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Aims and Scope

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- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

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- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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MUKHTARBAY OTELBAEV

(to the 75th birthday)

On October 3, 2017 was the 75th birthday of Mukhtarbay Otelbaev, Doctor of Physical and Mathematical Sciences (1978), Professor (1983), academician of the National Academy of Sciences of the Republic of Kazakhstan (2004), Honored Worker of the Republic of Kazakhstan (2012), laureate of the State Prize of the Republic of Kazakhstan in the field of science and technology (2007), Director of the Eurasian Mathematical Institute (since 2001), Professor of the Department вЂњFundamental MathematicsвЂќ of the L.N. Gumilyov Eurasian National University, the editor-in-chief of the Eurasian Mathematical Journal (together with V.I. Burenkov and V.A. Sadovnichy).

M. Otelbaev was born in the village of Karakemer of the Kurdai district, Zhambyl region. He graduated from the M.V. Lomonosov Moscow State University (1969) and then completed his postgraduate studies at the same university (1972). There he defended his doctor of sciences thesis (1978).

Professor Otelbaev's scientific interests are related to functional analysis, differential equations, computational mathematics, and theoretical physics.

He introduced the q-averaging, which is now called the Otelbaev function; using it he obtained a number of fundamental results. For embedding of the Sobolev weighted spaces and the resolvent of the Schrödinger operator, he established criterions for the compactness and finiteness of the type, as well as estimates of the eigenvalues of the Schrödinger and Dirac operators that are exact in order. He was the first to establish that there is no universal asymptotic formula for the distribution function of the Sturm-Liouville operator. He obtained effective conditions for the separation of the differential operators with nonsmooth and oscillating coefficients, he developed an abstract theory of extension and contraction of operators which are not necessarily linear in linear topological spaces. M. Otelbaev proposed a new numerical method for solving boundary value problems, and a method for approximate calculation of eigenvalues and eigenvectors of compact operators. He obtained the fundamental results in the theory of nonlinear evolution equations and in theoretical physics.

He has published more than 70 scientific papers in leading international journals entering the rating lists of Thomson Reuters and Scopus. Under his supervision 70 postgraduate students have defended their candidate of sciences theses, 9 of them became doctors of sciences. In 2006 and 2011 he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal congratulates Mukhtarbay Otelbaev on the occasion of his 75th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Award for the Eurasian Mathematical Journal

Dear readers, authors, reviewers and members of the Editorial Board of the Eurasian Mathematical Journal,

we are happy to inform you that in November 2017 the Eurasian Mathematical Journal was awarded the title "Leader of Science 2017" by the National Center of State Scientific-Technical Expertise of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan in the nomination "Leader of Kazakhstan Scientific Publications" for the high level of publication activities and high level of citations in Web of Science Core Collection in 2014-2016.

Recall that the Eurasian Mathematical Journal was founded by the L.N. Gumilyov Eurasian National University in 2010 in co-operation with the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia and the University of Padua (see [1]).

The journal pulishes carefully selected original research papers in all areas of mathematics, survey papers, and short communications. It publishes 4 issues in a year. The language of the paper must be English only. Papers accepted for publication are edited from the point of view of English.

More than 280 papers were published written by mathematicians from more than 40 countries representing all continents.

In 2014 the journal was registered in Scopus and in September 2014 the Elsevier-Kazakhstan Research Excellence Forum was held at the L.N. Gumilyov Eurasian National University dedicated to this occasion in which the Elsevier Chairman Professor Y.S. Chi participated (see [3] for details).

In 2015 the Eurasian Mathematical Journal was included in the list of Scopus mathematical journals, quartile Q4, and it is on the way to entering quartile Q3 (see [3]).

Attached is the invitation letter to the Rector of the L.N. Gumilyov Eurasian National University Professor E.B. Sydykov to the ceremony of awarding, which took place in Almaty on November 8, 2017.

On behalf of the Editorial Board of the EMJ V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

References

[1] B. Abdrayim, *Opening address by the rector of the L.N. Gumilyov Eurasian National University,* Eurasian Math. J. 1 (2010), no. 1, p. 5.

[2] *Eurasian Mathematical Journal is indexed in Scopus,* Eurasian Math. J. 5 (2014), no. 3, 6-8.

[3] V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova, *EMJ: from Scopus Q4 to Scopus Q3 in two years?!,* Eurasian Math. J. 7 (2016), no. 3, p. 6.

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
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Ректору
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Уважаемый Ерлан Батташевич!

АО «Национальный центр государственной научно-технической экспертизы» (далее АО «НЦГНТЭ») и компания Clarivate Analytics имеют честь пригласить Вас на церемонию вручения независимой награды «Лидер науки-2017» за высокие показатели публикационной активности и цитируемости в Web of Science Core Collection в период 2014-2016 годы.

Ваш журнал *«Eurasian Mathematical Journal»* награждается в номинации «**Лидер казахстанских научных изданий**».

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Президент **Ибраев А.Ж.**

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ON THE GENERALIZED RIESZ-DUAL SEQUENCES IN HILBERT SPACES

F. Enayati, M.S. Asgari

Communicated by E. Kissin

Key words: G-frames, g-orthonormal bases, g-Riesz basis, g-Riesz-dual sequence.

AMS Mathematics Subject Classification: 41A58, 42C15, 42C40.

Abstract. In this paper we define the generalized Riesz-dual sequence from a g-Bessel sequence with respect to a pair of g-orthonormal bases as a generalization of the Riesz-dual sequence. We characterize exactly properties of the first sequence in terms of the associated one, which yields duality relations for the abstract g-frame setting.

1 Introduction

Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ be orthonormal bases for a separable Hilbert space $\mathcal{H}$ and let $f = \{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty$ for all $j \in I$. In [4], Casazza, Kutyniok, and Lammers introduced the Riesz-dual sequence (R-dual sequence) of $\{f_i\}_{i\in I}$ with respect to $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ as the sequence $\{w_j^f\}_{j\in I}$ given by

$$w_j^f = \sum_{i\in I} \langle f_i, e_j\rangle h_i, \qquad \forall j \in I. \tag{1.1}$$

The paper [4] demonstrates that there is a strong relationship between the frame theoretic properties of $\{w_j^f\}_{j\in I}$ and $\{f_i\}_{i\in I}$. For more details we refer to [6, 7, 11, 14]. The purpose of this paper is to introduce the concept of Riesz-dual sequence for g-frames. We give characterizations of g-R-dual sequences and prove that g-R-dual sequences share many useful properties with R-dual sequences.

In 2006, a new generalization of the frame named g-frame was introduced by Sun [12] in a complex Hilbert space. G-frames are natural generalizations of frames which cover many other recent generalizations of frames, e.g., bounded quasi-projectors, frames of subspaces, outer frames, oblique frames, pseudo-frames and a class of time-frequency localization operators. Sun showed that all of the above applications of frames are special cases of g-frames. For more information about the theory and applications of g-frames we refer to [3, 5, 10, 13].

Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces and let $\{V_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{K}$ and $B(\mathcal{H}, V_i)$ denote the collection of all bounded linear operators from $\mathcal{H}$ into V_i for all $i \in I$. Recall that a family $\Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if there exist constants $0 < C \le D < \infty$ such that:

$$C\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le D\|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{1.2}$$

The constants C and D are called g-frame bounds. If only the right-hand inequality of (1.2) is required, we call it a g-Bessel sequence. We denote the representation space associated with a g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ as follows:

$$\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}=\Big\{\{g_i'\}_{i\in I}|\ \ g_i'\in V_i,\ \ \sum_{i\in I}\|g_i'\|^2<\infty\Big\}.$$

In order to analyze a signal $f\in\mathcal{H}$, i.e., to map it into the representation space, the analysis operator $T_\Lambda:\mathcal{H}\to\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ given by $T_\Lambda f=\{\Lambda_i f\}_{i\in I}$ is applied. The associated synthesis operator, which provides a mapping from the representation space to $\mathcal{H}$, is defined to be the adjoint operator $T_\Lambda^*:\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}\to\mathcal{H}$, which is given by $T_\Lambda^*(\{g_i'\}_{i\in I})=\sum_{i\in I}\Lambda_i^*g_i'$. By composing T_Λ and T_Λ^* we obtain the g-frame operator $S_\Lambda:\mathcal{H}\to\mathcal{H}$, $S_\Lambda f=T_\Lambda^*T_\Lambda f=\sum_{i\in I}\Lambda_i^*\Lambda_i f$, which is a positive, self-adjoint and invertible operator and such that $CI_\mathcal{H}\le S_\Lambda\le DI_\mathcal{H}$. The canonical dual g-frame for $\{\Lambda_i\}_{i\in I}$ is defined by $\{\widehat{\Lambda}_i\}_{i\in I}$ where $\widehat{\Lambda}_i=\Lambda_i S_\Lambda^{-1}$ which is also a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with $\frac{1}{D}$ and $\frac{1}{C}$ as its lower and upper g-frame bounds, respectively. Also we have

$$f=\sum_{i\in I}\Lambda_i^*\widehat{\Lambda}_i f=\sum_{i\in I}\widehat{\Lambda}_i^*\Lambda_i f,\qquad \forall f\in\mathcal{H}.$$

Moreover, $\{\Lambda_i S_\Lambda^{-\frac{1}{2}}\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$ with respect $\{V_i\}_{i\in I}$.

Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in the following examples.

Example 1. Let $\mathcal{H}=\mathbb{C}^{n+1}$ and $V_1=V_2=\ldots=V_{n+1}=\mathbb{C}^n$. Define

$$\Lambda_1=\begin{bmatrix}-1&0&0&\ldots&0&0\\1&0&0&\ldots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\1&0&0&\ldots&0&0\end{bmatrix},\quad\cdots\quad\Lambda_n=\begin{bmatrix}0&0&0&\ldots&1&0\\0&0&0&\ldots&1&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\ldots&-1&0\end{bmatrix},$$

$$\Lambda_{n+1}=\begin{bmatrix}0&0&0&\ldots&0&1\\0&0&0&\ldots&0&1\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\ldots&0&1\end{bmatrix}.$$

Then, the set $\{\Lambda_i\}_{i=1}^{n+1}$ is a n-tight g-frame for $\mathbb{C}^{n+1}$ with respect to $\mathbb{C}^n$. To see this explicitly, note that for any $f=(z_1,z_2,\ldots,z_{n+1})\in\mathbb{C}^{n+1}$, we have

$$\sum_{i=1}^{n+1}\|\Lambda_i f\|^2=n(|z_1|^2+|z_2|^2+\ldots+|z_{n+1}|^2)=n\|f\|^2.$$

The structure of this paper is as follows: In the rest of this section we will briefly recall the required facts of the theory g-orthonormal bases. For more information about g-orthonormal bases we refer to [8, 9]. We also, define the g-R-dual sequence from a g-Bessel sequence with respect to a pair of g-orthonormal bases as a generalization of the Riesz-dual sequence. We invert the process and calculate the g-Bessel sequence from the g-R-dual sequence. In Section 2, first we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. We also characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent).

Definition 1. Let $\{\Xi_i\in B(\mathcal{H},W_i)|\ i\in I\}$ be a sequence of operators. Then (*i*) $\{\Xi_i\}_{i\in I}$ is a g-complete set for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, if $\mathcal{H} = \overline{\operatorname{span}}\{\Xi_i^*(W_i)\}_{i\in I}$.

(*ii*) $\{\Xi_i\}_{i\in I}$ is an g-orthonormal system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, if $\Xi_i\Xi_j^* = \delta_{ij}I_{W_j}$ for all $i, j \in I$.

(*iii*) A g-complete and g-orthonormal system $\{\Xi_i\}_{i\in I}$ is called an g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

The following well-known and useful characterization of g-orthonormal bases is taken from [2].

Lemma 1.1. *Let $\Xi = \{\Xi_i\}_{i\in I}$ be an g-orthonormal system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then the following conditions are equivalent:*

(*i*) *Ξ is an g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.*

(*ii*) $\sum_{i\in I}\Xi_i^*\Xi_i = I_{\mathcal{H}}$.

(*iii*) $\|f\|^2 = \sum_{i\in I}\|\Xi_i f\|^2 \quad \forall f \in \mathcal{H}$.

(*iv*) *If $\Xi_i f = 0$ for all $i \in I$, then $f = 0$.*

Let $\Xi = \{\Xi_i\}_{i\in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. If $f = \sum_{i\in I}\Xi_i^* g_i$, then the coordinate representation of $f \in \mathcal{H}$ relative to the g-orthonormal basis Ξ is $[f]_\Xi = \{g_i\}_{i\in I}$. In this case $\{g_i\}_{i\in I} \in \left(\sum_{i\in I}\oplus W_i\right)_{\ell^2}$ and $\|f\| = \left\|[f]_\Xi\right\|_{\ell^2}$.

Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Xi' = \{\Xi_i'\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ respectively. The transition matrix from Ξ to Ξ' is the matrix $B = [B_{ij}]$ whose (i, j)-entry is $B_{ij} = \Xi_i'\Xi_j^*$ for all $i, j \in I$. Then we have $B[f]_\Xi = [f]_{\Xi'}$ where, $[f]_\Xi$ is the coordinate representation of an arbitrary vector $f \in \mathcal{H}$ in the basis Ξ and similarly for Ξ'. The transition matrix from Ξ' to Ξ is $B^{-1} = B^*$. Thus, if $B^* = [B_{ij}^*]$ then $B_{ij}^* = (B_{ji})^* = \Xi_i\Xi_j'^*$ for all $i, j \in I$.

Example 2. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and let $\{W_j\}_{j\in\mathbb{N}}$ be a family of subspaces of $\mathcal{H}$ is defined by

$$W_j = \operatorname{span}\{e_{2j-1} + e_{2j}\}, \quad \text{and} \quad \Xi_j f = \frac{1}{2}\langle f, e_{2j-1} + e_{2j}\rangle(e_{2j-1} + e_{2j}) \quad \forall j \in \mathbb{N}.$$

A direct calculation shows that $\|\Xi_j\| = 1$ and $\Xi_i\Xi_j^* g_j = \delta_{ij}g_j$ for all $1 \le i, j \le n$ and $g_j \in W_j$. Since $\langle e_1 - e_2, e_{2j-1} + e_{2j}\rangle = 0$ for all $j \in \mathbb{N}$, $\mathcal{H} \neq \overline{\operatorname{span}}\{\Xi_j^*(W_j)\}_{j\in J}$. Therefore $\{\Xi_j\}_{j\in\mathbb{N}}$ is an g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, but it is not an g-orthonormal basis for $\mathcal{H}$.

Example 3. Let $N \in \mathbb{N}, \mathcal{H} = \mathbb{C}^{N+1}$ and let $\{e_k\}_{k=1}^{N+1}$ be the standard orthonormal basis of $\mathcal{H}$. Define

$$W_j = \operatorname{span}\left\{\sum_{\substack{k=1\\k\neq j}}^{N+1} e_k\right\}, \quad \text{and} \quad \Xi_j(\{c_i\}_{i=1}^{N+1}) = \frac{c_j}{\sqrt{N}}\sum_{\substack{k=1\\k\neq j}}^{N+1} e_k$$

Then $\Xi_j^*\big(\lambda\sum_{\substack{k=1\\k\neq j}}^{N+1} e_k\big) = \sqrt{N}\lambda e_j$ for all $1 \le j \le N+1$. This show that

$$\overline{\operatorname{span}}\{\Xi_j^*(W_j)\}_{j=1}^{N+1} = \overline{\operatorname{span}}\{e_j\}_{j=1}^{N+1} = \mathcal{H}, \quad \text{and} \quad \Xi_i\Xi_j^* = \delta_{ij}.$$

Therefore $\{\Xi_j\}_{j\in\mathbb{N}}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j=1}^{N+1}$.

Example 4. Let $\mathcal{H}=\mathbb{C}^{2N}$ and $W_1=W_2=\ldots=W_N=\mathbb{C}^2$. Define

$$\Xi_1=\begin{bmatrix}1&0&\ldots&0&0\\0&1&\ldots&0&0\end{bmatrix},\ldots,\Xi_N=\begin{bmatrix}0&0&\ldots&1&0\\0&0&\ldots&0&1\end{bmatrix}.$$

A direct calculation shows that $\|\Xi_k\|=1$ and $\Xi_k\Xi_\ell^*=\delta_{k\ell}$ for any $1\le k,\ell\le N$. We also have

$$\sum_{k=1}^N\|\Xi_kf\|^2=\sum_{k=1}^N(|z_{2k-1}|^2+|z_{2k}|^2)=\|f\|^2,\qquad \forall f=\{z_i\}_{i=1}^{2N}\in\mathbb{C}^{2N}.$$

Therefore $\Xi=\{\Xi_k\}_{k=1}^N$ is an g-orthonormal basis for $\mathbb{C}^{2N}$ with respect to $\mathbb{C}^2$. Similarly, the sequence $\Psi=\{\Psi_k\}_{k=1}^N$ defined by

$$\Psi_1=\begin{bmatrix}0&1&\ldots&0&0\\1&0&\ldots&0&0\end{bmatrix},\ldots,\Psi_N=\begin{bmatrix}0&0&\ldots&0&1\\0&0&\ldots&1&0\end{bmatrix},$$

is also an g-orthonormal basis for $\mathbb{C}^{2N}$ with respect to $\mathbb{C}^2$ and the matrix

$$B=\left[\Psi_i\Xi_j^*\right]_{N\times N}=\begin{bmatrix}A&&\bar{0}\\&\ddots&\\\bar{0}&&A\end{bmatrix},\quad\text{where}\quad A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

is the transition matrix from Ξ to Ψ. Hence, for any $f\in\mathbb{C}^{2N}$ we have $B[f]_\Xi=[f]_\Psi$.

Now, we define the generalized Riesz-dual sequence from a sequence of operators.

Definition 2. Let $\Xi=\{\Xi_i\}_{i\in I}$ and $\Psi=\{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ respectively. Let $\Lambda=\{\Lambda_i:\mathcal{H}\to V_i|\ i\in I\}$ be such that the series $\sum_{i\in I}\Lambda_i^*g_i'$ is convergent for all $\{g_i'\}_{i\in I}\in\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$. Define

$$\Gamma_j^\Lambda:\mathcal{H}\to W_j,\qquad \Gamma_j^\Lambda=\sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i,\quad \forall j\in I. \tag{1.3}$$

Then $\{\Gamma_j^\Lambda\}_{j\in I}$ is called the generalized Riesz-dual sequence (g-R-dual sequence) for the sequence Λ with respect to (Ξ,Ψ).

Notice that the hypothesis the series $\sum_{i\in I}\Lambda_i^*g_i'$ is convergent for all $\{g_i'\}_{i\in I}\in\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$ is always fulfilled if the sequence $\Lambda=\{\Lambda_i\}_{i\in I}$ is g-Bessel sequence with respect to $\{V_i\}_{i\in I}$.

Example 5. Let $\mathcal{H}=\mathbb{C}^{2N}$ and let $\{\Xi_i\}_{i=1}^N$, $\{\Psi_i\}_{i=1}^N$ be the g-orthonormal bases for $\mathcal{H}$ with respect to $\mathbb{C}^2$ defined in Example 4. Define

$$\Lambda_1=\begin{bmatrix}1&1&\ldots&0&0\\0&1&\ldots&0&0\end{bmatrix},\ldots,\Lambda_N=\begin{bmatrix}0&0&\ldots&1&1\\0&0&\ldots&0&1\end{bmatrix}.$$

Then, $\Lambda=\{\Lambda_i\}_{i=1}^N$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\mathbb{C}^2$ with g-Bessel bound $B=3$. The g-R-dual sequence for the sequence Λ with respect to (Ξ,Ψ) is defined as follows:

$$\Gamma_1^\Lambda=\begin{bmatrix}0&1&\ldots&0&0\\1&1&\ldots&0&0\end{bmatrix},\ldots,\Gamma_N^\Lambda=\begin{bmatrix}0&0&\ldots&0&1\\0&0&\ldots&1&1\end{bmatrix},$$

which is also a g-Bessel sequence for $\mathcal{H}$ with respect to $\mathbb{C}^2$ with g-Bessel bound $B=3$.

Now, we need an algorithm to invert the process and calculate $\{\Lambda_i\}_{i\in I}$ from the sequence $\{\Gamma_j^\Lambda\}_{j\in I}$.

Theorem 1.1. *Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ respectively. Let $\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then, for all $i \in I$,*

$$\Lambda_i = \sum_{j\in I} \Psi_i(\Gamma_j^\Lambda)^*\Xi_j. \tag{1.4}$$

In particular, this shows that $\{\Lambda_i\}_{i\in I}$ is the g-R-dual sequence for $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ψ, Ξ).

Proof. The definition of $\{\Gamma_j^\Lambda\}_{j\in I}$ implies that for every $i, j \in I$

$$\begin{aligned}\Psi_i(\Gamma_j^\Lambda)^* &= \Psi_i\big(\sum_{k\in I} \Xi_j\Lambda_k^*\Psi_k\big)^* = \sum_{k\in I} \Psi_i\Psi_k^*\Lambda_k\Xi_j^* \\ &= \sum_{k\in I} \delta_{ik}\Lambda_k\Xi_j^* = \Lambda_i\Xi_j^*.\end{aligned}$$

Therefore $\Psi_i(\Gamma_j^\Lambda)^* = \Lambda_i\Xi_j^*$. Now, by Lemma 1.1 we have

$$\Lambda_i = \Lambda_i I_{\mathcal{H}} = \Lambda_i\big(\sum_{j\in I} \Xi_j^*\Xi_j\big) = \sum_{j\in I} \Lambda_i\Xi_j^*\Xi_j = \sum_{j\in I} \Psi_i(\Gamma_j^\Lambda)^*\Xi_j.$$

□

2 Characterizations of equivalence of the g-R-dual sequence

In this section we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. We also characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent). Recall that a family $\{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ if, it is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(V_i)\}_{i\in I}$ with respect to $\{V_i\}_{i\in I}$.

Definition 3. A sequence $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j)|\ j \in I\}$ is called a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, if $\{\Gamma_j\}_{j\in I}$ is a g-complete set for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and there exist constants $0 < A \le B < \infty$ such that:

$$A\sum_{j\in I} \|g_j\|^2 \le \big\|\sum_{j\in I} \Gamma_j^* g_j\big\|^2 \le B\sum_{j\in I} \|g_j\|^2, \tag{2.1}$$

for all sequences $\{g_j\}_{j\in I} \in \big(\sum_{j\in I} \oplus W_j\big)_{\ell^2}$. We define the g-Riesz basis bounds for $\{\Gamma_j\}_{j\in I}$ to be the largest number A and the smallest number B such that this inequality (2.1) holds. If $\{\Gamma_j\}_{j\in I}$ is a g-Riesz basis only for $\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}$, we call it is a g-Riesz basic sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$.

The following result is a characterization of g-Riesz bases for $\mathcal{H}$, see, e.g., [1] for a proof of this standard result.

Lemma 2.1. *Let $\{\Xi_j\}_{j\in I}$ be an g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Then the following holds.*

(i) $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j)|\ j \in I\}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, if and only if there exists a bounded bijective operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Gamma_j = \Xi_j U^*$ for all $j \in I$.

(*ii*) *Assume that* $\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I} = \mathcal{H}$ *and that* $\big\|\sum_{j\in I}\Gamma_j^* g_j\big\|^2 = \sum_{j\in I}\|g_j\|^2$, *for all sequences* $\{g_j\}_{j\in I} \in \big(\sum_{j\in I}\oplus W_j\big)_{\ell^2}$. *Then* $\{\Gamma_j\}_{j\in I}$ *is an g-orthonormal basis for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$.

The next result gives a characterization of g-frame sequences which keeps the information about the g-frame bounds.

Proposition 2.1. *Let* $\Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) : \ i \in I\}$. *Then the following conditions are equivalent.*

(*i*) $\Lambda = \{\Lambda_i\}_{i\in I}$ *is a g-frame sequence with respect to* $\{V_i\}_{i\in I}$ *with g-frame bounds* A *and* B.

(*ii*) *The synthesis operator* T_Λ^* *is well defined on* $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ *and satisfies the condition:*

$$A\|g'\|_{\ell^2}^2 \le \|T_\Lambda^* g'\|^2 \le B\|g'\|_{\ell^2}^2, \qquad \forall\, g' \in (\ker_{T_\Lambda^*})^\perp.$$

Proof. We note that if $f \in \overline{\text{span}}\{\Lambda_i^*(V_i)\}_{i\in I}^\perp$, then $\|\Lambda_i f\|^2 = \langle f, \Lambda_i^*\Lambda_i f\rangle = 0$ for all $i \in I$. This implies that the upper bound condition with bound B is equivalent to the right-hand inequality in (*ii*). We therefore assume that $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ and prove the equivalence of the lower bound condition with the left-hand inequality in (*ii*). First, assume that $\{\Lambda_i\}_{i\in I}$ satisfies the lower bound condition with bound A. Then $\mathcal{R}_{T_\Lambda^*}$ is closed because $\mathcal{R}_{T_\Lambda}$ is closed. Therefore $(\ker_{T_\Lambda^*})^\perp = \overline{\mathcal{R}_{T_\Lambda}} = \mathcal{R}_{T_\Lambda}$, i.e., $(\ker_{T_\Lambda^*})^\perp = \big\{T_\Lambda f : \ f \in \mathcal{H}\big\}$. Now, for any $f \in \mathcal{H}$ we have

$$\begin{aligned}\|T_\Lambda f\|_{\ell^2}^4 &= |\langle T_\Lambda^* T_\Lambda f, f\rangle|^2 = |\langle S_\Lambda f, f\rangle|^2 \le \|S_\Lambda f\|^2\|f\|^2\\ &\le \frac{1}{A}\|S_\Lambda f\|^2\sum_{i\in I}\|\Lambda_i f\|^2 = \frac{1}{A}\|S_\Lambda f\|^2\|T_\Lambda f\|_{\ell^2}^2.\end{aligned}$$

This implies that

$$A\|T_\Lambda f\|_{\ell^2}^2 \le \|S_\Lambda f\|^2 = \big\|T_\Lambda^*(T_\Lambda f)\big\|^2,$$

as desired. For the other implication, assume that the left-hand inequality in (*ii*) is satisfied. We prove that $\mathcal{R}_{T_\Lambda^*}$ is closed. Let $\{f_n\}_{n=1}^\infty \subset \mathcal{R}_{T_\Lambda^*}$ and $\lim_{n\to\infty} f_n = f$ for some $f \in \mathcal{H}$. There exists a sequence $\{g_n'\}_{n=1}^\infty \subset (\ker_{T_\Lambda^*})^\perp$ such that $T_\Lambda^* g_n' = f_n$. Now (*ii*) implies that $\{g_n'\}_{n=1}^\infty$ is a Cauchy sequence. Therefore $\{g_n'\}_{n=1}^\infty$ converges to some $g' \in \big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$, which by continuity of T_Λ^* we have $T_\Lambda^* g' = f$. Thus $\mathcal{R}_{T_\Lambda^*}$ is closed. Let $(T_\Lambda^*)^\dagger$ denote the pseudo-inverse of T_Λ^*, then we have $T_\Lambda^*(T_\Lambda^*)^\dagger T_\Lambda^* = T_\Lambda^*$ and the operator $(T_\Lambda^*)^\dagger T_\Lambda^*$ is the orthogonal projection onto $(\ker_{T_\Lambda^*})^\perp$, and the operator $T_\Lambda^*(T_\Lambda^*)^\dagger$ is the orthogonal projection onto $\mathcal{R}_{T_\Lambda^*}$. Thus, for any $g' \in \big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$, the inequality (*ii*) implies that

$$A\|(T_\Lambda^*)^\dagger T_\Lambda^* g'\|^2 \le \|T_\Lambda^*(T_\Lambda^*)^\dagger T_\Lambda^* g'\|^2 = \|T_\Lambda^* g'\|^2.$$

Since $\ker_{(T_\Lambda^*)^\dagger} = \mathcal{R}_{T_\Lambda^*}^\perp$, therefore $\|(T_\Lambda^*)^\dagger\|^2 \le A^{-1}$. But $T_\Lambda^\dagger T_\Lambda$ is the orthogonal projection onto

$$\mathcal{R}_{T_\Lambda^\dagger} = (\ker_{(T_\Lambda^\dagger)^*})^\perp = (\ker_{(T_\Lambda^*)^\dagger})^\perp = \mathcal{R}_{T_\Lambda^*},$$

so for all $f \in \overline{\text{span}}\{\Lambda_i^*(V_i)\}_{i\in I} = \mathcal{R}_{T_\Lambda^*}$ we obtain

$$\|f\|^2 = \|T_\Lambda^\dagger T_\Lambda f\|^2 \le \frac{1}{A}\|T_\Lambda f\|^2 = \frac{1}{A}\sum_{i\in I}\|\Lambda_i f\|^2.$$

This shows that $\Lambda = \{\Lambda_i\}_{i\in I}$ satisfies the lower bound condition as desired. □

The next result shows a basic connection between a sequence of operators and its g-R-dual sequence which will be used frequently in what follows.

Proposition 2.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect $\{V_i\}_{i\in I}$. For every $\{g_j\}_{j\in I} \in \big(\sum_{j\in I} \oplus W_j\big)_{\ell^2}$, $\{g'_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$, let $f = \sum_{j\in I} \Xi_j^* g_j$ and $h = \sum_{i\in I} \Psi_i^* g'_i$. Then*

$$\Big\| \sum_{j\in I} (\Gamma_j^\Lambda)^* g_j \Big\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2 \quad \text{and} \quad \Big\| \sum_{i\in I} \Lambda_i^* g'_i \Big\|^2 = \sum_{j\in I} \|\Gamma_j^\Lambda h\|^2.$$

In particular, $\|T_{\Gamma^\Lambda}^*\big([f]_\Xi\big)\| = \|T_\Lambda f\|_{\ell^2}$ and $\|T_\Lambda^*\big([f]_\Psi\big)\| = \|T_{\Gamma^\Lambda} f\|_{\ell^2}$, for every $f \in \mathcal{H}$.

Proof. It is easy to check that

$$\begin{aligned}\Big\| \sum_{j\in I} (\Gamma_j^\Lambda)^* g_j \Big\|^2 &= \Big\| \sum_{j\in I} \big(\sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i \big)^* g_j \Big\|^2 = \Big\| \sum_{i\in I} \Psi_i^* \Lambda_i f \Big\|^2 \\ &= \big\langle \sum_{i\in I} \Psi_i^* \Lambda_i f, \sum_{j\in I} \Psi_j^* \Lambda_j f \big\rangle = \sum_{i\in I} \sum_{j\in I} \langle \Lambda_i f, \Psi_i \Psi_j^* \Lambda_j f \rangle \\ &= \sum_{i\in I} \sum_{j\in I} \langle \Lambda_i f, \delta_{ij} \Lambda_j f \rangle = \sum_{i\in I} \|\Lambda_i f\|^2.\end{aligned}$$

Similarly, the second claim follows from Theorem 1.1. □

There exists an interesting relation between the synthesis operator of $\Lambda = \{\Lambda_i\}_{i\in I}$ and the span of $\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}$, which will turn out to be very useful in the sequel.

Proposition 2.3. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with g-R-dual sequence $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ξ, Ψ). Then the following statements hold.*

(*i*) *$f \in \big(\overline{\text{span}}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}\big)^\perp$ if and only if $[f]_\Psi \in \ker T_\Lambda^*$.*

(*ii*) *$f \in \big(\overline{\text{span}}\{\Lambda_j^*(V_j)\}_{j\in I}\big)^\perp$ if and only if $[f]_\Xi \in \ker T_{\Gamma^\Lambda}^*$.*

In particular,

$$\dim \big(\overline{\text{span}}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}\big)^\perp = \dim\ker T_\Lambda^* \quad \text{and} \quad \dim \big(\overline{\text{span}}\{\Lambda_j^*(V_j)\}_{j\in I}\big)^\perp = \dim\ker T_{\Gamma^\Lambda}^*.$$

Proof. Let $f \in \mathcal{H}$. First for each $j \in J$ and $g_j \in W_j$ we observe that

$$\langle f, (\Gamma_j^\Lambda)^* g_j \rangle = \sum_{i\in J} \langle f, \Psi_i^* \Lambda_i \Xi_j^* g_j \rangle = \big\langle \sum_{i\in J} \Lambda_i^* \Psi_i f, \Xi_j^* g_j \big\rangle = \big\langle T_\Lambda^*([f]_\Psi), \Xi_j^* g_j \big\rangle.$$

Since $\Xi = \{\Xi_j\}_{j\in J}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, $\big\langle T_\Lambda^*([f]_\Psi), \Xi_j^* g_j \big\rangle = 0$ for all $j \in I$ and $g_j \in W_j$, if and only if $T_\Lambda^*([f]_\Psi) = 0$. Thus, $f \in \big(\text{span}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}\big)^\perp$ is equivalent to $[f]_\Psi \in \ker T_\Lambda^*$. Similarly, the second claim follows from Theorem 1.1. □

The next result shows a kind of equilibrium between a sequence of operators and its R-dual sequence. It can be viewed as a general version of [4, Proposition 13].

Corollary 2.1. *The following conditions are equivalent.*

(*i*) *$\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A, B.*

(*ii*) *$\{\Gamma_j^\Lambda\}_{j\in I}$ is a g-frame sequence with respect to $\{W_j\}_{j\in I}$ with g-frame bounds A, B.*

(iii) $\{\Gamma_j^\Lambda\}_{j\in I}$ *is a g-Riesz basic sequence with respect to* $\{W_j\}_{j\in I}$ *with g-frame bounds* A, B.

Proof. $(i) \Leftrightarrow (ii)$ Proposition 2.1 and Proposition 2.3 imply that $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A, B if and only if

$$A\|[f]_\Psi\|_{\ell^2}^2 \leq \|T_\Lambda^*([f]_\Psi)\|^2 \leq B\|[f]_\Psi\|_{\ell^2}^2,$$

for all $f \in \overline{\text{span}}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}$. Now, Proposition 2.2 implies

$$A\|f\|^2 \leq \|T_{\Gamma^\Lambda} f\|_{\ell^2}^2 \leq B\|f\|^2.$$

$(i) \Leftrightarrow (iii)$ This equivalence follows immediately from Proposition 2.2. □

The dimension condition in Proposition 2.3 will play a crucial role for the g-R-dual sequence. Using Proposition 2.3 we can derive a simple characterization of an g-Riesz basic sequence being an g-R-dual sequence of a g-frame in the tight case.

Theorem 2.1. *Let* $\Lambda = \{\Lambda_i\}_{i\in I}$ *be a* A*-tight g-frames for* $\mathcal{H}$ *with respect to* $\{V_i\}_{i\in I}$ *and let* $\{\Gamma_j\}_{j\in I}$ *be a* A*-tight g-Riesz basic sequence in* $\mathcal{H}$ *with respect to* $\{W_j\}_{j\in I}$*. Then* $\{\Gamma_j\}_{j\in I}$ *is a g-R-dual sequence of* $\{\Lambda_i\}_{i\in I}$ *with respect to* (Ξ, Ψ)*, if and only if*

$$\dim\left(\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^\perp = \dim\ker T_\Lambda^*. \tag{2.2}$$

Proof. The necessity of the condition in (2.2) follows from Proposition 2.3. Now, assume that (2.2) holds. Then, according to Lemma 2.1 the sequence $\{\frac{1}{\sqrt{A}}\Gamma_j\}_{j\in I}$ is an g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Suppose that $\Xi = \{\Xi_j\}_{j\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ are g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $\{V_i\}_{i\in I}$ respectively. Consider the g-R-dual $\{\Theta_j\}_{j\in I}$ of $\Lambda = \{\Lambda_i\}_{i\in I}$ with respect to (Ξ, Ψ), i.e. $\Theta_j = \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i, \quad j \in I$. By Corollary 2.1 $\{\Theta_j\}_{j\in I}$ is a A-tight g-Riesz basic sequence with respect to $\{W_j\}_{j\in I}$ and hence $\{\frac{1}{\sqrt{A}}\Theta_j\}_{j\in I}$ is also an g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. By Proposition 2.3 and (2.2),

$$\dim\left(\overline{\text{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^\perp = \dim\ker T_\Lambda^* = \dim\left(\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^\perp. \tag{2.3}$$

In case $\left(\overline{\text{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^\perp = \left(\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^\perp = \{0\}$, the g-orthonormality of the sequences $\{\frac{1}{\sqrt{A}}\Theta_i\}_{i\in I}$ and $\{\frac{1}{\sqrt{A}}\Gamma_i\}_{i\in I}$ implies that there exists unitary operator

$$U : \mathcal{H} \to \mathcal{H}, \quad \text{by} \quad \Gamma_j = \Theta_j U^*, \ \ \forall j \in I.$$

In case $\left(\overline{\text{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^\perp \neq \{0\}$, letting $\{\Phi_j\}_{j\in I}$ and $\{\Omega_j\}_{j\in I}$ be g-orthonormal bases for

$$\left(\overline{\text{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^\perp \quad \text{and} \quad \left(\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^\perp,$$

with respect to $\{W_j\}_{j\in I}$ respectively. (2.3) implies that there exists unitary operator

$$U : \mathcal{H} \to \mathcal{H}, \quad \text{by} \quad \Gamma_j = \Theta_j U^*, \quad \Omega_j = \Phi_j U^* \ \ \forall j \in I.$$

In both cases, we have

$$\Gamma_j = \Theta_j U^* = \Big(\sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i\Big)U^* = \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i U^*, \ \ \forall j \in I,$$

which shows that $\{\Gamma_j\}_{j\in I}$ is a g-R-dual sequence of $\{\Lambda_i\}_{i\in I}$ with respect to $\{\Xi_j\}_{j\in I}$ and $\{\Psi_i U^*\}_{i\in I}$. □

The following result is about different types of equivalence of g-frames, which is taken from [10]. This result will moreover be employed in several proofs in the sequel.

Proposition 2.4. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Lambda' = \{\Lambda'_i\}_{i\in I}$ be Parseval g-frames for $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to $\{V_i\}_{i\in I}$, respectively. Then Λ is unitarily equivalent to Λ' if and only if the analysis operators T_Λ and $T_{\Lambda'}$ have the same range. Likewise, two g-frames with respect to $\{V_i\}_{i\in I}$ are equivalent if and only if their analysis operators have the same range.*

In the following we characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent).

Theorem 2.2. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda'_i\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then*

(*i*) *$\{\Lambda_i\}_{i\in I}$ is equivalent to $\{\Lambda'_i\}_{i\in I}$ in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if*

$$\overline{\text{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I} = \overline{\text{span}}\{(\Gamma_j^{\Lambda'})^*(W_j)\}_{j\in I}.$$

(*ii*) *$\{\Lambda_i\}_{i\in I}$ is unitarily equivalent to $\{\Lambda'_i\}_{i\in I}$ in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if $S_{\Gamma^\Lambda} = S_{\Gamma^{\Lambda'}}$.*

(*iii*) *$\{\Gamma_j^{\Lambda}\}_{j\in I}$ is unitarily equivalent to $\{\Gamma_j^{\Lambda'}\}_{j\in I}$ in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if and only if $S_\Lambda = S_{\Lambda'}$.*

Proof. (*i*) By Proposition 2.4 $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda'_i\}_{i\in I}$ are equivalent in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, if and only if $\mathcal{R}_{T_\Lambda} = \mathcal{R}_{T_{\Lambda'}}$. Therefore, $\ker T_\Lambda^* = \ker T_{\Lambda'}^*$. Now the conclusion follows by Proposition 2.3.

(*ii*) Using Propositions 2.1 and 2.4, $\{\Lambda_i\}_{i\in I}$ is unitarily equivalent to $\{\Lambda'_i\}_{i\in I}$ if and only if

$$\Big\|\sum_{i\in I}\Lambda_i^* g'_i\Big\|^2 = \Big\|\sum_{i\in I}\Lambda_i'^* g'_i\Big\|^2, \quad \forall\, \{g'_i\}_{i\in I} \in (\ker T_\Lambda^*)^\perp.$$

turn equivalent to

$$\langle S_{\Gamma^\Lambda} f, f\rangle = \sum_{j\in I}\|\Gamma_j^{\Lambda} f\|^2 = \sum_{j\in I}\|\Gamma_j^{\Lambda'} f\|^2 = \langle S_{\Gamma^{\Lambda'}} f, f\rangle,$$

for all $f \in \mathcal{H}$ and $g'_i = \Psi_i f$ $(i \in I)$. Hence $S_{\Gamma^\Lambda} = S_{\Gamma^{\Lambda'}}$, as required.

(*iii*) The proof follows immediately from (*ii*) and Theorem 1.1. □

Corollary 2.2. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then*

$$\overline{\text{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I} = \overline{\text{span}}\{(\Gamma_j^{\widehat{\Lambda}})^*(W_j)\}_{j\in I},$$

where $\{\widehat{\Lambda}_i\}_{i\in I}$ is the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Proof. Since $\{\widehat{\Lambda}_i\}_{i\in I}$ is equivalent to $\{\Lambda_i\}_{i\in I}$. Therefore the conclusion follows by Theorem 2.2. □

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References

[1] M.S. Asgari, *Operator-valued bases on Hilbert spaces.* J. Linear and Topological Algebra. 2 (2013), no. 4., 201-218.

[2] M.S. Asgari, H. Rahimi, *Generalized frames for operators in Hilbert spaces.* Infin. Dimens. Anal. Quantum. Probab. Relat. Top. 17 (2014), no. 2., 1450013 (20 pages).

[3] P.G. Casazza, G. Kutyniok, *Frames of subspaces*, in Wavelets, Frames and Operator Theory, College Park, MD, 2003, Contemp. Math. 345 (Amer. Math. Soc.) (2004), 87-113.

[4] P.G. Casazza, G. Kutyniok, M.C. Lammers, *Duality principles in frame theory.* J. Fourier Anal. Appl. 10 (2004), 383-408.

[5] O. Christensen, *An introduction to frames and Riesz bases.* Springer, Switzerland, 2016.

[6] O. Christensen, H.O. Kim, R.Y. Kim, *On the duality principle by Casazza, Kutyniok, and Lammers.* J. Fourier Anal. Appl. 17 (2011), 640-655.

[7] O. Christensen, X.C. Xiao, Y.C. Zhu, *Characterizing R-duality in Banach spaces.* Acta Mathematica Sinica, Eng. Series. 29 (2013), no.1., 75вЂ“84.

[8] X. Guo, *G-bases in Hilbert spaces.* Abstract and Applied Analysis. (2012), Article ID 923729, 14 pages, doi:10.1155/2012/923729.

[9] X. Guo, *Operator parameterizations of g-frames.* Taiwanese J. Math. Vol. 18 (2014), no. 1., 313-328.

[10] A. Najati, A. Rahimi, *Generalized frames in Hilbert spaces.* Bull. Iranian Math. Soc. 35 (2009), no. 1., 97-109.

[11] D.T. Stoeva, O. Christensen, *On R-Duals and the Duality Principle in Gabor Analysis.* J. Fourier Anal. Appl. 21 (2015), no.2., 383-400.

[12] W. Sun, *G-frames and g-Riesz bases.* J. Math. Anal. Appl. 322 (2006), 437-452.

[13] W. Sun, *Stability of g-frames.* J. Math. Anal. Appl. 326 (2007), 858-868.

[14] X.M. Xian, Y.C. Zhu, *Duality principles of frames in Banach spaces.* Acta Math. Sci. Ser. A Chin. Ed. 29 (2009), no.1., 94-102.

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