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- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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MUKHTARBAY OTELBAEV

(to the 75th birthday)

On October 3, 2017 was the 75th birthday of Mukhtarbay Otelbaev, Doctor of Physical and Mathematical Sciences (1978), Professor (1983), academician of the National Academy of Sciences of the Republic of Kazakhstan (2004), Honored Worker of the Republic of Kazakhstan (2012), laureate of the State Prize of the Republic of Kazakhstan in the field of science and technology (2007), Director of the Eurasian Mathematical Institute (since 2001), Professor of the Department вЂњFundamental MathematicsвЂќ of the L.N. Gumilyov Eurasian National University, the editor-in-chief of the Eurasian Mathematical Journal (together with V.I. Burenkov and V.A. Sadovnichy).

M. Otelbaev was born in the village of Karakemer of the Kurdai district, Zhambyl region. He graduated from the M.V. Lomonosov Moscow State University (1969) and then completed his postgraduate studies at the same university (1972). There he defended his doctor of sciences thesis (1978).

Professor Otelbaev's scientific interests are related to functional analysis, differential equations, computational mathematics, and theoretical physics.

He introduced the q-averaging, which is now called the Otelbaev function; using it he obtained a number of fundamental results. For embedding of the Sobolev weighted spaces and the resolvent of the Schrödinger operator, he established criterions for the compactness and finiteness of the type, as well as estimates of the eigenvalues of the Schrödinger and Dirac operators that are exact in order. He was the first to establish that there is no universal asymptotic formula for the distribution function of the Sturm-Liouville operator. He obtained effective conditions for the separation of the differential operators with nonsmooth and oscillating coefficients, he developed an abstract theory of extension and contraction of operators which are not necessarily linear in linear topological spaces. M. Otelbaev proposed a new numerical method for solving boundary value problems, and a method for approximate calculation of eigenvalues and eigenvectors of compact operators. He obtained the fundamental results in the theory of nonlinear evolution equations and in theoretical physics.

He has published more than 70 scientific papers in leading international journals entering the rating lists of Thomson Reuters and Scopus. Under his supervision 70 postgraduate students have defended their candidate of sciences theses, 9 of them became doctors of sciences. In 2006 and 2011 he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal congratulates Mukhtarbay Otelbaev on the occasion of his 75th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Award for the Eurasian Mathematical Journal

Dear readers, authors, reviewers and members of the Editorial Board of the Eurasian Mathematical Journal,

we are happy to inform you that in November 2017 the Eurasian Mathematical Journal was awarded the title "Leader of Science 2017" by the National Center of State Scientific-Technical Expertise of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan in the nomination "Leader of Kazakhstan Scientific Publications" for the high level of publication activities and high level of citations in Web of Science Core Collection in 2014-2016.

Recall that the Eurasian Mathematical Journal was founded by the L.N. Gumilyov Eurasian National University in 2010 in co-operation with the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia and the University of Padua (see [1]).

The journal pulishes carefully selected original research papers in all areas of mathematics, survey papers, and short communications. It publishes 4 issues in a year. The language of the paper must be English only. Papers accepted for publication are edited from the point of view of English.

More than 280 papers were published written by mathematicians from more than 40 countries representing all continents.

In 2014 the journal was registered in Scopus and in September 2014 the Elsevier-Kazakhstan Research Excellence Forum was held at the L.N. Gumilyov Eurasian National University dedicated to this occasion in which the Elsevier Chairman Professor Y.S. Chi participated (see [3] for details).

In 2015 the Eurasian Mathematical Journal was included in the list of Scopus mathematical journals, quartile Q4, and it is on the way to entering quartile Q3 (see [3]).

Attached is the invitation letter to the Rector of the L.N. Gumilyov Eurasian National University Professor E.B. Sydykov to the ceremony of awarding, which took place in Almaty on November 8, 2017.

On behalf of the Editorial Board of the EMJ V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

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ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
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Уважаемый Ерлан Батташевич!

АО «Национальный центр государственной научно-технической экспертизы» (далее АО «НЦГНТЭ») и компания Clarivate Analytics имеют честь пригласить Вас на церемонию вручения независимой награды «Лидер науки-2017» за высокие показатели публикационной активности и цитируемости в Web of Science Core Collection в период 2014-2016 годы.

Ваш журнал *«Eurasian Mathematical Journal»* награждается в номинации «**Лидер казахстанских научных изданий**».

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Президент **Ибраев А.Ж.**

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COMPARATIVE GROWTH ANALYSIS OF ENTIRE AND MEROMORPHIC FUNCTIONS USING THEIR RELATIVE TYPES AND RELATIVE WEAK TYPES

S.K. Datta, T. Biswas

Communicated by E.S. Smailov

Key words: meromorphic function, entire function, relative order, relative lower order, relative type, relative weak type.

AMS Mathematics Subject Classification: 30D20, 30D30, 30D35.

Abstract. In this paper we study some comparative growth properties of entire and meromorphic functions on the basis of their relative type and relative weak type.

1 Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. The maximum modulus function corresponding to entire g is defined as $M_g(r) = \max\{|g(z)| : |z| = r\}$. The order (lower order) of an entire function g which is generally used for computational purposes is defined in terms of the growth of g with respect to the function $\exp z$ which is defined as follows:

$$\rho_g = \limsup_{r\to\infty} \frac{\log\log M_g(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log\log M_g(r)}{\log(r)}$$

$$\left(\lambda_g = \liminf_{r\to\infty} \frac{\log\log M_g(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log\log M_g(r)}{\log(r)}\right).$$

An entire function for which the *order* and *lower order* are the same is said to be of *regular growth*. Functions which are not of *regular growth* are said to be of *irregular growth*.

For meromrophic function f, $M_f(r)$ cannot be defined since f is not analytic. In this case one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as the maximum modulus function, in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)$ $\left(\bar{N}_f(r,a)\right)$ known as the counting function of a-points (distinct a-points) of a meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r\right),$$

moreover we denote by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

Also the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of f is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is an entire function, then Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) .$$

Further, if f is a non-constant entire function, then $T_f(r)$ is a strictly increasing and continuous function of r. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. However, in the case of meromorphic functions, the growth indicators such as order and lower order which are classical in complex analysis are defined in terms of their growths with respect to the function $\exp z$ as follows:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

A meromorphic function for which the *order* and *lower order* are the same is said to be of *regular growth*. Functions which are not of *regular growth* are said to be of *irregular growth*.

To compare the relative growth of two meromorphic functions having same non zero finite *order* with respect to another meromorphic function, one can recall the definition of *type* of a meromorphic function which is another type of classical growth indicator. Next we give the definitions of the *type* and *weak type* of meromorphic functions which are as follows:

Definition 1. The *type* σ_f and *lower type* $\overline{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty .$$

Datta and Jha [4] introduced the definition of the *weak type* of a meromorphic function of finite positive lower order in the following way:

Definition 2. [4] The *weak type* τ_f and the growth indicator $\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f are defined by

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty .$$

Extending the notion of the relative order introduced by Bernal {[1], [2]}, Lahiri and Banerjee [7] gave the definition of the *relative order* of a meromorphic function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [7] if $g(z) = \exp z$.

In the same way, one can define the *relative lower order* of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In the case of meromorphic functions, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *relative order* or *relative lower order* with respect to an entire function. Datta and Biswas also [5] gave such definitions of *relative type* and *relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 3. [5] The *relative type* $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the *lower relative type* $\overline{\sigma}_g(f)$ in the following way:

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 4. [5] The *relative weak type* $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

In a similar manner, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 3 and Definition 4 coincide with the classical definitions of the type (lower type) and weak type of a meromorphic function.

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already continued their studies in the area of comparative growth properties of entire and meromorphic functions in different directions using the growth indicators such as *order*, *type* and *weak type*. But at that time, the concepts of the *relative order* and consequently *relative type* and *relative weak type* of meromorphic function with respect to another entire function which have already been discussed above were mostly unknown in complex analysis and researchers were not aware of the technical advantages given by such notion which gives an idea to avoid comparing growth just with exp function to calculate *order*, *type* and *weak type* respectively. In the paper we study some relative growth properties of entire and meromorphic functions with respect to another entire function on the basis of relative type and relative weak type. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [6] and [9]. Hence we do not explain those in detail.

2 Lemmas

First of all let us recall the following theorem due to Debnath et al. [3]
Theorem A. Let f be a meromorphic function and g be an entire functions with non-zero finite order and lower order. Then

$$\frac{\lambda_f}{\rho_g} \leq \lambda_g(f) \leq \min\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \leq \max\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \leq \rho_g(f) \leq \frac{\rho_f}{\lambda_g}.$$

From the conclusion of the above theorem, we present the following two lemmas which will be needed in the sequel.

Lemma 2.1. [3] *Let f be a meromorphic function with $0 < \lambda_f \leq \rho_f < \infty$ and g be an entire function of regular growth with non-zero finite order. Then*

$$\rho_g(f) = \frac{\rho_f}{\rho_g} \quad \text{and} \quad \lambda_g(f) = \frac{\lambda_f}{\lambda_g}.$$

Lemma 2.2. [3] *Let f be a meromorphic function of regular growth with non-zero finite order and g be an entire function with $0 < \lambda_g \leq \rho_g < \infty$. Then*

$$\rho_g(f) = \frac{\lambda_f}{\lambda_g} \quad \text{and} \quad \lambda_g(f) = \frac{\rho_f}{\rho_g}.$$

3 Main results

In this section we state the main results of the paper.

Theorem 3.1. *Let f be a meromorphic function with $0 < \rho_f < \infty$ and g be an entire function of regular growth with non-zero finite order. Then*

$$\left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \leq \overline{\sigma}_g(f) \leq \min\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \max\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \sigma_g(f) \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.$$

Proof. From the definitions of σ_f and $\overline{\sigma}_f$, we have that for any $\varepsilon > 0$ that all sufficiently large values of r that

$$T_f(r) \leq (\sigma_f + \varepsilon) \cdot r^{\rho_f}, \tag{3.1}$$
$$T_f(r) \geq (\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f} \tag{3.2}$$

and we also get that for a sequence of values of r tending to infinity

$$T_f(r) \geq (\sigma_f - \varepsilon) \cdot r^{\rho_f}, \tag{3.3}$$
$$T_f(r) \leq (\overline{\sigma}_f + \varepsilon) \cdot r^{\rho_f}. \tag{3.4}$$

Similarly from the definitions of σ_g and $\overline{\sigma}_g$, it follows that for all sufficiently large values of r

$$\begin{aligned} T_g(r) &\leq (\sigma_g+\varepsilon)\cdot r^{\rho_g} \\ i.e.,\ r &\leq T_g^{-1}\left[(\sigma_g+\varepsilon)\cdot r^{\rho_g}\right] \\ i.e.,\ T_g^{-1}(r) &\geq \left[\left(\frac{r}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right], \end{aligned} \tag{3.5}$$

$$\begin{aligned} T_g(r) &\geq \{(\overline{\sigma}_g-\varepsilon)\cdot r^{\rho_g}\} \\ i.e.,\ r &\geq T_g^{-1}\left[\{(\overline{\sigma}_g-\varepsilon)\cdot r^{\rho_g}\}\right] \\ i.e.,\ T_g^{-1}(r) &\leq \left[\left(\frac{r}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \end{aligned} \tag{3.6}$$

and that for a sequence of values of r tending to infinity

$$\begin{aligned} T_g(r) &\geq \{(\sigma_g-\varepsilon)\cdot r^{\rho_g}\} \\ i.e.,\ r &\geq T_g^{-1}\{(\sigma_g-\varepsilon)\cdot r^{\rho_g}\} \\ i.e.,\ T_g^{-1}(r) &\leq \left[\left(\frac{r}{(\sigma_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right], \end{aligned} \tag{3.7}$$

$$\begin{aligned} T_g(r) &\leq \{(\overline{\sigma}_g+\varepsilon)\cdot r^{\rho_g}\} \\ i.e.,\ r &\leq T_g^{-1}\left[\{(\overline{\sigma}_g+\varepsilon)\cdot r^{\rho_g}\}\right] \\ i.e.,\ T_g^{-1}(r) &\geq \left[\left(\frac{r}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]. \end{aligned} \tag{3.8}$$

Now from (3.3) and in view of (3.5), we get that for a sequence of values of r tending to infinity

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[\{(\sigma_f-\varepsilon)\cdot r^{\rho_f}\}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left(\frac{\{(\sigma_f-\varepsilon)\cdot r^{\rho_f}\}}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$\begin{aligned} i.e.,\ T_g^{-1}T_f(r) &\geq \left[\frac{(\sigma_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}} \\ i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} &\geq \left[\frac{(\sigma_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}. \end{aligned} \tag{3.9}$$

As $\varepsilon\,(>0)$ is arbitrary, in view of Lemma 2.1 it follows that

$$\begin{aligned} \limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} &\geq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \\ i.e.,\ \sigma_g(f) &\geq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}. \end{aligned} \tag{3.10}$$

Analogously from (3.2) and in view of (3.8), it follows that for a sequence of values of r tending to infinity

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[\left\{(\overline{\sigma}_f-\varepsilon)\cdot r^{\rho_f}\right\}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left(\frac{\{(\overline{\sigma}_f-\varepsilon)\cdot r^{\rho_f}\}}{(\overline{\sigma}_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\overline{\sigma}_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \geq \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\overline{\sigma}_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{3.11}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above and Lemma 2.1 that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \sigma_g(f) \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{3.12}$$

Again in view of (3.6), we have from (3.1) that for all sufficiently large values of r

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[\{(\sigma_f+\varepsilon)\cdot r^{\rho_f}\}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left(\frac{\{(\sigma_f+\varepsilon)\cdot r^{\rho_f}\}}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left[\frac{(\sigma_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\sigma_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{3.13}$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain in view of Lemma 2.1 that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \sigma_g(f) \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{3.14}$$

Again from (3.2) and in view of (3.5), we get that for all sufficiently large values of r

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[\{(\overline{\sigma}_f-\varepsilon)\cdot r^{\rho_f}\}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left(\frac{\{(\overline{\sigma}_f-\varepsilon)\cdot r^{\rho_f}\}}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \geq \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{3.15}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from the above and Lemma 2.1 that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e.,\ \overline{\sigma}_g(f) \geq \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}. \tag{3.16}$$

Also in view of (3.7), we get from (3.1) that for a sequence of values of r tending to infinity

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[\{(\sigma_f+\varepsilon)\cdot r^{\rho_f}\}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left(\frac{\{(\sigma_f+\varepsilon)\cdot r^{\rho_f}\}}{(\sigma_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left[\frac{(\sigma_f+\varepsilon)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$
$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\sigma_f+\varepsilon)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{3.17}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from Lemma 2.1 and the above that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e.,\ \overline{\sigma}_g(f) \leq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}. \tag{3.18}$$

Similarly from (3.4) and in view of (3.6), it follows that for a sequence of values of r tending to infinity

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[\{(\overline{\sigma}_f+\varepsilon)\cdot r^{\rho_f}\}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left(\frac{\{(\overline{\sigma}_f+\varepsilon)\cdot r^{\rho_f}\}}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left[\frac{(\overline{\sigma}_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$
$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\overline{\sigma}_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{3.19}$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from Lemma 2.1 and the above

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e.,\ \overline{\sigma}_g(f) \leq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{3.20}$$

Thus the theorem follows from (3.10), (3.12), (3.14), (3.16), (3.18) and (3.20). □

Theorem 3.2. *Let f be a meromorphic function of regular growth with non zero finite order and g be an entire function with $0 < \lambda_g < \infty$. Then*

$$\left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \leq \overline{\sigma}_g(f) \leq \min\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \max\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \sigma_g(f) \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}.$$

Proof. From the definitions of $\overline{\tau}_f$ and τ_f, we have that for all sufficiently large values of r

$$\begin{aligned} T_f(r) &\leq (\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}, \\ T_f(r) &\geq (\tau_f - \varepsilon) \cdot r^{\lambda_f} \end{aligned}$$

and also that for a sequence of values of r tending to infinity

$$\begin{aligned} T_f(r) &\geq (\overline{\tau}_f - \varepsilon) \cdot r^{\lambda_f}, \\ T_f(r) &\leq (\tau_f + \varepsilon) \cdot r^{\lambda_f}. \end{aligned}$$

Similarly from the definitions of $\overline{\tau}_g$ and τ_g, it follows that for all sufficiently large values of r

$$\begin{aligned} T_g(r) &\leq (\overline{\tau}_g + \varepsilon) \cdot r^{\lambda_g} \\ i.e.,\ r &\leq T_g^{-1}\left[(\overline{\tau}_g + \varepsilon) \cdot r^{\lambda_g}\right] \\ i.e.,\ T_g^{-1}(r) &\geq \left[\left(\frac{r}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right], \end{aligned}$$

$$\begin{aligned} T_g(r) &\geq (\tau_g - \varepsilon) \cdot r^{\lambda_g} \\ i.e.,\ r &\geq T_g^{-1}\left[(\tau_g - \varepsilon) \cdot r^{\lambda_g}\right] \\ i.e.,\ T_g^{-1}(r) &\leq \left[\left(\frac{r}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right] \end{aligned}$$

and that for a sequence of values of r tending to infinity

$$\begin{aligned} T_g(r) &\geq (\overline{\tau}_g - \varepsilon) \cdot r^{\lambda_g} \\ i.e.,\ r &\geq T_g^{-1}\left[(\overline{\tau}_g - \varepsilon) \cdot r^{\lambda_g}\right] \\ i.e.,\ T_g^{-1}(r) &\leq \left[\left(\frac{r}{(\overline{\tau}_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right], \end{aligned}$$

$$\begin{aligned} T_g(r) &\leq (\tau_g + \varepsilon) \cdot r^{\lambda_g} \\ i.e.,\ r &\leq T_g^{-1}\left[(\tau_g + \varepsilon) \cdot r^{\lambda_g}\right] \\ i.e.,\ T_g^{-1}(r) &\geq \left[\left(\frac{r}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]. \end{aligned}$$

Now using the same technique of Theorem 3.1, one can easily prove the conclusion of the present theorem by the help of Lemma 2.2 and the above inequalities. Therefore the remaining part of the proof of the present theorem is omitted. □

Similarly in the line of Theorem 3.1 and Theorem 3.2 and with the help of Lemma 2.1 and Lemma 2.2, one may easily prove the following two theorems and therefore their proofs are omitted.

Theorem 3.3. *Let f be a meromorphic function with $0 < \lambda_f < \infty$ and g be an entire function of regular growth with non zero finite order. Then*

$$\left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \leq \tau_g(f) \leq \min\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \max\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_{g(m,p)}}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \overline{\tau}_g(f) \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}.$$

Theorem 3.4. *Let f be a meromorphic function of regular growth with non zero finite order and g be an entire function with $0 < \rho_g < \infty$. Then*

$$\left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \leq \tau_g(f) \leq \min\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \max\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \overline{\tau}_g(f) \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.$$

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