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MUKHTARBAY OTELBAEV

(to the 75th birthday)

On October 3, 2017 was the 75th birthday of Mukhtarbay Otelbaev, Doctor of Physical and Mathematical Sciences (1978), Professor (1983), academician of the National Academy of Sciences of the Republic of Kazakhstan (2004), Honored Worker of the Republic of Kazakhstan (2012), laureate of the State Prize of the Republic of Kazakhstan in the field of science and technology (2007), Director of the Eurasian Mathematical Institute (since 2001), Professor of the Department вЂњFundamental MathematicsвЂќ of the L.N. Gumilyov Eurasian National University, the editor-in-chief of the Eurasian Mathematical Journal (together with V.I. Burenkov and V.A. Sadovnichy).

M. Otelbaev was born in the village of Karakemer of the Kurdai district, Zhambyl region. He graduated from the M.V. Lomonosov Moscow State University (1969) and then completed his postgraduate studies at the same university (1972). There he defended his doctor of sciences thesis (1978).

Professor Otelbaev's scientific interests are related to functional analysis, differential equations, computational mathematics, and theoretical physics.

He introduced the q-averaging, which is now called the Otelbaev function; using it he obtained a number of fundamental results. For embedding of the Sobolev weighted spaces and the resolvent of the Schrödinger operator, he established criterions for the compactness and finiteness of the type, as well as estimates of the eigenvalues of the Schrödinger and Dirac operators that are exact in order. He was the first to establish that there is no universal asymptotic formula for the distribution function of the Sturm-Liouville operator. He obtained effective conditions for the separation of the differential operators with nonsmooth and oscillating coefficients, he developed an abstract theory of extension and contraction of operators which are not necessarily linear in linear topological spaces. M. Otelbaev proposed a new numerical method for solving boundary value problems, and a method for approximate calculation of eigenvalues and eigenvectors of compact operators. He obtained the fundamental results in the theory of nonlinear evolution equations and in theoretical physics.

He has published more than 70 scientific papers in leading international journals entering the rating lists of Thomson Reuters and Scopus. Under his supervision 70 postgraduate students have defended their candidate of sciences theses, 9 of them became doctors of sciences. In 2006 and 2011 he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal congratulates Mukhtarbay Otelbaev on the occasion of his 75th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Award for the Eurasian Mathematical Journal

Dear readers, authors, reviewers and members of the Editorial Board of the Eurasian Mathematical Journal,

we are happy to inform you that in November 2017 the Eurasian Mathematical Journal was awarded the title "Leader of Science 2017" by the National Center of State Scientific-Technical Expertise of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan in the nomination "Leader of Kazakhstan Scientific Publications" for the high level of publication activities and high level of citations in Web of Science Core Collection in 2014-2016.

Recall that the Eurasian Mathematical Journal was founded by the L.N. Gumilyov Eurasian National University in 2010 in co-operation with the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia and the University of Padua (see [1]).

The journal pulishes carefully selected original research papers in all areas of mathematics, survey papers, and short communications. It publishes 4 issues in a year. The language of the paper must be English only. Papers accepted for publication are edited from the point of view of English.

More than 280 papers were published written by mathematicians from more than 40 countries representing all continents.

In 2014 the journal was registered in Scopus and in September 2014 the Elsevier-Kazakhstan Research Excellence Forum was held at the L.N. Gumilyov Eurasian National University dedicated to this occasion in which the Elsevier Chairman Professor Y.S. Chi participated (see [3] for details).

In 2015 the Eurasian Mathematical Journal was included in the list of Scopus mathematical journals, quartile Q4, and it is on the way to entering quartile Q3 (see [3]).

Attached is the invitation letter to the Rector of the L.N. Gumilyov Eurasian National University Professor E.B. Sydykov to the ceremony of awarding, which took place in Almaty on November 8, 2017.

On behalf of the Editorial Board of the EMJ V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

References

[1] B. Abdrayim, *Opening address by the rector of the L.N. Gumilyov Eurasian National University,* Eurasian Math. J. 1 (2010), no. 1, p. 5.

[2] *Eurasian Mathematical Journal is indexed in Scopus,* Eurasian Math. J. 5 (2014), no. 3, 6-8.

[3] V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova, *EMJ: from Scopus Q4 to Scopus Q3 in two years?!,* Eurasian Math. J. 7 (2016), no. 3, p. 6.

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Уважаемый Ерлан Батташевич!

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Президент **Ибраев А.Ж.**

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ON THE UNIFORM ZERO-TWO LAW FOR POSITIVE CONTRACTIONS OF JORDAN ALGEBRAS

F. Mukhamedov

Communicated by T. Bekjan

Key words: zero-two law, positive contraction, Jordan algebra.

AMS Mathematics Subject Classification: 47A35, 17C65, 46L70, 46L52.

Abstract. Following an idea of Ornstein and Sucheston, Foguel proved the so-called uniform "zero-two" law: let $T : L^1(X, \mathcal{F}, \mu) \to L^1(X, \mathcal{F}, \mu)$ be a positive contraction. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

In this paper we prove a non-associative version of the unform "zero-two" law for positive contractions of L_1-spaces associated with JBW-algebras.

1 Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space with a positive σ-additive measure μ. In what follows, for the sake of shortness, we denote by L_1 the usual $L_1(X, \mathcal{F}, \mu)$ space associated with $(X, \mathcal{F}, \mu)$. A linear operator $T : L_1 \to L_1$ is called a *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$.

Jamison and Orey [13] proved that if P is a Markov operator recurrent in the sense of Harris, with σ-finite invariant measure μ, then $\|P^n g\|_1 \to 0$ for every $g \in L_1$ with $\int g\, d\mu = 0$ if (and only if) the chain is aperiodic. Clearly, when the chain is not aperiodic, taking f with positive and negative parts supported in different sets of the cyclic decomposition, we have $\lim_{n\to\infty} \|P^n f\|_1 = 2\|f\|_1$.

Ornstein and Sucheston [21] obtained an analytic proof of the Jamison-Orey result, and in their work they proved the following theorem [21, Theorem 1.1].

Theorem 1.1. *Let $T : L_1 \to L_1$ be a positive contraction. Then either*

$$\sup_{\|f\|_1 \leq 1} \lim_{n\to\infty} \|T^{n+1} f - T^n f\|_1 = 2. \tag{1.1}$$

or $\|T^{n+1} f - T^n f\|_1 \to 0$ for every $f \in L_1$.

This result was later called a *strong zero-two law*. Consequently, [21, Theorem 1.3], if T is ergodic with $T^*\mathbf{1} = \mathbf{1}$ (e.g. T is ergodic and conservative), then either (1.1) holds, or $\|T^n g\|_1 \to 0$ for every $g \in L_1$ with $\int g\, d\mu = 0$. Some extensions of the strong zero-two law can be found in [1, 23].

Interchanging "sup" and "lim" in the strong zero-two law we have the following *uniform zero-two law*, proved by Foguel [10] using ideas of [21] and [9].

Theorem 1.2. *Let $T : L_1 \to L_1$ be a positive contraction. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

In [24] Zahoropol was able to reduce the proof of Theorem 1.2 to the following

Theorem 1.3. [24] *Let $T : L_1 \to L_1$ be a positive contraction. Then for the following statements:*

(i) there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - T^m\| < 2$;

(ii) there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - (T^{m+1} \wedge T^m)\| < 1$;

(iii) one has

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

the implications hold: (i)⇒(ii)⇒ (iii).

To establish the implication (ii)⇒(iii) the following auxiliary fact is needed.

Theorem 1.4. [24] *Let $T, S : L_1 \to L_1$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$ then $\|S^n - T^n\| < 1$ for all $n \in \mathbb{N}$.*

The aim of this paper is to prove a non-associative version of the uniform "zero-two" law for positive contractions of L_1-spaces associated with JBW-algebras. We emphasize that Theorem 1.2 will be included in the main result as a particular case. To prove our main result we use a Jordan analogue of Theorem 1.4 which was proved in [19].

We remark that a "zero-two" law for Markov processes was proved in [7], which allowed to study random walks on locally compact groups. Other extensions and generalizations of the formulated law have been investigated by many authors [15, 9]. In all these investigations, the generalization was in the direction of replacement of the L^1-space by an abstract Banach lattice (see for example [14, 16]). In [17, 18] we have proposed another kind of generalization of the uniform zero-two law in L^1-spaces.

Note that Jordan Banach algebras [11],[22] are non-associative real analogues of von Neumann algebras. The existence of exceptional JBW-algebras does not allow one to use the ideas and methods of von Neumann algebras [11]. A large number of papers (see for example, [2, 3]) are devoted to ergodic type theorems in Jordan algebras. It is worth mentioning that book [8] is devoted to asymptotic analysis of L_1-contractions on commutative and non-commutative settings. The motivation for these investigations arose from the problems of quantum statistical mechanics and quantum field theory (see [6]). We hope that our result will serve to prove certain limiting theorems for quantum random walks, since nowadays such activities have been reviewed by many authors (see for example, [12]) motivated by various physical reasons.

2 Preliminaries

In this section we recall some well known facts concerning Jordan algebras.

Let A be a linear space over the filed of real numbers $\mathbb{R}$. A pair $(A, \circ)$, where $\circ$ is a binary operation (i.e. multiplication), is called *Jordan algebra* if the following conditions are satisfied:

(i) $a \circ (b + c) = a \circ b + a \circ c$; $(b + c) \circ a = b \circ a + c \circ a$ for any $a, b, c \in A$;

(ii) $\lambda(a \circ b) = (\lambda a) \circ b = a \circ (\lambda b)$ for any $\lambda \in \mathbb{R}$, $a, b \in A$;

(iii) $a \circ b = b \circ a$ for any $a, b \in A$;

(iv) $a^2 \circ (b \circ a) = (a^2 \circ b) \circ a$ for any $a, b \in A$.

Let A be a Jordan algebra with unity $\mathbb{I}$ and simultaneously be a Banach space over the reals. If the norm on A is such that $\|a^2\| = \|a\|^2$ and $\|a^2\| \leq \|a^2 + b^2\|$ for all $a,\ b \in A$, then A is called a *JB-algebra* (see [4],[5],[11]). Note that in each JB-algebra A, the set $A^+ = \{a^2 : a \in A\}$ is a regular convex cone and defines a partial ordering in A, compatible with the algebraic operations. A JB-algebra A is called a *JBW-algebra* if there exists a Banach space N, which is said to be pre-dual to A, such that A is isometrically isomorphic to the space N^* of continuous linear functionals on N. So, one can introduce the $\sigma(A, N)$-weak topology on the JBW-algebra A. It is known that the pre-dual space N of a JBW-algebra A can be identified with the space of all $\sigma(A, N)$-weakly continuous linear functionals (which is denoted by A_*) on A.

Recall that a *trace* on a JBW-algebra is a map $\tau : A^+ \to [0,\ \infty]$ such that

(1) $\tau(a + \lambda b) = \tau(a) + \lambda\tau(b)$ for all $a,\ b \in A^+$ and $\lambda \in \mathbb{R}_+$, provided that $0 \cdot (\infty) = 0$,

(2) $\tau(U_s a) = \tau(a)$ for all $a \in A^+$ and $s \in A$, $s^2 = \mathbb{I}$, where $U_s x = 2s \circ (s \circ x) - s^2 \circ x$.

A trace τ is said to be *faithful* if $\tau(a) > 0$ for all $a \in A^+$, $a \neq 0$; it is normal if for each increasing net x_α in A^+ that is bounded above one has $\tau(\sup x_\alpha) = \sup \tau(x_\alpha)$; it is *semi-finite* if there exists a net $\{b_\alpha\} \subset A^+$ increasing to $\mathbb{I}$ such that $\tau(b_\alpha) < \infty$ for all α.

Throughout the paper we will consider a JBW-algebra A with a faithful semi-finite normal trace τ. Therefore, we omit this condition from the formulation of theorems.

Given $1 \leq p < \infty$, let $A_p = \{x \in A : \tau(|x|^p) < \infty\}$, here $|x|$ denotes the modules of an element x. Define the map $\|\cdot\|_p : A \to [0,\ \infty)$ by the formula $\|\cdot\|_p = (\tau(|a|^p))^{1/p}$. Then the pair $(A_p, \|\cdot\|_p)$ is a normed space (see [4]). Its completion in the norm $\|\cdot\|_p$ will be denoted by $L_p(A, \tau)$. As usual, we set $L_\infty(A, \tau) = A$ is equipped with the norm of A. It is shown [4] that the spaces $L_1(A, \tau)$ and A_* are isometrically isomorphic, therefore they can be indentified. Further we will use this fact without noting.

For more information about Jordan algebras we refer a reader to [4],[5],[11].

In the sequel we shall work with mappings of L_1-space. Therefore, we recall that a linear bounded operator $T : L_1(A, \tau) \to L_1(A, \tau)$ is *positive* if $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a *contraction* if $\|T\| \leq 1$. Here $\|T\|$ is defined as usual, i.e. $\|T\| = \sup\{\|Tx\| :\ \|x\| = 1,\ x \in L_1(A, \tau)\}$.

In [19] we have proved the following fact.

Theorem 2.1. [19] *Let $T, S : L_1(A, \tau) \longrightarrow L_1(A, \tau)$ be two positive contractions such that $T \leq S$. If there is an $n_0 \in \mathbb{N}$ such that $\|S^{n_0} - T^{n_0}\| < 1$. Then $\|S^n - T^n\| < 1$ for every $n \geq n_0$.*

This result generalizes the Zaharopol's result to a non-associative setting.

Remark 1. Unfortunately, we remark that Theorem 2.1 is no longer true if one replaces L_1-space by an L_p-space, $1 < p < \infty$. The corresponding examples were provided in [24, 19].

3 Main result

Before the formulation of main result, we need an auxiliary well-known fact.

Lemma 3.1. [15, 21] *One has*

$$\frac{1}{2^\ell}\sum_{k=0}^{\ell-1}\left|\binom{\ell}{k} - \binom{\ell}{k+1}\right| = O\Big(\frac{1}{\sqrt{\ell}}\Big)$$

Now we are ready to state a main result of the paper.

Theorem 3.1. *Let $T : L_1(A,\tau) \to L_1(A,\tau)$ be a positive contraction such that $T^*\mathbb{I} = \mathbb{I}$ (here T^* stands for the conjugate operator of T). If there are natural numbers $m, k \in \mathbb{N}$ and a positive operator $S : L_1(A,\tau) \to L_1(A,\tau)$ such that*

$$T^{m+k} \geq S, \quad T^m \geq S \quad \text{with} \tag{3.1}$$

$$\|T^{m+k} - S\| < 1, \quad \|T^m - S\| < 1. \tag{3.2}$$

Then one has

$$\lim_{n\to\infty} \|T^{n+k} - T^n\| = 0. \tag{3.3}$$

Remark 2. We stress that if one replaces A with an associative algebra, then the corresponding L_1-space coincides with the usual L_1-spaces. In this case, the operator S equals to $T^{m+k} \wedge T^m$, since in the considered case the L_1-space is a Banach lattice. In the last case, (3.1) is automatically satisfied. Hence, Theorem 3.1 is reduced to Theorem 1.3 which implies the "zero-two".

Proof. First we define

$$Q_1 = \frac{1}{2}(T^{m+k} - S) + \frac{1}{2}T^k(T^m - S).$$

It then follows from (3.1),(3.2) that Q_1 is positive and $\|Q_1\| < 1$. Moreover, one has

$$T^{m+k} = \left(\frac{I + T^k}{2}\right) S + Q_1$$

where I stands for the identity mapping.

Now put

$$Q_{\ell+1} = \left(\frac{I + T^k}{2}\right)^{\ell} Q_1 S^{\ell} + T^{m+k} Q_{\ell}, \quad \ell \in \mathbb{N}.$$

Taking into account the positivity of S and Q_1, for each $\ell \in \mathbb{N}$ we clearly get that Q_ℓ is a positive operator on $L_1(A,\tau)$. Moreover, one has

$$T^{\ell(m+k)} = \left(\frac{I + T^k}{2}\right)^{\ell} S^{\ell} + Q_{\ell}, \quad \ell \in \mathbb{N}. \tag{3.4}$$

Let us prove (3.4) by induction. Clearly, it is valid for $\ell = 1$. Assume that (3.4) is true for ℓ, and we will prove it for $\ell + 1$. Indeed, one finds

$$\begin{aligned}
T^{(\ell+1)(m+k)} &= T^{m+k}T^{\ell(m+k)} = \left(\frac{I + T^k}{2}\right)^{\ell} T^{m+k}S^{\ell} + T^{m+k}Q_{\ell} \\
&= \left(\frac{I + T^k}{2}\right)^{\ell} \left(\left(\frac{I + T^k}{2}\right) S + Q_1\right) S^{\ell} + T^{m+k}Q_{\ell} \\
&= \left(\frac{I + T^k}{2}\right)^{\ell+1} S^{\ell+1} + \left(\frac{I + T^k}{2}\right)^{\ell} Q_1 S^{\ell} + T^{m+k}Q_{\ell} \\
&= \left(\frac{I + T^k}{2}\right)^{\ell+1} S^{\ell+1} + Q_{\ell+1}
\end{aligned}$$

which proves the required equality.

Let us put $V_\ell^{(1)} = S^\ell$, and

$$V_\ell^{(d+1)} = T^{\ell(m+k)}V_\ell^{(d)} + V_\ell^{(1)}Q_\ell^d, \quad d \in \mathbb{N}.$$

One can see that for every $d, \ell \in \mathbb{N}$ the operator $V_\ell^{(d)}$ is positive. Then one has

$$T^{d\ell(m+k)} = \left(\frac{I+T^k}{2}\right)^\ell V_\ell^{(d)} + Q_\ell^d, \quad d, \ell \in \mathbb{N}. \tag{3.5}$$

Again let us prove the last equality by induction. Keeping in mind that (3.5) is true for d, it is enough to establish (3.5) for $d+1$. Indeed, we have

$$\begin{aligned}
T^{(d+1)\ell(m+k)} &= T^{\ell(m+k)}T^{d(m+k)} = T^{\ell(m+k)}\left(\left(\frac{I+T^k}{2}\right)^\ell V_\ell^{(d)} + Q_\ell^d\right) \\
&= \left(\frac{I+T^k}{2}\right)^\ell T^{\ell(m+k)}V_\ell^{(d)} + \left(\left(\frac{I+T^k}{2}\right)^\ell S^\ell + Q_\ell\right)Q_\ell^d \\
&= \left(\frac{I+T^k}{2}\right)^\ell \left(T^{\ell(m+k)}V_\ell^{(d)} + V_\ell^{(1)}Q_\ell^d\right) + Q_\ell^{d+1} \\
&= \left(\frac{I+T^k}{2}\right)^\ell V_\ell^{(d+1)} + Q_\ell^{d+1}
\end{aligned}$$

which proves (3.5).

Taking into account $T^*(\mathbb{I}) = \mathbb{I}$ and the positivity of $V_\ell^{(d)}$ and Q_ℓ, from (3.5) we immediately obtain

$$V_\ell^{(d)*}(\mathbb{I}) + Q_\ell^{*d}(\mathbb{I}) = \mathbb{I}$$

which implies that $\|V_\ell^{(d)}\| \leq 1$ and $\|Q_\ell\| \leq 1$.

From (3.1) and (3.2), due to Theorem 2.1, one finds that $\|T^{\ell m} - S^\ell\| < 1$ for all $\ell \in \mathbb{N}$. Using this inequality with $T^*(\mathbb{I}) = \mathbb{I}$ and the positivity of $T^{\ell m} - S^\ell$ we find that

$$\|\mathbb{I} - S^{*\ell}(\mathbb{I})\| = \|(T^*)^{\ell m} - S^{*\ell}\| = \|T^{\ell m} - S^\ell\| < 1, \tag{3.6}$$

here we have used a well-known fact that $\|A\| = \|A^*\|$ for every linear bounded operator on Banach space.

Now it follows from (3.4) that

$$Q_\ell^*(\mathbb{I}) = \mathbb{I} - S^{*\ell}(\mathbb{I}). \tag{3.7}$$

which with (3.6) and the positivity of Q_ℓ implies that

$$\|Q_\ell\| = \|Q_\ell^*(\mathbb{I})\| < 1 \ \text{ for all } \ \ell \in \mathbb{N}.$$

On the other hand, since T is a positive contraction, then using Lemma 3.1 we have

$$\begin{aligned}\left\|\left(\frac{I+T^k}{2}\right)^\ell - T^k\left(\frac{I+T^k}{2}\right)^\ell\right\| &= \frac{1}{2^\ell}\left\|\sum_{m=0}^{\ell}\binom{\ell}{m}(T^{km}-T^{k(m+1)})\right\| \\ &= \frac{1}{2^\ell}\left\|\sum_{m=0}^{\ell-1}\left(\binom{\ell}{m+1}-\binom{\ell}{m}\right)T^{k(m+1)} + I - T^{k(\ell+1)}\right\| \\ &\leq \frac{1}{2^\ell}\left(\sum_{m=0}^{\ell-1}\left|\binom{\ell}{m}-\binom{\ell}{m+1}\right| + 2\right) \\ &= O\left(\frac{1}{\sqrt{\ell}}\right).\end{aligned}$$

Hence, given an arbitrary $\varepsilon > 0$, we choose $\ell_\varepsilon \in \mathbb{N}$ such that

$$\left\|\left(\frac{I+T^k}{2}\right)^{\ell_\varepsilon} - T^k\left(\frac{I+T^k}{2}\right)^{\ell_\varepsilon}\right\| \leq \frac{\varepsilon}{2} \tag{3.8}$$

According to $\|Q_{\ell_\varepsilon}\| < 1$ (see (3.7)) one finds a number $d_\varepsilon \in \mathbb{N}$ such that

$$\|Q_{\ell_\varepsilon}^{d_\varepsilon}\| < \frac{\varepsilon}{4} \tag{3.9}$$

Then putting $n_0 = d_\varepsilon \ell_\varepsilon (m+k)$, from (3.5) with (3.9),(3.8) we obtain

$$\begin{aligned}\|T^{n_0+k} - T^{n_0}\| &= \|T^{d_\varepsilon \ell_\varepsilon (m+k)+k} - T^{d_\varepsilon \ell_\varepsilon (m+k)}\| \\ &\leq \left\|\left(T^k\left(\frac{I+T^k}{2}\right)^{\ell_\varepsilon} - \left(\frac{I+T^k}{2}\right)^{\ell_\varepsilon}\right)\right)V_{\ell_\varepsilon}^{(d_\varepsilon)}\right\| \\ &\quad + \|Q_{\ell_\varepsilon}^{d_\varepsilon}(T^k - I)\| \\ &\leq \frac{\varepsilon}{2} + 2\cdot\frac{\varepsilon}{4} < \varepsilon\end{aligned}$$

Take any $n \geq n_0$ then from the last inequality one gets

$$\|T^{n+k} - T^n\| = \|T^{n-n_0}(T^{n_0+k} - T^{n_0})\| \leq \|T^{n_0+k} - T^{n_0}\| < \varepsilon$$

which completes the proof.

□

Remark 3. We remark that the proved Theorem 3.1 holds if we replace L_1-space by JBW-algebra A since the dual of $L_1(A, \tau)$ is A.

Remark 4. Let M be a von Neumann algebra with a normal faithful semi-finite trace τ (see [6] for definitions). By M_{sa} we denote the set of all self-adjoint elements of M. It is known that M_{sa} is a JBW-algebra, hence, the proved theorem will be valid on preduals of von Neumann algebras. We note that a similar kind of result has been proved in [20] for positive contractions of C^*-algebras.

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References

[1] M. Akcoglu, J. Baxter, *Tail field representations and the zero-two law,* Israel J. Math. 123 (2001), 253–272.

[2] Sh.A. Ayupov, *Ergodic theorems for Markov operators in Jordan algebras. II.* Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. 5 (1982), 7–12 (in Russian).

[3] Sh.A. Ayupov, *Statistical ergodic theorems in Jordan algebras,* Russian Math. Surveys. 36 (1981), 201–202.

[4] Sh.A. Ayupov, *Classification and representation of ordered Jordan algebras*, Tashkent. Fan. 1985 (in Russian).

[5] Sh.A. Ayupov, A. Rakhimov, Sh. Usmanov, Jordan, real and Lie structures in operator algebras, Dordrecht. Kluwer Academic Publishers. 1997.

[6] O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics.* Part I. New York-Heidelberg -Berlin. Springer-Verlag. 1979.

[7] Y. Derriennic, *Lois "zéro ou deux" pour les processes de Markov. Applications aux marches aléatoires.* Ann. Inst. H. Poincaré Sec. B 12 (1976), 111–129.

[8] E.Ya. Emel'yanov, *Non-spectral asymptotic analysis of oneparameter operator semigroups.* In: Operator Theory: Advances and Applications, 173. Basel. Birkhauser Verlag. 2007.

[9] S.R. Foguel, *On the "zero-two" law,* Israel J. Math. 10 (1971), 275–280.

[10] S.R. Foguel, *More on the "zero-two" law,* Proc. Amer. Math. Soc. 61 (1976), 262–264.

[11] H. Hanche-Olsen, E. Stormer, *Jordan operator algebras*, Boston, Pitman Advanced Publs. Program. 1984.

[12] R. Jajte *Strong linit theorems in non-commutative probability*, Lecture Notes in Math, vol. 1110, Berlin-Heidelberg. Springer-Verlag. 1984.

[13] B. Jamison, S. Orey, *Markov chains recurrent in the sense of Harris,* Z. Wahrsch. Verw. Geb. 8 (1967), 41–48.

[14] Y. Katznelson, L. Tzafriri, *On power bounded operators,* J. Funct. Anal. 68 (1986), 313–328.

[15] M. Lin, *On the 0–2 law for conservative Markov operators,* Z. Wahrscheinlichkeitstheor. Verw. Geb. 61 (1982), 513–525.

[16] M. Lin, *The uniform zero-two law for positive operators in Banach lattices,* Studia Math. 131 (1998), 149–153.

[17] F. Mukhamedov, *On dominant contractions and a generalization of the zero-two law.* Positivity. 15 (2011), 497-–508.

[18] F. Mukhamedov, *On a multi-parametric generalization of the uniform zero-two law in L^1-spaces,* Bull. Korean Math. Soc. 52 (2015), 1819–1826.

[19] F. Mukhamedov, S. Temir, H. Akin, *A note on dominant contractions of Jordan algebras,* Turk. J. Math. 34 (2010), 85–93.

[20] S. Neshveyev, L. Tuset, *The Martin boundary of a discrete quantum group,* J. Reine und Angewandte Math 538 (2004), 23–70.

[21] D. Ornstein, L. Sucheston, *An operator theorem on L_1 convergence to zero with applications to Markov operators,* Ann. Math. Statist. 41 (1971), 1631–1639.

[22] F.W. Shultz, *On normed Jordan algebras which are Banach dual spaces.* J. Funct. Anal. 31 (1979), 360–376.

[23] R. Wittmann, *Analogues of the "zero-two" law for positive linear contractions in L_p and $C(X)$,* Israel J. Math. 59 (1987), 8–28.

[24] R. Zaharopol, *On the 'zero-two' law for positive contraction,* Proc. Edin. Math. Soc. 32 (1989), 363–370.

[25] R. Zaharopol, *The modulus of a regular liniear operators and the "zero-two" law in L_p-spaces ($1 < p < \infty$, $p \neq 2$),* J. Funct. Anal. 68 (1986), 300–312.

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