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Aims and Scope

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- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

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- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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MUKHTARBAY OTELBAEV

(to the 75th birthday)

On October 3, 2017 was the 75th birthday of Mukhtarbay Otelbaev, Doctor of Physical and Mathematical Sciences (1978), Professor (1983), academician of the National Academy of Sciences of the Republic of Kazakhstan (2004), Honored Worker of the Republic of Kazakhstan (2012), laureate of the State Prize of the Republic of Kazakhstan in the field of science and technology (2007), Director of the Eurasian Mathematical Institute (since 2001), Professor of the Department вЂњFundamental MathematicsвЂќ of the L.N. Gumilyov Eurasian National University, the editor-in-chief of the Eurasian Mathematical Journal (together with V.I. Burenkov and V.A. Sadovnichy).

M. Otelbaev was born in the village of Karakemer of the Kurdai district, Zhambyl region. He graduated from the M.V. Lomonosov Moscow State University (1969) and then completed his postgraduate studies at the same university (1972). There he defended his doctor of sciences thesis (1978).

Professor Otelbaev's scientific interests are related to functional analysis, differential equations, computational mathematics, and theoretical physics.

He introduced the q-averaging, which is now called the Otelbaev function; using it he obtained a number of fundamental results. For embedding of the Sobolev weighted spaces and the resolvent of the Schrödinger operator, he established criterions for the compactness and finiteness of the type, as well as estimates of the eigenvalues of the Schrödinger and Dirac operators that are exact in order. He was the first to establish that there is no universal asymptotic formula for the distribution function of the Sturm-Liouville operator. He obtained effective conditions for the separation of the differential operators with nonsmooth and oscillating coefficients, he developed an abstract theory of extension and contraction of operators which are not necessarily linear in linear topological spaces. M. Otelbaev proposed a new numerical method for solving boundary value problems, and a method for approximate calculation of eigenvalues and eigenvectors of compact operators. He obtained the fundamental results in the theory of nonlinear evolution equations and in theoretical physics.

He has published more than 70 scientific papers in leading international journals entering the rating lists of Thomson Reuters and Scopus. Under his supervision 70 postgraduate students have defended their candidate of sciences theses, 9 of them became doctors of sciences. In 2006 and 2011 he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal congratulates Mukhtarbay Otelbaev on the occasion of his 75th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Award for the Eurasian Mathematical Journal

Dear readers, authors, reviewers and members of the Editorial Board of the Eurasian Mathematical Journal,

we are happy to inform you that in November 2017 the Eurasian Mathematical Journal was awarded the title "Leader of Science 2017" by the National Center of State Scientific-Technical Expertise of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan in the nomination "Leader of Kazakhstan Scientific Publications" for the high level of publication activities and high level of citations in Web of Science Core Collection in 2014-2016.

Recall that the Eurasian Mathematical Journal was founded by the L.N. Gumilyov Eurasian National University in 2010 in co-operation with the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia and the University of Padua (see [1]).

The journal pulishes carefully selected original research papers in all areas of mathematics, survey papers, and short communications. It publishes 4 issues in a year. The language of the paper must be English only. Papers accepted for publication are edited from the point of view of English.

More than 280 papers were published written by mathematicians from more than 40 countries representing all continents.

In 2014 the journal was registered in Scopus and in September 2014 the Elsevier-Kazakhstan Research Excellence Forum was held at the L.N. Gumilyov Eurasian National University dedicated to this occasion in which the Elsevier Chairman Professor Y.S. Chi participated (see [3] for details).

In 2015 the Eurasian Mathematical Journal was included in the list of Scopus mathematical journals, quartile Q4, and it is on the way to entering quartile Q3 (see [3]).

Attached is the invitation letter to the Rector of the L.N. Gumilyov Eurasian National University Professor E.B. Sydykov to the ceremony of awarding, which took place in Almaty on November 8, 2017.

On behalf of the Editorial Board of the EMJ V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

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ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
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Ректору
Евразийского национального
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Уважаемый Ерлан Батташевич!

АО «Национальный центр государственной научно-технической экспертизы» (далее АО «НЦГНТЭ») и компания Clarivate Analytics имеют честь пригласить Вас на церемонию вручения независимой награды «Лидер науки-2017» за высокие показатели публикационной активности и цитируемости в Web of Science Core Collection в период 2014-2016 годы.

Ваш журнал *«Eurasian Mathematical Journal»* награждается в номинации «**Лидер казахстанских научных изданий**».

Торжественное мероприятие состоится 8 ноября 2017 года по адресу: г. Алматы, ул. Богенбай батыра 221, Актовый зал, начало в 10.00, регистрация с 09.00 ч.

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Президент **Ибраев А.Ж.**

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SOME RESULTS ON RIEMANNIAN $g-$ NATURAL METRICS GENERATED BY CLASSICAL LIFTS ON THE TANGENT BUNDLE

L. Bilen, A. Gezer

Communicated by J.A. Tussupov

Key words: affine Killing and Killing vector fields, conformal curvature tensor, Riemannian $g-$natural metric, metric connection, tangent bundle.

AMS Mathematics Subject Classification: 53C07, 53B20, 53C21.

Abstract. Let (M, g) be an n-dimensional Riemannian manifold and TM its tangent bundle equipped with Riemannian $g-$natural metrics which are linear combinations of the three classical lifts of the base metric with constant coefficients. The purpose of the present paper is three-fold. Firstly, to study conditions for the tangent bundle TM to be locally conformally flat. Secondly, to define a metric connection on the tangent bundle TM with respect to the Riemannian $g-$natural metric and study some its properties. Finally, to classify affine Killing and Killing vector fields on the tangent bundle TM.

1 Introduction

Let (M, g) be a Remannian manifold. On the tangent bundle TM of M one can construct several (pseudo-) Riemannian metrics obtained by lifting the metric g from the base manifold M to the tangent bundle TM. The first known Riemannian metric which is called the Sasaki metric on the tangent bundle was constructed by S. Sasaki in [16]. It was shown in many papers that the study of some geometric properties of the tangent bundle endowed with the Sasaki metric led to the flatness of the base manifold (for recent survey related to the Sasaki metric, see [7]). In the next years, using, in particular, various kinds of classical lifts of the metric g from M to TM, some authors were interested in finding other lifted metrics on the tangent bundles, with quite interesting properties (e.g. [4, 5, 6, 15, 18, 20]). According to this concept of lift, the Sasaki metric is no other than the diagonal lift of the base metric. V. Oproiu and his collaborators constructed natural metrics on the tangent bundles of Riemannian manifolds possessing interesting geometric properties ([11, 12, 13, 14]). All metrics mentioned above belong to a wide class of the so-called *g-natural metrics* on the tangent bundle, initially classified by O. Kowalski and M. Sekizawa [10] and fully characterized by M.T.K Abbassi and M. Sarih [1, 2, 3] (see also [9] for other presentation of the basic result from [10]).

In this paper we consider a tangent bundle TM equipped with a Riemanian $g-$natural metrics of the form $G = a^S g + b^H g + c^V g$, where a, b and c are constants satisfying $a > 0$ and $a(a + c) - b^2 > 0$, generated by the classical lifts: the Sasaki metric ${}^S g$, the horizontal lift ${}^H g$ and the vertical lift ${}^V g$ of g. The Riemanian g-natural metric G is a particular subclass of all Riemannian $g-$natural metrics on TM in [1]. In [4], M.T.K. Abbassi and M. Sarih proved that the Riemanian $g-$natural metric G is as rigid as the Sasaki metric in the following sense: if (TM, G) is a space of constant scalar curvature, then (M, g) is flat. Our aim is to study some properties of the Riemanian $g-$natural metric G in terms of adapted frame which allows the tensor calculus to be efficiently done in TM.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_q^p(TM)$ the corresponding set on the tangent bundle TM.

2 Preliminaries

Let M be an n-dimensional Riemannian manifold and denote by $\pi : TM \to M$ its tangent bundle which fibres the tangent spaces to M. Then TM is a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely a system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i\right)$, $\bar{i} = n+i = n+1, ..., 2n$, where (y^i) is the cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}|_P\right\}$, P being an arbitrary point in U whose coordinates are (x^i). Summation over repeated indices is always assumed.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of X are given, with respect to the induced coordinates, by

$$ {}^V X = X^i \partial_{\bar{i}}, \tag{2.1} $$

$$ {}^H X = X^i \partial_i - y^s \Gamma_{sk}^i X^k \partial_{\bar{i}}, \tag{2.2} $$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ_{sk}^i are the coefficients of the Levi-Civita connection ∇ of g.

Let S be a $(p,q)-$tensor field on $M, q > 1$. We then consider the tensor field $\gamma S \in \Im_{q-1}^p(TM)$ on $\pi^{-1}(U)$ defined by

$$ \gamma S = (y^s S_{si_2...i_q}^{j_1...j_p}) \partial_{\overline{j_1}} \otimes ... \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes ... \otimes dx^{i_q} $$

with respect to the induced coordinates (x^i, y^i)([20], p. 12). The tensor field γS defined on each $\pi^{-1}(U)$ determines global tensor field on TM. For any $C \in \Im_1^1(M)$, we easily see that γC has components, with respect to the induced coordinates (x^i, y^i),

$$ (\gamma C) = y^s C_s^i \partial_{\bar{i}}. \tag{2.3} $$

Also, note that $(\gamma C)({}^V f) = 0$, $f \in \Im_0^0(M)$, i.e. γC is a vertical vector field on TM.

With the connection ∇_g of g on M, we can introduce on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM a frame field which is very useful in our computation. The adapted frame on $\pi^{-1}(U)$ consists of the following $2n$ linearly independent vector fields:

$$ \begin{aligned} E_j &= \partial_j - y^s \Gamma_{sj}^h \partial_{\overline{h}}, \\ E_{\bar{j}} &= \partial_{\bar{j}}. \end{aligned} $$

We write the adapted frame as $\{E_\beta\} = \left\{E_j, E_{\bar{j}}\right\}$. Straightforward calculations give the following lemma.

Lemma 2.1. [19, 20] *The Lie brackets of the adapted frame of TM satisfy the following identities:*

$$ \begin{cases} [E_j, E_i] = y^b R_{ijb}^{\ \ \ a} E_{\bar{a}}, \\ [E_j, E_{\bar{i}}] = \Gamma_{ji}^a E_{\bar{a}}, \\ \left[E_{\bar{j}}, E_{\bar{i}}\right] = 0, \end{cases} $$

where $R_{ijb}^{\ \ \ a}$ denote the components of the curvature tensor of M.

Using (2.1), (2.2) and (2.3), we have

$$ {}^{H}X = X^{j}E_{j}, \tag{2.4} $$

$$ {}^{V}X = X^{j}E_{\bar{j}}, \tag{2.5} $$

and

$$ \gamma C = y^{s}C_{s}^{j}E_{\bar{j}} \tag{2.6} $$

with respect to the adapted frame $\{E_{\beta}\}$ (for details, see [20]).

3 On locally conformally flat tangent bundles with Riemannian $g-$natural metrics of the form $G = a\,{}^{S}g + b\,{}^{H}g + c\,{}^{V}g$

The Riemannian $g-$natural metric $G = a\,{}^{S}g + b\,{}^{H}g + c\,{}^{V}g$ on the tangent bundle TM over the Riemannian manifold (M, g) is defined by the following three equations:

$$ G({}^{H}X, {}^{H}Y) = (a+c)^{V}(g(X,Y)), \tag{3.1} $$

$$ G({}^{V}X, {}^{H}Y) = G({}^{H}X, {}^{V}Y) = b^{V}(g(X,Y)), \tag{3.2} $$

$$ G({}^{V}X, {}^{V}Y) = a^{V}(g(X,Y)) \tag{3.3} $$

for all $X, Y \in \Im_{0}^{1}(M)$, where the inequalities $a > 0$ and $a(a+c) - b^{2} > 0$ hold [4].

If $g = g_{ij}dx^{i}dx^{j}$ is the expression of the Riemannian metric g on M, then from (3.1)-(3.3), the Riemannian $g-$natural metric G is expressed in the adapted local frame by

$$ G = (G_{\alpha\beta}) = \begin{pmatrix} (a+c)g_{ij} & bg_{ij} \\ bg_{ij} & ag_{ij} \end{pmatrix}. \tag{3.4} $$

For the Levi-Civita connection of the Riemannian $g-$natural metric G, we have the following.

Proposition 3.1. [4] *Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor. Then the corresponding Levi-Civita connection $\widetilde{\nabla}$ of TM with the Riemannian g-natural metric of the form $G = a\,{}^{S}g + b\,{}^{H}g + c\,{}^{V}g$ is characterized by the following equalities*

$$ \begin{aligned} i)\ \widetilde{\nabla}_{{}^{H}X}\,{}^{H}Y &= {}^{H}(\nabla_{X}Y) + \frac{ab}{2\alpha}\,{}^{H}[R(y,X)Y + R(y,Y)X] \\ &\quad + \frac{b^{2}}{\alpha}\,{}^{V}(R(X,y)Y - \frac{a(a+c)}{2\alpha}\,{}^{V}(R(X,Y)y), \\ ii)\ \widetilde{\nabla}_{{}^{H}X}\,{}^{V}Y &= -\frac{a^{2}}{2\alpha}\,{}^{H}(R(Y,y)X) + {}^{V}(\nabla_{X}Y) + \frac{ab}{2\alpha}\,{}^{V}(R(Y,y)X), \\ iii)\ \widetilde{\nabla}_{{}^{V}X}\,{}^{H}Y &= -\frac{a^{2}}{2\alpha}\,{}^{H}(R(X,y)Y + \frac{ab}{2\alpha}\,{}^{V}(R(X,y)Y), \\ iv)\ \widetilde{\nabla}_{{}^{V}X}\,{}^{V}Y &= 0 \end{aligned} \tag{3.5} $$

for all $X, Y \in \Im_{0}^{1}(M)$, where $\alpha = a(a+c) - b^{2}$ and the inequalities $a > 0$ and $a(a+c) - b^{2} > 0$ hold.

For the tangent bundle (TM, G), the conformal curvature tensor $\widetilde{C}$ is given by:

$$\begin{aligned}\widetilde{C}_{\alpha\gamma\beta\sigma} &= \widetilde{R}_{\alpha\gamma\beta\sigma} - \frac{1}{2(n-1)}(G_{\gamma\sigma}\widetilde{R}_{\alpha\beta} - G_{\alpha\sigma}\widetilde{R}_{\gamma\beta} + G_{\alpha\beta}\ \widetilde{R}_{\gamma\sigma} - \ G_{\gamma\beta}\ \widetilde{R}_{\alpha\sigma}) \\ &+ \frac{\widetilde{S}}{2(n-1)(2n-1)}(G_{\alpha\beta}G_{\gamma\sigma} - G_{\alpha\sigma}G_{\gamma\beta}).\end{aligned}$$

A Riemannian manifold (M, g) ($\dim M \geq 4$) is called locally conformally flat if the conformal curvature tensor $\widetilde{C} = 0$. In this section, we shall prove the following theorem.

Theorem 3.1. *Let M be an $n-$dimensional Riemannian manifold and TM its tangent bundle with the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$, such that $a > 0$ and $a(a+c) - b^2 > 0$. The tangent bundle TM is locally conformally flat if and only if M is locally flat and the Riemannian $g-$natural metric G is of the form $G = a(^S g +^H g) + c^V g$.*

Now, let us consider the formula $\widetilde{\nabla}_{E_\alpha} E_\beta = \widetilde{\Gamma}^\gamma_{\alpha\beta} E_\gamma$ with respect to the adapted frame $\{E_\beta\}$, where $\widetilde{\Gamma}^\gamma_{\alpha\beta}$ denote the components of the Levi-Civita connection $\widetilde{\nabla}$ of G. On taking account of (3.5) for the cases ${}^V X = E_{\bar{i}}$, ${}^H X = E_i$ and ${}^V Y = E_{\bar{j}}$, ${}^H Y = E_j$, we have the following.

Lemma 3.1. *The Levi-Civita connection $\widetilde{\nabla}$ of (TM, G) is characterized by the following equalities*

$$\begin{cases} \widetilde{\nabla}_{E_i} E_j = \{\Gamma^h_{ij} - \frac{ab}{2\alpha} y^s (R_{isj}{}^h + R_{jsi}{}^h)\} E_h + \{\frac{b^2}{\alpha} y^s R_{isj}{}^h - \frac{a(a+c)}{2\alpha} y^s R_{ijs}{}^h\} E_{\bar{h}}, \\ \widetilde{\nabla}_{E_i} E_{\bar{j}} = \{-\frac{a^2}{2\alpha} y^s R_{jsi}{}^h\} E_h + \{\Gamma^h_{ij} + \frac{ab}{2\alpha} y^s R_{jsi}{}^h\} E_{\bar{h}}, \\ \widetilde{\nabla}_{E_{\bar{i}}} E_j = \{-\frac{a^2}{2\alpha} y^s R_{isj}{}^h\} E_h + \{\frac{ab}{2\alpha} y^s R_{isj}{}^h\} E_{\bar{h}}, \\ \widetilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0, \end{cases} \tag{3.6}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ^h_{ij}, and $R_{hji}{}^s$ respectively, denote the components of the Levi-Civita connection ∇ and the curvature tensor field R of g on M with respect to the natural frame $\{\partial_i\}$.

The Riemannian curvature tensor $\widetilde{R}$ of (TM, G) is obtained from the well-known formula

$$\widetilde{R}\left(\widetilde{X}, \widetilde{Y}\right)\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(TM)$. For pairs $\widetilde{X} = E_m, E_{\bar{m}}$ and $\widetilde{Y} = E_i, E_{\bar{i}}$ and $\widetilde{Z} = E_j, E_{\bar{j}}$, by Lemma 2.1 and 3.1, we get the following lemma.

Lemma 3.2. *The components of the curvature tensor $\widetilde{R}$ of (TM,G) are as follows:*

$$\begin{aligned}
(i)\quad & \widetilde{R}(E_m,E_i)E_j \\
= & \Big\{R_{mij}{}^{k}+\frac{ab}{2\alpha}y^{s}\left[\nabla_i\left(R_{msj}{}^{k}+R_{jsm}{}^{k}\right)-\nabla_m\left(R_{isj}{}^{k}+R_{jsi}{}^{k}\right)\right] \\
& +\frac{a^2}{4\alpha}y^py^s\left[R_{hsm}{}^{k}R_{ijp}{}^{h}-R_{hsi}{}^{k}R_{mjp}{}^{h}-2R_{mis}{}^{h}R_{hpj}{}^{k}\right] \\
& +\frac{a^2b^2}{4\alpha^2}y^py^s\left[R_{msh}{}^{k}\left(R_{ipj}{}^{h}+R_{jpi}{}^{h}\right)-R_{ish}{}^{k}\left(R_{mpj}{}^{h}+R_{jpm}{}^{h}\right)\right]\Big\}E_k \\
& +\Big\{\frac{b^2}{\alpha}y^s\left[\nabla_mR_{isj}{}^{k}-\nabla_iR_{msj}{}^{k}\right]+\frac{a\left(a+c\right)}{2\alpha}y^s\left[\nabla_iR_{mjs}{}^{k}-\nabla_mR_{ijs}{}^{k}\right] \\
& +\frac{ab^3}{4\alpha^2}y^py^s\left[R_{smh}{}^{k}\left(R_{ipj}{}^{h}+R_{jpi}{}^{h}\right)+R_{ish}{}^{k}\left(R_{mpj}{}^{h}+R_{jpm}{}^{h}\right)\right] \\
& +\frac{ab}{4\alpha}y^py^s[R_{mhs}{}^{k}\left(R_{ipj}{}^{h}+R_{jpi}{}^{h}\right)-R_{ihs}{}^{k}\left(R_{mpj}{}^{h}+R_{jpm}{}^{h}\right) \\
& +R_{hsi}{}^{k}R_{mjp}{}^{h}-R_{hsm}{}^{k}R_{ijp}{}^{h}+2R_{mis}{}^{h}R_{hpj}{}^{k}]\Big\}E_{\overline{k}}
\end{aligned}$$

$$\begin{aligned}
(ii)\quad & \widetilde{R}(E_m,E_i)E_{\overline{j}} \\
= & \Big\{\frac{a^2}{2\alpha}y^s\left(\nabla_iR_{jsm}{}^{k}-\nabla_mR_{jsi}{}^{k}\right)+\frac{a^3b}{4\alpha^2}y^py^s\left[R_{jpi}{}^{h}R_{msh}{}^{k}-R_{jpm}{}^{h}R_{ish}{}^{k}\right]\Big\}E_k \\
& +\Big\{R_{mij}{}^{k}+\frac{ab}{2\alpha}y^s\left(\nabla_mR_{jsi}{}^{k}-\nabla_iR_{jsm}{}^{k}\right)+\frac{a^2b^2}{4\alpha^2}y^py^s\left[R_{jpi}{}^{h}R_{smh}{}^{k}+R_{jpm}{}^{h}R_{ish}{}^{k}\right] \\
& +\frac{a^2}{4\alpha}y^py^s\left[R_{mhs}{}^{k}R_{jpi}{}^{h}-R_{ihs}{}^{k}R_{jpm}{}^{h}\right]\Big\}E_{\overline{k}}
\end{aligned}$$

$$\begin{aligned}
(iii)\quad & \widetilde{R}(E_m,E_{\overline{i}})E_j \\
= & \Big\{\frac{ab}{2\alpha}(R_{mij}{}^{k}+R_{jim}{}^{k})-\frac{a^2}{2\alpha}y^s\nabla_mR_{isj}{}^{k} \\
& +\frac{a^3b}{4\alpha^2}y^py^s\left[R_{ipj}{}^{h}R_{msh}{}^{k}-R_{ish}{}^{k}\left(R_{mpj}{}^{h}+R_{jpm}{}^{h}\right)\right]\Big\}E_k \\
& +\Big\{\frac{-b^2}{\alpha}R_{mij}{}^{k}+\frac{a(a+c)}{2\alpha}R_{mji}{}^{k}+\frac{ab}{2\alpha}y^s\nabla_mR_{isj}{}^{k}+\frac{a^2}{4\alpha}y^py^sR_{mhs}{}^{k}R_{ipj}{}^{h} \\
& +\frac{a^2b^2}{4\alpha^2}y^py^s\left[R_{ish}{}^{k}\left(R_{mpj}{}^{h}+R_{jpm}{}^{h}\right)+R_{ipj}{}^{h}R_{smh}{}^{k}\right]\Big\}E_{\overline{k}}
\end{aligned}$$

$$\begin{aligned}
(iv)\quad & \widetilde{R}(E_{\overline{m}},E_i)E_j \\
= & \Big\{\frac{-ab}{2\alpha}(R_{imj}{}^{k}+R_{jmi}{}^{k})+\frac{a^2}{2\alpha}y^s\nabla_iR_{msj}{}^{k} \\
& +\frac{a^3b}{4\alpha^2}y^py^s\left[R_{mph}{}^{k}\left(R_{jsi}{}^{h}+R_{isj}{}^{h}\right)-R_{ish}{}^{k}R_{mpj}{}^{h}\right]\Big\}E_k \\
& +\Big\{\frac{b^2}{\alpha}R_{imj}{}^{k}-\frac{a(a+c)}{2\alpha}R_{ijm}{}^{k}-\frac{ab}{2\alpha}y^s\nabla_iR_{msj}{}^{k} \\
& -\frac{a^2b^2}{4\alpha^2}y^py^s\left[R_{msh}{}^{k}\left(R_{jpi}{}^{h}+R_{ipj}{}^{h}\right)+R_{mpj}{}^{h}R_{sih}{}^{k}\right] \\
& -\frac{a^2}{4\alpha}y^py^sR_{ish}{}^{k}R_{mpj}{}^{h}\Big\}E_{\overline{k}}
\end{aligned}$$

$$(v)\ \widetilde{R}(E_m, E_{\bar{i}})E_{\bar{j}} = \left\{\frac{a^2}{2\alpha}R_{jim}{}^k - \frac{a^4}{4\alpha^2}y^p y^s R_{ish}{}^k R_{jpm}{}^h\right\} E_k + \left\{\frac{-ab}{2\alpha}R_{jim}{}^k + \frac{a^3 b}{4\alpha^2}y^p y^s R_{ish}{}^k R_{jpm}{}^h\right\} E_{\bar{k}}$$

$$(vi)\ \widetilde{R}(E_{\overline{m}}, E_i)E_{\bar{j}} = \left\{-\frac{a^2}{2\alpha}R_{jmi}{}^k + \frac{a^4}{4\alpha^2}y^p y^s R_{msh}{}^k R_{jpi}{}^h\right\} E_k + \left\{\frac{ab}{2\alpha}R_{jmi}{}^k - \frac{a^3 b}{4\alpha^2}y^p y^s R_{msh}{}^k R_{jpi}{}^h\right\} E_{\bar{k}}$$

$$(vii)\quad \widetilde{R}(E_{\overline{m}}, E_{\bar{i}})E_j = \left\{\frac{a^2}{\alpha}R_{mij}{}^k + \frac{a^4}{4\alpha^2}y^p y^s\left[R_{msh}{}^k R_{ipj}{}^h - R_{ish}{}^k R_{mpj}{}^h\right]\right\} E_k + \left\{\frac{ab}{\alpha}R_{imj}{}^k - \frac{a^3 b}{4\alpha^2}y^p y^s\left[R_{msh}{}^k R_{ipj}{}^h - R_{ish}{}^k R_{mpj}{}^h\right]\right\} E_{\bar{k}}$$

$$(viii)\ \widetilde{R}(E_{\overline{m}}, E_{\bar{i}})E_{\bar{j}} = 0$$

with respect to the adapted frame $\{E_\beta\}$, where $\alpha = a(a+c) - b^2$ (for invariant forms of $\widetilde{R}$, see [4]).

Let $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\sigma\alpha\beta}{}^{\sigma}$ denote the components of Ricci tensor of (TM, G). From Lemma 3.2 we have

Lemma 3.3. *The components $\widetilde{R}_{\alpha\beta}$ of the Ricci tensor of (TM, G) are as follows:*

$$(i)\ \widetilde{R}_{\bar{i}\bar{j}} = -\frac{a^4}{4\alpha^2}y^p y^s R_{ish}{}^m R_{jpm}{}^h$$

$$(ii)\ \widetilde{R}_{i\bar{j}} = -\frac{ab}{2\alpha}R_{ji} + \frac{a^2}{2\alpha}y^s(\nabla_s R_{ji} - \nabla_j R_{si}) - \frac{a^3 b}{4\alpha^2}y^p y^s R_{pjm}{}^h R_{sih}{}^m,$$

$$(iii)\ \widetilde{R}_{\bar{i}j} = -\frac{ab}{2\alpha}R_{ij} + \frac{a^2}{2\alpha}y^s(\nabla_s R_{ij} - \nabla_i R_{sj}) - \frac{a^3 b}{4\alpha^2}y^p y^s R_{pim}{}^h R_{sjh}{}^m,$$

$$(iv)\ \widetilde{R}_{ij} = (1 - \frac{b^2}{\alpha})R_{ij} + \frac{ab}{2\alpha}y^s(2\nabla_s R_{ij} - \nabla_i R_{sj} - \nabla_j R_{is}) + \frac{a^2}{4\alpha}y^p y^s(R_{shi}{}^m R_{mjp}{}^h + R_{his}{}^m R_{mpj}{}^h - 2R_{mis}{}^h R_{hpj}{}^m) - \frac{a^2 b^2}{4\alpha^2}y^p y^s(R_{ish}{}^m R_{mpj}{}^h + R_{ish}{}^m R_{jpm}{}^h)$$

with respect to the adapted frame $\{E_\beta\}$.

The scalar curvature $\widetilde{S}$ of (TM, G) is defined by $\widetilde{S} = G^{\alpha\beta}\widetilde{R}_{\alpha\beta}$, where $G^{\alpha\beta}$ denote the components of the inverse matrix of $(G_{\alpha\beta})$ in (3.4) which has the following local expression:

$$G^{-1} = (G^{\beta\gamma}) = \begin{pmatrix} \frac{a}{\alpha}g^{jk} & -\frac{b}{\alpha}g^{jk} \\ -\frac{b}{\alpha}g^{jk} & \frac{(a+c)}{\alpha}g^{jk} \end{pmatrix}.$$

In view of Lemma 3.3, the following result is obtained.

Lemma 3.4. *The scalar curvature* $\widetilde{S}$ *of* (TM, G) *is given by*

$$\widetilde{S} = \frac{a}{\alpha}S - \frac{a^3}{2\alpha^4}y^p y^s R_{pijm}R_s{}^{ijm},$$

where S *denote the scalar curvature of* (M, g).

Proof of Theorem 3.1. The tangent bundle (TM, G) is locally conformally flat if and only if the components of the curvature tensor $\widetilde{R}$ of (TM, G) satisfy the following equations:

$$\begin{aligned}\widetilde{R}_{\alpha\gamma\beta\sigma} &= \frac{1}{2(n-1)}(G_{\gamma\sigma}\widetilde{R}_{\alpha\beta} - G_{\alpha\sigma}\widetilde{R}_{\gamma\beta} + G_{\alpha\beta}\widetilde{R}_{\gamma\sigma} - G_{\gamma\beta}\widetilde{R}_{\alpha\sigma}) \\ &\quad - \frac{\widetilde{S}}{2(2n-1)(n-1)}(G_{\alpha\beta}G_{\gamma\sigma} - G_{\alpha\sigma}G_{\gamma\beta}),\end{aligned} \tag{3.7}$$

where $\widetilde{R}_{\alpha\gamma\beta\sigma} = G_{\sigma\varepsilon}\widetilde{R}_{\alpha\gamma\beta}{}^{\varepsilon}$.

In the cases of $\alpha = \overline{m}$, $\gamma = \overline{i}$, $\beta = \overline{j}$, $\sigma = l$ and $\alpha = \overline{m}$, $\gamma = \overline{i}$, $\beta = \overline{j}$, $\sigma = \overline{l}$ in (3.7), we get

$$\begin{aligned}\widetilde{R}_{\overline{m}\overline{i}\overline{j}l} &= \frac{1}{2(n-1)}(bg_{il}\widetilde{R}_{\overline{m}\overline{j}} - bg_{ml}\widetilde{R}_{\overline{i}\overline{j}} + ag_{mj}\widetilde{R}_{\overline{i}l} - ag_{ij}\widetilde{R}_{\overline{m}l}) \\ &\quad - \frac{\widetilde{S}}{2(2n-1)(n-1)}(abg_{mj}g_{il} - abg_{ml}g_{ij})\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\widetilde{R}_{\overline{m}\overline{i}\overline{j}\overline{l}} &= \frac{1}{2(n-1)}(ag_{il}\widetilde{R}_{\overline{m}\overline{j}} - ag_{ml}\widetilde{R}_{\overline{i}\overline{j}} + ag_{mj}\widetilde{R}_{\overline{i}\overline{l}} - ag_{ij}\widetilde{R}_{\overline{m}\overline{l}}) \\ &\quad - \frac{\widetilde{S}}{2(2n-1)(n-1)}(a^2 g_{mj}g_{il} - a^2 g_{ml}g_{ij}).\end{aligned} \tag{3.9}$$

By using $(viii)$ of Lemma 3.2 and (3.4) we have $\widetilde{R}_{\overline{m}\overline{i}\overline{j}k} = 0$ and $\widetilde{R}_{\overline{m}\overline{i}\overline{j}\overline{k}} = 0$. Thus the equations (3.8) and (3.9) reduce to the followings:

$$\begin{aligned}&\frac{\widetilde{S}}{2(2n-1)(n-1)}(abg_{mj}g_{il} - abg_{ml}g_{ij}) \\ = &\frac{1}{2(n-1)}(bg_{il}\widetilde{R}_{\overline{m}\overline{j}} - bg_{ml}\widetilde{R}_{\overline{i}\overline{j}} + ag_{mj}\widetilde{R}_{\overline{i}l} - ag_{ij}\widetilde{R}_{\overline{m}l})\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\frac{\widetilde{S}}{2(2n-1)(n-1)}(a^2 g_{mj}g_{il} - a^2 g_{ml}g_{ij}) \\ = &\frac{1}{2(n-1)}(ag_{il}\widetilde{R}_{\overline{m}\overline{j}} - ag_{ml}\widetilde{R}_{\overline{i}\overline{j}} + ag_{mj}\widetilde{R}_{\overline{i}\overline{l}} - ag_{ij}\widetilde{R}_{\overline{m}\overline{l}}).\end{aligned} \tag{3.11}$$

Comparing (3.10) with (3.11), we obtain $a = b$ and $\widetilde{R}_{\overline{i}l} = \widetilde{R}_{\overline{i}\overline{l}}$. From here, by means of (i) and (iii) of Lemma 3.3, we get

$$R_{ij} = 0$$

and

$$\widetilde{R}_{\overline{i}\overline{l}} = -\frac{a^4}{4\alpha^2}y^p y^s R_{pih}{}^m R_{slm}{}^h. \tag{3.12}$$

Transvecting (3.11) by g^{il} and then by g^{mj} respectively, we have

$$\frac{an}{2n-1}\widetilde{S} = 2g^{il}\widetilde{R}_{\bar{i}\bar{l}}. \tag{3.13}$$

By (3.12), we calculate

$$\begin{aligned} g^{il}\widetilde{R}_{\bar{i}\bar{l}} &= -\frac{a^4}{4\alpha^2}y^p y^s g^{il} R_{pih}{}^m R_{slm}{}^h \\ &= \frac{a^4}{4\alpha^2}y^p y^s R_{pihj} R_s{}^{ihj} \\ &= -\frac{a\alpha^2}{2}\widetilde{S}. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain

$$(\frac{an}{2n-1} + a\alpha^2)\widetilde{S} = 0$$

from which $\widetilde{S} = 0$, i.e. $R_{pihj}R_s{}^{ihj} = 0$ which gives $R_{pihj} = 0$. This completes the proof. □

4 Metric connection with torsion of Riemannian g-natural metrics of the form $G = a^S g + b^H g + c^V g$

It is well-known that a linear connection ∇ on a Riemannian manifold (M, g) is metric connection with repect to g if $\nabla g = 0$. The (unique) metric connection ∇ which is torsion-free is called the Levi-Civita connection of g. But there exist other metric connections whose torsion tensor is non-zero on a Riemannian manifold (M, g). In this section we consider a metric connection on (TM, G) with a non-zero torsion tensor.

The horizontal lift ${}^H\nabla$ of a linear connection on M to TM is the unique linear connection defined on TM by the following conditions

$$\begin{aligned} {}^H\nabla_{{}^HX}{}^HY &= {}^H(\nabla_X Y),\ {}^H\nabla_{{}^HX}{}^VY = {}^V(\nabla_X Y) \\ {}^H\nabla_{{}^VX}{}^HY &= 0,\ {}^H\nabla_{{}^VX}{}^VY = 0 \end{aligned}$$

for all $X, Y \in \Im_0^1(M)$. The torsion tensor $\widetilde{T}$ of ${}^H\nabla$ satisfies the conditions

$$\begin{aligned} \widetilde{T}({}^VX, {}^VY) &= 0,\ \widetilde{T}({}^VX, {}^HY) = {}^V(T(X,Y)), \\ \widetilde{T}({}^HX, {}^HY) &= {}^H(T(X,Y)) - \gamma R(X,Y) \end{aligned}$$

where T and R are respectively the torsion and curvature tensor fields of the linear connection ∇ on M (for details, see, [20]). From the last identities above, we say that the connection ${}^H\nabla$ has non-zero torsion tensor even if ∇ is selected as the Levi-Civita connection ∇_g of g on the Riemannian manifold (M, g). By using the definition of the horizontal lift ${}^H\nabla$ and (3.1)-(3.3), on calculating

$$({}^H\nabla_{\widetilde{X}}G)(\widetilde{Y}, \widetilde{Z}) = \widetilde{X}(G(\widetilde{Y}, \widetilde{Z})) - G({}^H\nabla_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) - G(\widetilde{Y}, {}^H\nabla_{\widetilde{X}}\widetilde{Z})$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im_0^1(TM)$, we get

$$\begin{aligned} ({}^H\nabla_{{}^VX}G)({}^VY, {}^VZ) &= 0,\ ({}^H\nabla_{{}^HX}G)({}^VY, {}^VZ) = a\ {}^V((\nabla_X g)(Y,Z)), \\ ({}^H\nabla_{{}^VX}G)({}^VY, {}^HZ) &= 0,\ ({}^H\nabla_{{}^HX}G)({}^VY, {}^HZ) = b\ {}^V((\nabla_X g)(Y,Z)), \\ ({}^H\nabla_{{}^VX}G)({}^HY, {}^VZ) &= 0,\ ({}^H\nabla_{{}^HX}G)({}^HY, {}^VZ) = b\ {}^V((\nabla_X g)(Y,Z)), \\ ({}^H\nabla_{{}^VX}G)({}^HY, {}^HZ) &= 0,\ ({}^H\nabla_{{}^HX}G)({}^HY, {}^HZ) = (a+c)\ {}^V((\nabla_X g)(Y,Z)). \end{aligned}$$

If $\nabla = \nabla_g$, where ∇_g is the Levi-Civita connection of g on the Riemannian manifold (M, g), then ${}^H\nabla G = 0$. This gives the following result.

Theorem 4.1. *The horizontal lift ${}^H(\nabla_g)$ of the Levi-Civita connection ∇_g of a Riemannian manifold (M, g) is a metric connection with a non-zero torsion on TM with respect to the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$, such that $a > 0$ and $a(a + c) - b^2 > 0$.*

The metric connection ${}^H(\nabla_g)$ is given by

$$\begin{cases} \nabla_{E_i} E_j = \Gamma_{ij}^h E_h, \\ \widetilde{\nabla}_{E_i} E_{\overline{j}} = \Gamma_{ij}^h E_{\overline{h}}, \\ \widetilde{\nabla}_{E_{\overline{i}}} E_j = 0, \ \widetilde{\nabla}_{E_{\overline{i}}} E_{\overline{j}} = 0, \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$. Also we can say that the metric connection ${}^H(\nabla_g)$ and the Levi-Civita connection $\widetilde{\nabla}$ of G coincide if and only if (M, g) is flat. On computing the contracted curvature tensor (Ricci tensor) of ${}^H(\nabla_g)$, we have the components ${}^H R_{\alpha\beta} = ^H R_{\gamma\alpha\beta}^{\ \ \ \gamma}$ such that

$${}^H R_{ij} = R_{ij}, \ {}^H R_{\overline{i}j} = 0, \ {}^H R_{i\overline{j}} = 0, \ {}^H R_{\overline{i}\overline{j}} = 0$$

with respect to the adapted frame, where R_{ij} is the Ricci tensor of ∇_g on M ([20], p.154). For the scalar curvature of ${}^H(\nabla_g)$ with respect to G, we get

$$\begin{aligned} {}^H S &= G^{\alpha\beta}\, {}^H R_{\alpha\beta} = \frac{a}{\alpha} g^{ij} R_{ij} \\ &= \frac{a}{\alpha} S. \end{aligned}$$

From the last identity, we can state the following theorem.

Theorem 4.2. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$, such that $a > 0$ and $a(a+c) - b^2 > 0$. The scalar curvature ${}^H S$ of the tangent bundle TM with the metric connection ${}^H(\nabla_g)$ with respect to G is zero if and only if the scalar curvature S of ∇_g on (M, g) is zero.*

5 Affine Killing and Killing Vectors with respect to Riemannian g-natural metrics of the form $G = a^S g + b^H g + c^V g$

Let A is a $(1, 1)-$tensor field on M with the components (A_j^i), then ${}^*A$ defined by

$${}^*A = \{A_s^i y^f\} E_i,$$

with respect to the adapted frame $\{E_\beta\}$ is a smooth vector field on TM [17].

Let $L_{\widetilde{X}}$ be the Lie derivation with respect to the vector field $\widetilde{X}$. We shall first state following lemma which are needed later on.

Lemma 5.1. (see [8]) *The Lie derivations of the adapted frame and its dual basis with respect to $\widetilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ are given as follows:*

$$\begin{aligned} &(1) \quad L_{\widetilde{X}} E_h = -(E_h v^a) E_a + \left\{ y^b v^c R_{hcb}^{\ \ \ a} - v^{\overline{b}} \Gamma_{bh}^a - (E_h v^{\overline{a}}) \right\} E_{\overline{a}} \\ &(2) \quad L_{\widetilde{X}} E_{\overline{h}} = -(E_{\overline{h}} v^a) E_a + \left\{ v^b \Gamma_{bh}^a - (E_{\overline{h}} v^{\overline{a}}) \right\} E_{\overline{a}} \\ &(3) \quad L_{\widetilde{X}} dx^h = (E_a v^h) dx^a + (E_{\overline{a}} v^h) \delta y^a \\ &(4) \quad L_{\widetilde{X}} \delta y^h = \left\{ y^c v^b R_{bac}^{\ \ \ h} + v^{\overline{b}} \Gamma_{ba}^h + (E_a v^{\overline{h}}) \right\} dx^a \\ &\qquad\qquad - \left\{ v^b \Gamma_{ba}^h - (E_{\overline{a}} v^{\overline{h}}) \right\} \delta y^a. \end{aligned}$$

The general forms of affine Killing vector fields on (TM, G) are given by

Theorem 5.1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$, such that $a > 0$ and $a(a+c) - b^2 > 0$. Then the vector field $\widetilde{X}$ is an affine Killing vector field on (TM, G) if and only if the vector field $\widetilde{X}$ defined by*

$$\widetilde{X} = {}^H B + {}^V D + \gamma C + {}^* A,$$

where $B = (B^h)$, $D = (D^h) \in \Im^1_0(M)$ and $A = (A^h_i)$, $C = (C^h_i) \in \Im^1_1(M)$ satisfying

(i) $\nabla_i A^h_j = \frac{a^2}{2\alpha} R_{jli}{}^h D^l$,
(ii) $\nabla_i C^h_j = -R_{lij}{}^h B^l - \frac{ab}{2\alpha} R_{jli}{}^h D^l$,
(iii) $L_B \Gamma^h_{ji} = \frac{ab}{2\alpha}(R_{jli}{}^h + R_{ilj}{}^h) D^l$,
(iv) $L_D \Gamma^h_{ji} = -\frac{a(a+c)}{\alpha}(R_{jli}{}^h - \frac{1}{2} R_{jil}{}^h) D^l$,
(v) $R_{ajl}{}^h A^l_i = 0$,
(vi) $\frac{a^2}{2\alpha} B^l \nabla_l R_{jsi}{}^h = \frac{a^2}{2\alpha} R_{jsi}{}^l \nabla_l B^h - \frac{a^2}{2\alpha} R_{jsl}{}^h \nabla_i B^l - \frac{a^2}{2\alpha} R_{lsi}{}^h C^l_j$
$-\frac{a^2}{2\alpha} R_{jli}{}^h C^l_s - \frac{ab}{2\alpha} R_{jsi}{}^l A^h_l$,
(vii) $R_{jsi}{}^l (\frac{ab}{2\alpha} \nabla_l B^h - \frac{b^2}{2\alpha} A^h_l - \frac{ab}{2\alpha} C^h_l + \frac{a^2}{2\alpha} \nabla_l D^h) = 0$,
(viii) $\frac{ab}{2\alpha} D^l \nabla_j R_{lis}{}^h = R_{lsi}{}^h (\frac{b^2}{\alpha} \nabla_j B^l + \frac{ab}{2\alpha} \nabla_j D^l - \frac{b^2}{\alpha} C^l_j) + \frac{ab}{2\alpha} R_{lsj}{}^h \nabla_i D^l$
$-\frac{a(a+c)}{2\alpha}[R_{jil}{}^h (C^l_s + \nabla_s B^l) + R_{jls}{}^h (C^l_i + \nabla_i B^l) + R_{lis}{}^h (C^l_j + \nabla_j B^l)]$
$-\frac{ab^2 + a^2(a+c)}{2b\alpha} R_{jis}{}^l \nabla_l D^h$.

Proof. Let $\widetilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ be an affine Killing vector on TM:

$$(L_{\widetilde{X}} \widetilde{\nabla})(\widetilde{Y}, \widetilde{Z}) = L_{\widetilde{X}}(\widetilde{\nabla}_{\widetilde{Y}} \widetilde{Z}) - \widetilde{\nabla}_{\widetilde{Y}}(L_{\widetilde{X}} \widetilde{Z}) - \widetilde{\nabla}_{(L_{\widetilde{X}} \widetilde{Y})} \widetilde{Z} = 0 \tag{5.1}$$

for any $\tilde{X}, \tilde{Y}, \widetilde{Z} \in \Im^1_0(TM)$.

Putting $\widetilde{Y} = E_{\bar{j}}$ and $\widetilde{Z} = E_{\bar{i}}$ in (5.1), by virtue of Lemma 3.1 and 5.1 we have

$$\begin{aligned}
&(L_{\widetilde{X}} \widetilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) \\
= \;& L_{\widetilde{X}}(\widetilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}}) - \widetilde{\nabla}_{E_{\bar{j}}}(L_{\widetilde{X}} E_{\bar{i}}) - \widetilde{\nabla}_{(L_{\widetilde{X}} E_{\bar{j}})} E_{\bar{i}} \\
= \;& L_{\widetilde{X}} \widetilde{\Gamma}^h_{\bar{j}\bar{i}} E_h + L_{\widetilde{X}} \widetilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} E_{\bar{h}} \\
= \;& \{\partial_{\bar{j}} \partial_{\bar{i}} v^h + \frac{a^2}{2\alpha} y^b (R_{bic}{}^h \partial_{\bar{j}} v^c + R_{bjc}{}^h \partial_{\bar{i}} v^c)\} E_h \\
& + \{\partial_{\bar{j}} \partial_{\bar{i}} v^{\bar{h}} - \frac{ab}{2\alpha} y^b (R_{bic}{}^h \partial_{\bar{j}} v^c + R_{bjc}{}^h \partial_{\bar{i}} v^c)\} E_{\bar{h}} \\
= \;& 0.
\end{aligned}$$

From $L_{\widetilde{X}} \widetilde{\Gamma}^h_{\bar{j}\bar{i}} = L_{\widetilde{X}} \widetilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} = 0$, we respectively obtain

$$\partial_{\bar{j}} \partial_{\bar{i}} v^h + \frac{a^2}{2\alpha} y^b (R_{bic}{}^h \partial_{\bar{j}} v^c + R_{bjc}{}^h \partial_{\bar{i}} v^c) = 0 \tag{5.2}$$

and

$$\partial_{\bar{j}} \partial_{\bar{i}} v^{\bar{h}} - \frac{ab}{2\alpha} y^b (R_{bic}{}^h \partial_{\bar{j}} v^c + R_{bjc}{}^h \partial_{\bar{i}} v^c) = 0. \tag{5.3}$$

The equation (5.2) is rewritten as follow:

$$\frac{2\alpha}{a^2} \partial_{\bar{j}} \partial_{\bar{i}} v^h = -\partial_{\bar{j}}(y^b R_{bic}{}^h v^c) - \partial_{\bar{i}}(y^b R_{bjc}{}^h v^c). \tag{5.4}$$

Differentiating (5.4) partially, we have

$$\begin{aligned}\tfrac{2\alpha}{a^2}\partial_{\bar{k}}\partial_{\bar{j}}\partial_{\bar{i}}v^h &= -\partial_{\bar{k}}\partial_{\bar{j}}(y^b R_{bic}{}^h v^c) - \partial_{\bar{k}}\partial_{\bar{i}}(y^b R_{bjc}{}^h v^c)\\ &= -\partial_{\bar{j}}\partial_{\bar{i}}(y^b R_{bkc}{}^h v^c) - \partial_{\bar{j}}\partial_{\bar{k}}(y^b R_{bic}{}^h v^c)\\ &= -\partial_{\bar{i}}\partial_{\bar{k}}(y^b R_{bjc}{}^h v^c) - \partial_{\bar{i}}\partial_{\bar{j}}(y^b R_{bkc}{}^h v^c).\end{aligned}$$

Therefore we get $\partial_{\bar{k}}\partial_{\bar{j}}(\frac{\alpha}{a^2}\partial_{\bar{i}}v^h + y^b R_{bic}{}^h v^c) = 0$, hence we can put

$$\partial_{\bar{j}}(\frac{\alpha}{a^2}\partial_{\bar{i}}v^h + y^b R_{bic}{}^h v^c) =: P_{ji}^h \tag{5.5}$$

and

$$\frac{\alpha}{a^2}\partial_{\bar{i}}v^h + y^b R_{bic}{}^h v^c = A_i^h + y^a P_{ai}^h, \tag{5.6}$$

where A_i^h and $P_{ji}{}^h$ are certain functions which depend only on the variables (x^h). The coordinate transformation rule implies that $A = (A_i^h) \in \Im_1^1(M)$ and $P = (P_{ji}^h) \in \Im_2^1(M)$.

From (5.2), we have

$$P_{ij}^h + P_{ji}^h = 2\partial_{\bar{i}}\partial_{\bar{j}}v^h + \frac{a^2}{\alpha}y^b\{R_{bic}{}^h(\partial_{\bar{j}}v^c) + R_{bjc}{}^h(\partial_{\bar{i}}v^c)\} = 0,$$

from which

$$\partial_{\bar{i}}(y^b R_{bjc}{}^h\, v^c) - \partial_{\bar{j}}(y^b R_{bic}{}^h\, v^c) = P_{ij}^h - P_{ji}^h = 2P_{ij}^h.$$

Thus we have

$$\begin{aligned}2y^a P_{ai}^h &= y^a\partial_{\bar{a}}(y^b R_{bic}{}^h\, v^c) - y^a\partial_{\bar{i}}(y^b R_{bac}{}^h\, v^c)\\ &= -2y^a R_{iac}{}^h\, v^c + y^b y^a R_{aic}{}^h\, \partial_{\bar{b}}v^c.\end{aligned} \tag{5.7}$$

From (5.6) and (5.7), we get

$$\frac{\alpha}{a^2}\partial_{\bar{i}}v^h - \frac{1}{2}y^b y^a R_{aic}{}^h\, \partial_{\bar{b}}v^c = A_i^h, \tag{5.8}$$

which gives

$$\frac{\alpha}{a^2}y^a\partial_{\bar{a}}v^h = y^a A_a^h. \tag{5.9}$$

Therefore, substituting (5.9) into (5.8), it follows that

$$\frac{\alpha}{a^2}\partial_{\bar{i}}v^h = A_i^h + \frac{1}{2}y^b y^a R_{aic}{}^h A_b^c \tag{5.10}$$

from which

$$\partial_{\bar{j}}\partial_{\bar{i}}v^h = \frac{a^2}{2\alpha}y^a(R_{aic}{}^h A_j^c + R_{jic}{}^h A_a^c). \tag{5.11}$$

On the other hand, substituting (5.10) into (5.2), we obtain

$$\begin{aligned}\partial_{\bar{j}}\partial_{\bar{i}}v^h = &-\tfrac{a^4}{2\alpha^2}y^a(R_{aic}{}^h A_j^c + R_{ajc}{}^h A_i^c)\\ &-\tfrac{a^4}{4\alpha^2}y^a y^b y^k(R_{aic}{}^h R_{kjl}{}^c A_b^l + R_{ajc}{}^h R_{kil}{}^c A_b^l).\end{aligned} \tag{5.12}$$

Comparing (5.11) with (5.12), we get

$$\left(\frac{a^2}{2\alpha} + \frac{a^4}{2\alpha^2}\right) R_{aic}{}^h A_j^c + \frac{a^2}{2\alpha}R_{jic}{}^h A_a^c + \frac{a^4}{2\alpha^2}R_{ajc}{}^h A_i^c = 0,$$

from which, changing the roles j and i, and adding together, we have

$$\left(\frac{a^2}{2\alpha}+\frac{a^4}{\alpha^2}\right)[R_{aic}{}^hA_j^c+R_{ajc}{}^hA_i^c]=0,$$

that is,

$$R_{aic}{}^hA_j^c+R_{ajc}{}^hA_i^c=0.$$

Furthermore, by virtue of the first Bianchi identity we obtain

$$R_{ajc}{}^hA_i^c=0. \tag{5.13}$$

From (5.11) and (5.13), we have

$$\partial_{\bar{i}}v^h=A_i^h.$$

Hence we can put

$$v^h=y^aA_a^h+B^h, \tag{5.14}$$

where B^h are certain functions which depend only on (x^h). The coordinate transformation rule implies that $B=(B^h)\in\Im_0^1(M)$. Here, substituting (5.14) into (5.5) and using (5.13), we get

$$P_{ji}^h=R_{jic}^h\,B^c. \tag{5.15}$$

Substituting (5.14) into (5.3), by using (5.13), we get

$$v^{\bar{h}}=y^aC_a^h+D^h, \tag{5.16}$$

where C_a^h and D^h are certain functions which depend only on the variables (x^h) and $C=(C_a^h)\in\Im_1^1(M)$ and $D=(D^h)\in\Im_0^1(M)$.

Putting $\widetilde{Y}=E_{\bar{j}}$ and $\widetilde{Z}=E_i$ in (5.1), using (5.13), (5.14), (5.16), Lemma 3.1 and 5.1, we obtain

$$\begin{aligned}
&(L_{\widetilde{X}}\widetilde{\nabla})(E_{\bar{j}},E_i) \qquad (5.17)\\
=\;& L_{\widetilde{X}}(\widetilde{\nabla}_{E_{\bar{j}}}E_i)-\widetilde{\nabla}_{E_{\bar{j}}}(L_{\widetilde{X}}E_i)-\widetilde{\nabla}_{(L_{\widetilde{X}}E_{\bar{j}})}E_i\\
=\;& L_{\widetilde{X}}\widetilde{\Gamma}_{\bar{j}i}^hE_h+L_{\widetilde{X}}\widetilde{\Gamma}_{\bar{j}i}^{\bar{h}}E_{\bar{h}}\\
=\;& \Big\{\nabla_iA_j^h-\frac{a^2}{2\alpha}R_{jli}{}^hD^l+\Big[\frac{a^2}{2\alpha}(-B^l\nabla_lR_{jsi}{}^h+R_{jsi}{}^l\nabla_lB^h-R_{jsl}{}^h\nabla_iB^l\\
&-R_{lsi}{}^hC_j^l-R_{jli}{}^hC_s^l)-\frac{ab}{2\alpha}R_{jsi}{}^lA_l^h\Big]y^s+\Big[\frac{a^2}{2\alpha}(-A_k^l\nabla_lR_{jsi}{}^h\\
&+R_{jsi}{}^l\nabla_lA_k^h-R_{jsl}{}^h\nabla_iA_k^l)\Big]y^sy^k\Big\}E_h\\
&+\Big\{\nabla_iC_j^h+R_{lij}{}^hB^l+\frac{ab}{2\alpha}R_{jli}{}^hD^l+\Big[\frac{a^2}{2\alpha}R_{jsi}{}^l\nabla_lD^h+\frac{ab}{2\alpha}(R_{jsl}{}^h\nabla_iB^l\\
&+B^l\nabla_lR_{jsi}{}^h+R_{lsi}{}^hC_j^l+R_{jli}{}^hC_s^l-R_{jsi}{}^lC_l^h)\Big]y^s+\Big[\frac{ab}{2\alpha}(A_k^l\nabla_lR_{jsi}{}^h\\
&+R_{jkl}{}^h\nabla_iA_s^l)+\frac{a^2}{2\alpha}(R_{jsi}{}^l\nabla_lC_k^h-R_{jsi}{}^cR_{clk}{}^hB^l)\Big]y^sy^k\Big\}E_{\bar{h}}.
\end{aligned}$$

From $L_{\widetilde{X}}\Gamma_{\bar{j}i}^h=0$ in (5.17), we obtain

$$\nabla_iA_j^h=\frac{a^2}{2\alpha}R_{jli}{}^hD^l, \tag{5.18}$$

$$\frac{a^2}{2\alpha}B^l\nabla_l R_{jsi}{}^h = \frac{a^2}{2\alpha}R_{jsi}{}^l\nabla_l B^h - \frac{a^2}{2\alpha}R_{jsl}{}^h\nabla_i B^l - \frac{a^2}{2\alpha}R_{lsi}{}^h C_j^l - \frac{a^2}{2\alpha}R_{jli}{}^h C_s^l - \frac{ab}{2\alpha}R_{jsi}{}^l A_l^h \tag{5.19}$$

and

$$-A_k^l\nabla_l R_{jsi}{}^h + R_{jsi}{}^l\nabla_l A_k^h - R_{jsl}{}^h\nabla_i A_k^l = 0.$$

By means of (5.13), (5.18) and the second Bianchi identity, the last equation holds.

From $L_{\widetilde{X}}\Gamma_{\overline{j}\overline{i}}^{\overline{h}} = 0$ in (5.17), it follows that

$$\nabla_i C_j^h = -R_{lij}{}^h B^l - \frac{ab}{2\alpha}R_{jli}{}^h D^l \tag{5.20}$$

$$\frac{a^2}{2\alpha}R_{jsi}{}^l\nabla_l D^h + \frac{ab}{2\alpha}(R_{jsl}{}^h\nabla_i B^l + B^l\nabla_l R_{jsi}{}^h + R_{lsi}{}^h C_j^l + R_{jli}{}^h C_s^l - R_{jsi}{}^l C_l^h) = 0 \tag{5.21}$$

and

$$\frac{ab}{2\alpha}(A_k^l\nabla_l R_{jsi}{}^h + R_{jkl}{}^h\nabla_i A_s^l) + \frac{a^2}{2\alpha}(R_{jsi}{}^l\nabla_l C_k^h - R_{jsi}{}^c R_{clk}{}^h B^l) = 0. \tag{5.22}$$

Contracting j and h in (5.22), then (5.22) can be written as

$$\frac{ab}{2\alpha}(A_k^l\nabla_l R_{si} + R_{kl}\nabla_i A_s^l) + \frac{a^2}{2\alpha}(R_{jsi}{}^l\nabla_l C_k^j - R_{jsi}{}^c R_{clk}{}^j B^l) = 0. \tag{5.23}$$

The equation (5.23) holds. In fact, by virtue of (5.13) and (5.18) and (5.20) and the second Bianchi identity, we get

$$\begin{aligned}&\frac{ab}{2\alpha}(A_s^l\nabla_i R_{ls} + R_{ls}\nabla_i A_s^l + A_s^l\nabla_j R_{lik}{}^j) + \frac{a^2}{2\alpha}(R_{jsi}{}^l(\nabla_l C_k^j + R_{clk}{}^j B^c)) \\ = &-\frac{a^3 b}{4\alpha^2}(R_{jis}{}^l + R_{jsi}{}^l)R_{kcl}{}^j D^c = 0.\end{aligned}$$

From (5.19) and (5.21), we obtain

$$R_{jsi}{}^l\left(\frac{ab}{2\alpha}\nabla_l B^h - \frac{b^2}{2\alpha}A_l^h - \frac{ab}{2\alpha}C_l^h + \frac{a^2}{2\alpha}\nabla_l D^h\right) = 0. \tag{5.24}$$

Putting $\widetilde{Y} = E_j$ and $\widetilde{Z} = E_i$ in (5.1), using (5.13), (5.14), (5.16), (5.18), (5.19), (5.20) and (5.24), after tremendous calculations give the followings:

$$L_B\Gamma_{ji}^h - \frac{ab}{2\alpha}(R_{jli}{}^h + R_{ilj}{}^h)D^l = 0,$$

$$L_D\Gamma_{ji}^h + \frac{a(a+c)}{\alpha}(R_{jli}{}^h - \frac{1}{2}R_{jil}{}^h)D^l = 0,$$

$$\begin{aligned}&\frac{ab}{2\alpha}D^l\nabla_j R_{lis}{}^h \\ = &R_{lsi}{}^h(\frac{b^2}{\alpha}\nabla_j B^l + \frac{ab}{2\alpha}\nabla_j D^l - \frac{b^2}{\alpha}C_j^l) + \frac{ab}{2\alpha}R_{lsj}{}^h\nabla_i D^l \\ &-\frac{a(a+c)}{2\alpha}[R_{jil}{}^h(C_s^l + \nabla_s B^l) + R_{jls}{}^h(C_i^l + \nabla_i B^l) + R_{lis}{}^h(C_j^l + \nabla_j B^l)] \\ &-\frac{ab^2 + a^2(a+c)}{2b\alpha}R_{jis}{}^l\nabla_l D^h.\end{aligned}$$

Conversely, if B^h, D^h and A_i^h, C_i^h are given so that they satisfy $(i)-(viii)$, reversing the above steps, we see that $\widetilde{X} = {}^H B + {}^V D + \gamma C + {}^* A$ is an affine Killing vector field on (TM, G). Hence, the proof is complete. □

Let $\widetilde{X}$ be a vector field on TM with components $(v^h, v^{\overline{h}})$ with respect to the adapted frame $\{E_h, E_{\overline{h}}\}$. Then $\widetilde{X}$ is a fibre-preserving vector field on TM if and only if v^h depend only on the variables (x^h). In the case, the vector field $\widetilde{X}$ in Theorem 5.1 reduces $\widetilde{X} = {}^H B +^V D + \gamma C$. Hence, as a corollary to Theorem 5.1, we obtain the following conclusion.

Corollary 5.1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$, such that $a > 0$ and $a(a + c) - b^2 > 0$. If TM admits a fibre-preserving affine Killing vector field $\widetilde{X}$, then the fibre-preserving vector field $\widetilde{X}$ defined by*

$$\widetilde{X} = {}^H B +^V D + \gamma C,$$

where $B = (B^h)$, $D = (D^h) \in \Im^1_0(M)$ and $C = (C^h_i) \in \Im^1_1(M)$ satisfying
(i) $\nabla_i C^h_j = -R_{lij}{}^h B^l$,
(ii) $L_B \Gamma^h_{ji} = 0$,
(iii) $L_D \Gamma^h_{ji} = 0$,
(iv) $\frac{a(a+c)}{2\alpha}[R_{jil}{}^h(C^l_s + \nabla_s B^l) + R_{jls}{}^h(C^l_i + \nabla_i B^l) + R_{lis}{}^h(C^l_j + \nabla_j B^l)]$
$+\frac{a^2(a+c)}{2b\alpha} R_{jis}{}^l \nabla_l D^h = 0.$

As an application of Theorem 5.1, we state a result related to a classification of Killing vector fields on TM with respect to the Riemannian $g-$natural metric G. Firstly we give the following lemma.

Lemma 5.2. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$. The Lie derivatives $L_{\widetilde{X}} G$ with respect to the vector field $\widetilde{X} = v^a E_a + v^{\overline{a}} E_{\overline{a}}$ are given as follows:*

$$\begin{aligned}
& L_{\widetilde{X}} G \\
= \; & \{(a + c)\left[v^h E_h g_{ij} + g_{hj}\left(E_i v^h\right) + g_{ih}\left(E_j v^h\right)\right] + b g_{ih}[y^c v^b R_{bjc}{}^h \\
& + v^{\overline{b}} \Gamma^h_{bj} + (E_j v^{\overline{h}})] + b g_{hj}[y^c v^b R_{bic}{}^h + v^{\overline{b}} \Gamma^h_{bi} + (E_i v^{\overline{h}})]\Big\} dx^i dx^j \\
& + \{(a + c) g_{hi}(E_{\overline{j}} v^h) + b[v^h E_h g_{ij} + g_{hj}\left(E_i v^h\right) - g_{ih} v^b \Gamma^h_{bj} \\
& + g_{ih}(E_{\overline{j}} v^{\overline{h}})] + a g_{hj}[y^c v^b R_{bic}{}^h + v^{\overline{b}} \Gamma^h_{bi} + (E_i v^{\overline{h}})]\Big\} dx^i \delta y^j \\
& + \Big\{a[v^h E_h g_{ij} - g_{hj} v^b \Gamma^h_{bi} - g_{ih} v^b \Gamma^h_{bj} + g_{hj}(E_{\overline{i}} v^{\overline{h}}) + g_{ih}(E_{\overline{j}} v^{\overline{h}})] \\
& + b[g_{hj}(E_{\overline{i}} v^h) + g_{ih}(E_{\overline{j}} v^h)]\} \delta y^i \delta y^j .
\end{aligned}$$

Proof. The proof is similar to that of the Proposition 2.3 [19], so we omit it. □

The general forms of Killing vector fields on (TM, G) are given by

Theorem 5.2. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the Riemannian $g-$natural metric $G = a^S g + b^H g + c^V g$, such that $a > 0$ and $a(a + c) - b^2 > 0$. Then the vector field $\widetilde{X}$ is a Killing vector field on (TM, G) if and only if the vector field $\widetilde{X}$ defined by*

$$\widetilde{X} = {}^H B +^V D + \gamma C + {}^* A$$

where $B = (B^h)$, $D = (D^h) \in \Im^1_0(M)$ and $A = (A^h_i)$, $C = (C^h_i) \in \Im^1_1(M)$ satisfying
(i) $\nabla_i A^h_j = \frac{a^2}{2\alpha} R_{jli}{}^h D^l$,
(ii) $\nabla_i C^h_j = -R_{lij}{}^h B^l - \frac{ab}{2\alpha} R_{jli}{}^h D^l$,
(iii) $L_B \Gamma^h_{ji} - \frac{ab}{2\alpha}(R_{jli}{}^h + R_{ilj}{}^h) D^l = 0$,

(iv) $L_D\Gamma^h_{ji} + \frac{a(a+c)}{\alpha}(R_{jli}{}^h - \frac{1}{2}R_{jil}{}^h)D^l = 0$
(v) $R_{ajc}{}^h A^c_i = 0,$
(vi) $(a+c)\,L_B g_{ij} + bL_D g_{ij} = 0,$
(vii) $g_{hi}((a+c)\,A^h_j + bC^h_j) + g_{hj}(b\nabla_i B^h + a\nabla_i D^h) = 0,$
(viii) $a(g_{hj}C^h_i + g_{ih}C^h_j) + b(g_{hj}A^h_i + g_{ih}A^h_j) = 0.$

Proof. Let TM be the tangent bundle over M with the Riemannian $g-$natural metric of the form $G = a^S g + b^H g + c^V g$. Let $\widetilde{X}$ be a Killing vector field on (TM, G) such that $L_{\widetilde{X}}G = 0$. Since it is also an affine Killing vector field, according to Theorem 5.1, it can be expressed the following form

$$\widetilde{X} = {}^H B + {}^V D + \gamma C + {}^*A \tag{5.25}$$

where $B = (B^h)$, $D = (D^h) \in \Im^1_0(M)$ and $A = (A^h_i)$, $C = (C^h_i) \in \Im^1_1(M)$ satisfy the conditions listed in Theorem 5.1.

Substituting (5.25) into Lemma 5.2, by means of $(i) - (v)$ in Theorem 5.1, we get

$$\begin{aligned} & L_{\widetilde{X}}\widetilde{G} \\ = & \{(a+c)\,L_B g_{ij} + bL_D g_{ij}\}dx^i dx^j \\ & +\{g_{hi}((a+c)\,A^h_j + bC^h_j) + g_{hj}(b\nabla_i B^h + a\nabla_i D^h)\}dx^i \delta y^j \\ & +\{a(g_{hj}C^h_i + g_{ih}C^h_j) + b(g_{hj}A^h_i + g_{ih}A^h_j)\}\delta y^i \delta y^j \\ = & \ 0 \end{aligned}$$

from which

$$(a+c)\,L_B g_{ij} + bL_D g_{ij} = 0,$$

$$g_{hi}((a+c)\,A^h_j + bC^h_j) + g_{hj}(b\nabla_i B^h + a\nabla_i D^h) = 0,$$

and

$$a(g_{hj}C^h_i + g_{ih}C^h_j) + b(g_{hj}A^h_i + g_{ih}A^h_j) = 0.$$

Conversely, it is easily seen that $\widetilde{X} = {}^H B + {}^V D + \gamma C + {}^*A$ is a Killing vector fields on (TM, G) under which of the conditions $(i) - (viii)$. This completes the proof. □

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