries representing all continents.

In 2014 the journal was registered in Scopus and in September 2014 the Elsevier-Kazakhstan Research Excellence Forum was held at the L.N. Gumilyov Eurasian National University dedicated to this occasion in which the Elsevier Chairman Professor Y.S. Chi participated (see [3] for details).

In 2015 the Eurasian Mathematical Journal was included in the list of Scopus mathematical journals, quartile Q4, and it is on the way to entering quartile Q3 (see [3]).

Attached is the invitation letter to the Rector of the L.N. Gumilyov Eurasian National University Professor E.B. Sydykov to the ceremony of awarding, which took place in Almaty on November 8, 2017.

On behalf of the Editorial Board of the EMJ V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

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Уважаемый Ерлан Батташевич!

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THIRD HANKEL DETERMINANT FOR THE RECIPROCALS OF BOUNDED TURNING FUNCTIONS

B. Venkateswarlu, D. Vamshee Krishna, N. Rani, T. RamReddy

Communicated by B.E. Kanguzhin

Key words: univalent function, function whose reciprocal derivative has a positive real part, third Hankel determinant, positive real function, Toeplitz determinants.

AMS Mathematics Subject Classification: 30C45, 30C50.

Abstract. The objective of this paper is to introduce a certain new subclass of analytic functions and obtain an upper bound for the third Hankel determinant for the functions belonging to this class, using Toeplitz determinants.

1 Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of all univalent functions. For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients give information about geometric properties of these functions. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [14] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (a_1 = 1)$$

This determinant has been considered by many authors in the literature. For example, Noor [12] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [9]. In the recent years several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions. In particular for $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For our discussion, in this paper, we consider the Hankel determinant in the case $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{1.2}$$

For $f \in A$, $a_1 = 1$, so that, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

and by applying the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.3}$$

The sharp upper bound for the second Hankel functional $H_2(2)$ for the subclass RT of S, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [11], was obtained by Janteng [8]. It was known that if $f \in RT$ then $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, ...\}$. Also the sharp upper bound for the functional $|a_3 - a_2^2|$ was $\frac{2}{3}$, stated in [2], for the class RT. Further, the best possible sharp upper bound for the functional $|a_2a_3 - a_4|$ was obtained by Babalola [2] and hence the sharp inequality for $|H_3(1)|$, for the class RT.

Motivated by the above mentioned results obtained by different authors in this direction and the result by Babalola [2], in the present paper, we introduce certain new subclass of analytic functions and seek a sharp upper bound to the functional $|a_2a_4 - a_3^2|$ and an upper bound to the third Hankel determinant, defined as follows.

Definition 1. A function $f \in A$ is said to be a function whose reciprocal derivative has a positive real part (also called the reciprocal of a bounded turning function), denoted by $f \in \widetilde{RT}$, if and only if

$$Re\left(\frac{1}{f'(z)}\right) > 0, \forall z \in E. \tag{1.4}$$

Some preliminary lemmas required for proving our results are in Section 2.

2 Preliminary results

Let $\mathscr{P}$ denote the class of all functions p of the form

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ... = 1 + \sum_{n=1}^{\infty} c_nz^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\text{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called the Caratheòdory function [3].

Lemma 2.1. [13, 15] *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. [6] *The power series for $p(z)$ given in (2.1) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad for \quad n = 1, 2, 3....$$

and $c_{-k}=\bar{c}_k$, are all non-negative. They are strictly positive except for

$$p(z)=\sum_{k=1}^{m}\rho_k p_0(e^{it_k}z),$$

$\rho_k>0$, t_k real and $t_k\neq t_j$, for $k\neq j$, where $p_0(z)=\frac{1+z}{1-z}$; in this case $D_n>0$ for $n<(m-1)$ and $D_n\doteq 0$ for $n\geq m$.

This necessary and sufficient condition found in [6] is due to Caratheòdory and Toeplitz. We may assume without restriction that $c_1>0$. On using Lemma 2.2, for $n=2$, we have

$$D_2=\begin{vmatrix} 2 & c_1 & c_2\\ \bar{c}_1 & 2 & c_1\\ \bar{c}_2 & \bar{c}_1 & 2\end{vmatrix}=[8+2Re\{c_1^2c_2\}-2\mid c_2\mid^2-4|c_1|^2]\geq 0,$$

which is equivalent to

$$2c_2 \;=\; c_1^2+x(4-c_1^2),\ \text{for some } x, |x|\leq 1. \tag{2.2}$$

For $n=3$,

$$D_3=\begin{vmatrix} 2 & c_1 & c_2 & c_3\\ \bar{c}_1 & 2 & c_1 & c_2\\ \bar{c}_2 & \bar{c}_1 & 2 & c_1\\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2\end{vmatrix}\geq\ 0.$$

and is equivalent to

$$|(4c_3-4c_1c_2+c_1^3)(4-c_1^2)+c_1(2c_2-c_1^2)^2|\leq 2(4-c_1^2)^2-2|(2c_2-c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3=c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z,$$
$$\text{for some } z,\ \text{with } |z|\leq 1. \tag{2.4}$$

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [10] and used by several authors in the literature.

3 Main results

Theorem 3.1. *If $f(z)\in\widetilde{RT}$ then*

$$|a_2a_4-a_3^2|\leq\frac{4}{9}$$

and the inequality is sharp.

Proof. For

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\widetilde{RT},$$

there exists an analytic function $p\in\mathscr{P}$ in the open unit disc E with $p(0)=1$ and $\text{Re}\{p(z)\}>0$ such that

$$\frac{1}{f'(z)}=p(z)\Leftrightarrow 1=f'(z)p(z). \tag{3.1}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$1=\Big(1+\sum_{n=2}^{\infty}na_nz^{n-1}\Big)\Big(1+\sum_{n=1}^{\infty}c_nz^n\Big).$$

upon simplification, we obtain

$$1=1+(c_1+2a_2)z+(c_2+2a_2c_1+3a_3)z^2+(c_3+2a_2c_2+3a_3c_1+4a_4)z^3+ (c_4+2a_2c_3+3a_3c_2+4a_4c_1+5a_5)z^4\cdots \quad (3.2)$$

Equating the coefficients of like powers of z, z^2, z^3 and z^4 respectively on both sides of (3.2), after simplifying, we get

$$a_2=-\frac{c_1}{2};\quad a_3=\frac{1}{3}(c_1^2-c_2);\quad a_4=-\frac{1}{4}(c_3-2c_1c_2+c_1^3);$$
$$a_5=-\frac{1}{5}(c_4-2c_1c_3+3c_1^2c_2-c_2^2-c_1^4). \quad (3.3)$$

Substituting the values of a_2, a_3 and a_4 from (3.3) in the functional $|a_2a_4-a_3^2|$ for the function $f\in\widetilde{RT}$, upon simplification, we obtain

$$|a_2a_4-a_3^2|=\frac{1}{72}\left|9c_1c_3-2c_1^2c_2-8c_2^2+c_1^4\right|, \quad (3.4)$$

which is equivalent to

$$|a_2a_4-a_3^2|=\frac{1}{72}\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|,$$
$$\text{where } d_1=9;\ d_2=-2;\ d_3=-8;\ d_4=1. \quad (3.5)$$

Substituting the values of c_2 and c_3 given in (2.2) and (2.4) respectively on the right-hand side of (3.5), we have

$$|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|=\Big|d_1c_1\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2 +2(4-c_1^2)(1-|x|^2)z\}+d_2c_1^2\times\frac{1}{2}\{c_1^2+x(4-c_1^2)\} +d_3\times\frac{1}{4}\{c_1^2+x(4-c_1^2)\}^2+d_4c_1^4\Big|. \quad (3.6)$$

Using triangle inequality and the fact that $|z|<1$, we get

$$4\mid d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\mid\le\Big|(d_1+2d_2+d_3+4d_4)c_1^4+2d_1c_1(4-c_1^2) +2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| -\big\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\big\}(4-c_1^2)|x|^2\Big|. \quad (3.7)$$

From (3.5), we can now write

$$d_1+2d_2+d_3+4d_4=1;\quad 2(d_1+d_2+d_3)=-2; \quad (3.8)$$
$$(d_1+d_3)c_1^2+2d_1c_1-4d_3=c_1^2+18c_1+32=(c_1+16)(c_1+2) \quad (3.9)$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a,b \geq 0$ in (3.9), we can have

$$-\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\} \leq -(c_1^2 - 18c_1 + 32). \tag{3.10}$$

Substituting the calculated values from (3.8) and (3.10) on the right-hand side of (3.7), we have

$$4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big| c_1^4 + 18c_1(4-c_1^2) - 2c_1^2(4-c_1^2)|x| \\ - (c_1^2 - 18c_1 + 32)(4-c_1^2)|x|^2 \Big|.$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality,we get

$$\begin{aligned} 4\ |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leq \Big[c^4 + 18c(4-c^2) + 2c^2(4-c^2)\mu \\ &\quad + (c^2 - 18c + 32)(4-c^2)\mu^2\Big] \\ &= F(c,\mu)\ ,\ 0 \leq \mu = |x| \leq 1 \ \text{ and } \ 0 \leq c \leq 2. \end{aligned} \tag{3.11}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in (3.11) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = [2c^2 + 2(c^2 - 18c + 32)\mu](4-c^2). \tag{3.12}$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.12), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. Moreover, for a fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$G(c) = 2(-c^4 - 10c^2 + 64). \tag{3.13}$$

$$G'(c) = -8c(c^3 + 5). \tag{3.14}$$

From (3.14), we observe that $G'(c) \leq 0$, for every $c \in [0,2]$. Therefore, $G(c)$ is a decreasing function of c in the interval $[0,2]$, whose maximum value occurs at $c = 0$ only. From (3.13), the maximum value of $G(c)$ at $c = 0$ is given by

$$G_{max} = G(0) = 128. \tag{3.15}$$

Simplifying the expressions (3.11) and (3.15), we get

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \ 32. \tag{3.16}$$

From the relations (3.4) and (3.16), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \ \leq \frac{4}{9}.$$

By setting $c_1 = c = 0$ and selecting $x = 1$ in the expressions (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$ respectively. Substituting these values in (3.16) together with the values in (3.5), we observe that equality is attained, which shows that our result is sharp. The extremal function in this case is given by

$$\frac{1}{f'(z)} = 1 + 2z^2 + 2z^4 + \cdots = \frac{1+z^2}{1-z^2}.$$

This completes the proof of our Theorem. □

Remark 1. It is observed that the sharp upper bound to the second Hankel determinant of a function whose derivative has a positive real part [8] and a function whose reciprocal derivative has a positive real part is the same.

Theorem 3.2. *If* $f(z) \in \widetilde{RT}$ *then* $|a_2a_3 - a_4| \le \frac{1}{6}\left[\frac{5}{3}\right]^{\frac{3}{2}}$ *and the inequality is sharp.*

Proof. Substituting the values of a_2, a_3 and a_4 from (3.3) in the functional $\mid a_2a_3 - a_4 \mid$ for the function $f \in \widetilde{RT}$, after simplifying, we get

$$\mid a_2a_3 - a_4 \mid = \frac{1}{12} \mid 3c_3 - 4c_1c_2 + c_1^3 \mid . \tag{3.17}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively on the right-hand side of (3.17), we have

$$\begin{aligned}|3c_3 - 4c_1c_2 + c_1^3| = \Big| & 3 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\ & + 2(4-c_1^2)(1- \mid x^2 \mid)z\} \\ & - 4c_1 \times \frac{1}{2}c_1^2 + x(4-c_1^2) + c_1^3\Big|.\end{aligned}$$

Using the fact $\mid z \mid < 1$, after simplifying, we get

$$4|3c_3 - 4c_1c_2 + c_1^3| \le | - c_1^3 + 6(4-c_1^2) - 2c_1(4-c_1^2)|x| + (-3c_1 - 6)(4-c_1^2)|x|^2|.$$

Since $c_1 = c \in [0,2]$, using the result $(c_1 + a) \ge (c_1 - a)$, where $a \ge 0$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned}4 \mid 3c_3 - 4c_1c_2 + c_1^3 \mid \le \mid & c^3 + 6\left(4-c^2\right) + 2c\left(4-c^2\right)\mu + 3\left(c-2\right)\left(4-c^2\right)\mu^2 \mid \\ = & F\left(c,\mu\right) \quad , \quad 0 \le \mu = \mid x \mid \le 1 \text{ and } 0 \le c \le 2.\end{aligned} \tag{3.18}$$

$$\text{Where } F(c,\mu) = c^3 + 6\left(4-c^2\right) + 2c\left(4-c^2\right)\mu + 3\left(c-2\right)\left(4-c^2\right)\mu^2.$$

Applying the same procedure described in Theorem 3.1, we observe that

$$\frac{\partial F}{\partial \mu} = [2c + 6(c-2)\mu](4-c^2) > 0 \tag{3.19}$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.19), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. Further, for a fixed $c \in [0,2]$, we have

$$\max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$G(c) = -4c^3 + 20c. \tag{3.20}$$

$$G'(c) = -12c^2 + 20. \tag{3.21}$$

$$G''(c) = -24c. \tag{3.22}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (3.21), we get

$$c = \pm\sqrt{\frac{5}{3}}.$$

Since $c \in [0,2]$, consider $c = \sqrt{\frac{5}{3}}$ only. Substituting this value in (3.21), we observe that

$$G''(c) = -24\sqrt{\frac{5}{3}} < 0.$$

By the second derivative test, $G(c)$ has maximum value at $c = \sqrt{\frac{5}{3}}$. Substituting the value of c^2 in the expression (3.20), upon simplification, we obtain the maximum value of $G(c)$ as

$$G_{max} = \frac{40}{3}\sqrt{\frac{5}{3}}. \tag{3.23}$$

Simplifying the expressions (3.18) and (3.23), we get

$$| \, 3c_3 - 4c_1c_2 + c_1^3 \, | \leq \frac{10}{3}\sqrt{\frac{5}{3}}. \tag{3.24}$$

From the relations (3.17) and (3.24), upon simplification, we obtain

$$|a_2a_3 - a_4| \leq \frac{1}{6}\left[\frac{5}{3}\right]^{\frac{3}{2}}.$$

By setting $c_1 = \sqrt{\frac{5}{3}}$ and selecting $x = 1$ in the expressions (2.2) and (2.4), we find that $c_2 = \frac{1}{3}$ and $c_3 = \frac{4}{3}\sqrt{\frac{5}{3}}$ respectively. Substituting these values in (3.24), we observe that equality is attained, which shows that our result is sharp. This completes the proof of our Theorem. □

Remark 2. It is observed that the sharp upper bound to the $|a_2a_3 - a_4|$ of a function whose derivative has a positive real part [2] and a function whose reciprocal derivative has a positive real part is the same.

The following Theorem is a straight forward verification on applying the same procedure as described in Theorems 3.1 and 3.3 and the result is sharp for the values $c_1 = 0$, $c_2 = 2$ and $x = 1$.

Theorem 3.3. *If* $f \in \widetilde{RT}$ *then* $|a_3 - a_2^2| \leq \frac{2}{3}$.

Using the fact that $|c_n| \leq 2$, $n \in N = \{1, 2, 3, \cdots\}$, with the help of c_2 and c_3 values given in 2.2 and 2.4 respectively together with the values in (3.3), we obtain $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, 4, 5, \cdots\}$.

Using the results of Theorems 3.1, 3.3, 3.5 and $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, 4, 5, \cdots\}$, we obtain the following corollary.

Corollary 3.1. *If* $f(z) \in \widetilde{RT}$ *then*

$$|H_3(1)| \;\leq\; \frac{1}{3}\left[\frac{76}{45} + \frac{1}{4}\left(\frac{5}{3}\right)^{\frac{3}{2}}\right].$$

Remark 3. It is observed that the sharp upper bound to the third Hankel determinant of a function whose derivative has a positive real part [2] and a function whose reciprocal derivative has a positive real part is the same.

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