ISSN 2077–9879

Eurasian Mathematical Journal

2017, Volume 8, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University

in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

MUKHTARBAY OTELBAEV

(to the 75th birthday)

On October 3, 2017 was the 75th birthday of Mukhtarbay Otelbaev, Doctor of Physical and Mathematical Sciences (1978), Professor (1983), academician of the National Academy of Sciences of the Republic of Kazakhstan (2004), Honored Worker of the Republic of Kazakhstan (2012), laureate of the State Prize of the Republic of Kazakhstan in the field of science and technology (2007), Director of the Eurasian Mathematical Institute (since 2001), Professor of the Department вЂњFundamental MathematicsвЂќ of the L.N. Gumilyov Eurasian National University, the editor-in-chief of the Eurasian Mathematical Journal (together with V.I. Burenkov and V.A. Sadovnichy).

M. Otelbaev was born in the village of Karakemer of the Kurdai district, Zhambyl region. He graduated from the M.V. Lomonosov Moscow State University (1969) and then completed his postgraduate studies at the same university (1972). There he defended his doctor of sciences thesis (1978).

Professor Otelbaev's scientific interests are related to functional analysis, differential equations, computational mathematics, and theoretical physics.

He introduced the q-averaging, which is now called the Otelbaev function; using it he obtained a number of fundamental results. For embedding of the Sobolev weighted spaces and the resolvent of the Schrödinger operator, he established criterions for the compactness and finiteness of the type, as well as estimates of the eigenvalues of the Schrödinger and Dirac operators that are exact in order. He was the first to establish that there is no universal asymptotic formula for the distribution function of the Sturm-Liouville operator. He obtained effective conditions for the separation of the differential operators with nonsmooth and oscillating coefficients, he developed an abstract theory of extension and contraction of operators which are not necessarily linear in linear topological spaces. M. Otelbaev proposed a new numerical method for solving boundary value problems, and a method for approximate calculation of eigenvalues and eigenvectors of compact operators. He obtained the fundamental results in the theory of nonlinear evolution equations and in theoretical physics.

He has published more than 70 scientific papers in leading international journals entering the rating lists of Thomson Reuters and Scopus. Under his supervision 70 postgraduate students have defended their candidate of sciences theses, 9 of them became doctors of sciences. In 2006 and 2011 he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal congratulates Mukhtarbay Otelbaev on the occasion of his 75th birthday and wishes him good health and new achievements in mathematics and mathematical education.

Award for the Eurasian Mathematical Journal

Dear readers, authors, reviewers and members of the Editorial Board of the Eurasian Mathematical Journal,

we are happy to inform you that in November 2017 the Eurasian Mathematical Journal was awarded the title "Leader of Science 2017" by the National Center of State Scientific-Technical Expertise of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan in the nomination "Leader of Kazakhstan Scientific Publications" for the high level of publication activities and high level of citations in Web of Science Core Collection in 2014-2016.

Recall that the Eurasian Mathematical Journal was founded by the L.N. Gumilyov Eurasian National University in 2010 in co-operation with the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia and the University of Padua (see [1]).

The journal pulishes carefully selected original research papers in all areas of mathematics, survey papers, and short communications. It publishes 4 issues in a year. The language of the paper must be English only. Papers accepted for publication are edited from the point of view of English.

More than 280 papers were published written by mathematicians from more than 40 countries representing all continents.

In 2014 the journal was registered in Scopus and in September 2014 the Elsevier-Kazakhstan Research Excellence Forum was held at the L.N. Gumilyov Eurasian National University dedicated to this occasion in which the Elsevier Chairman Professor Y.S. Chi participated (see [3] for details).

In 2015 the Eurasian Mathematical Journal was included in the list of Scopus mathematical journals, quartile Q4, and it is on the way to entering quartile Q3 (see [3]).

Attached is the invitation letter to the Rector of the L.N. Gumilyov Eurasian National University Professor E.B. Sydykov to the ceremony of awarding, which took place in Almaty on November 8, 2017.

On behalf of the Editorial Board of the EMJ V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova

References

[1] B. Abdrayim, *Opening address by the rector of the L.N. Gumilyov Eurasian National University,* Eurasian Math. J. 1 (2010), no. 1, p. 5.

[2] *Eurasian Mathematical Journal is indexed in Scopus,* Eurasian Math. J. 5 (2014), no. 3, 6-8.

[3] V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova, *EMJ: from Scopus Q4 to Scopus Q3 in two years?!,* Eurasian Math. J. 7 (2016), no. 3, p. 6.

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
БІЛІМ ЖӘНЕ ҒЫЛЫМ МИНИСТРЛІГІ
ҒЫЛЫМ КОМИТЕТІ
«ҰЛТТЫҚ МЕМЛЕКЕТТІК ҒЫЛЫМИ-ТЕХНИКАЛЫҚ САРАПТАМА ОРТАЛЫҒЫ»
АКЦИОНЕРЛІК ҚОҒАМЫ

МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ
РЕСПУБЛИКИ КАЗАХСТАН
КОМИТЕТ НАУКИ
АКЦИОНЕРНОЕ ОБЩЕСТВО
«НАЦИОНАЛЬНЫЙ ЦЕНТР ГОСУДАРСТВЕННОЙ
НАУЧНО-ТЕХНИЧЕСКОЙ ЭКСПЕРТИЗЫ»

050026, Қазақстан Республикасы
Алматы қ., Бөгенбай батыр көш., 221
Тел.: +7 (727) 378-05-09
Email: info@ncste.kz http://www.inti.kz

050026, Республика Казахстан
г. Алматы, ул. Богенбай батыра, 221
Тел.: +7 (727) 378-05-09
Email: info@ncste.kz http://www.inti.kz

Исх №: 3367/12-04
«30» 10. 2017г.

Ректору
Евразийского национального
университета имени Л.Н.Гумилева
г-ну Сыдыкову Е.Б.

Уважаемый Ерлан Батташевич!

АО «Национальный центр государственной научно-технической экспертизы» (далее АО «НЦГНТЭ») и компания Clarivate Analytics имеют честь пригласить Вас на церемонию вручения независимой награды «Лидер науки-2017» за высокие показатели публикационной активности и цитируемости в Web of Science Core Collection в период 2014-2016 годы.

Ваш журнал *«Eurasian Mathematical Journal»* награждается в номинации «**Лидер казахстанских научных изданий**».

Торжественное мероприятие состоится 8 ноября 2017 года по адресу: г. Алматы, ул. Богенбай батыра 221, Актовый зал, начало в 10.00, регистрация с 09.00 ч.

Для участия в мероприятии просим пройти онлайн регистрацию на сайте www.nauka.kz до 6 ноября 2017 года.

По всем вопросам обращаться по тел.:+7 727 378 05 78, e-mail: 3780544@bk.ru.

Президент **Ибраев А.Ж.**

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 8, Number 4 (2017), 11 – 17

INCLUSION AND CONVOLUTION PROPERTIES OF A CERTAIN CLASS OF ANALYTIC FUNCTIONS

M. Al-Kaseasbeh, M. Darus

Communicated by E.D. Nursultanov

Key words: inclusion property, convolution, differential operator.

AMS Mathematics Subject Classification: 30C45.

Abstract. In this article, the class $\mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$ of analytic functions defined by using the combination of generalised operators of Salagean and Ruscheweyh is introduced. Inclusion relations, convolution properties and other properties for the class $\mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$ are given.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of analytic functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the open disc $\mathbb{D} = \{z : |z| < 1\}$.

Let $f(z)$ and $g(z)$, respectively, be given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

The Hadamard product (convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

A large number of differential operators have been created. Undoubtedly, many of the operators are the generalised ones. Few are formed by combination of two different operators. In like manner, it is worth to mention some earlier differential operators and their power series expansion for the forthcoming constructions.

Al-Oboudi [5], for $f \in \mathcal{A}$, defined the operator D_λ^n $(n \in \mathbb{N}_0, \lambda \geq 0)$ as follows:

$$\begin{aligned} D_\lambda^0 f(z) &= f(z), \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z), \\ &\vdots \\ D_\lambda^n f(z) &= D_\lambda^1 (D_\lambda^{n-1} f(z)) \quad n \geq 2. \end{aligned}$$

Further, the operator D_λ^n can be expressed in terms of power series as follows:

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, a_k z^k. \tag{1.2}$$

Later, Darus and Al-Shaqsi [1] introduced the operator

$$R_{\alpha,\beta}^n : \mathcal{A} \to \mathcal{A}$$

where $\beta \geq 0$, n, $\alpha \in \mathbb{N}_0$, and

$$\begin{aligned} R_{\alpha,\beta}^0 f(z) &= f(z) \\ R_{\alpha,\beta}^1 f(z) &= zf'(z) + \beta z^2 f''(z) \\ &\vdots \\ R_{\alpha,\beta}^n f(z) &= R_{\alpha,\beta}^1 \left(R_{\alpha,\beta}^{n-1} f(z) \right) \quad n \geq 2. \end{aligned}$$

Also, the operator $R_{\alpha,\beta}^n$ can be expressed in terms of power series as follows:

$$R_{\alpha,\beta}^n f(z) = z + \sum_{k=2}^{\infty} [1+\beta(k-1)]^n C(\alpha,k) \, a_k z^k, \tag{1.3}$$

where $C(\alpha,k) = \binom{k+\alpha-1}{\alpha}$.

Next definition provides a linear combination of D_λ^n and $R_{\alpha,\beta}^n$ which was introduced in [4].

Definition 1. [4] Let $f \in \mathcal{A}$, $\lambda, \beta, \gamma \geq 0$, and $n, \alpha \in \mathbb{N}_0$. We define the following operator

$$D_{\lambda,\alpha,\beta,\gamma}^n f(z) = (1-\gamma) R_{\alpha,\beta}^n f(z) + \gamma D_\lambda^n f(z).$$

As particular cases these operators include the Ruscheweyh operator $D_{\lambda,\alpha,\beta,0}^1 \equiv D_\lambda$ [6], the Salagean operator $D_{1,\alpha,\beta,1}^n \equiv D^1$ [8], the generalised Ruscheweyh operator $D_{\lambda,\alpha,\beta,0}^n \equiv R_{\alpha,\beta}^n$ [1], and the generalised Salagean operator $D_{\lambda,\alpha,\beta,1}^n f(z) \equiv D_\lambda^n$ [5].

If f is given by (1.1), then

$$D_{\lambda,\alpha,\beta,\gamma}^n f(z) = z + \sum_{k=2}^{\infty} \Psi(n,k,\lambda,\beta,\gamma) a_k z^k,$$

where

$$\Psi(n,k,\lambda,\beta,\gamma) = \gamma[1+\lambda(k-1)]^n + (1-\gamma)[1+\beta(k-1)]^n. \tag{1.4}$$

Next is our definition for the class $\mathcal{R}^n(\lambda,\alpha,\beta,\gamma,\delta)$ given as follows:

Definition 2. We denote by $\mathcal{R}^n(\lambda,\alpha,\beta,\gamma,\delta)$, where $0 \leq \delta < 1$, the subclass of $\mathcal{A}$ of all functions f for which $\Re \left\{ \left(D_{\lambda,\alpha,\beta,\gamma}^n f(z) \right)' \right\} > \delta$ for all $z \in \mathbb{D}$.

The following lemmas are needed to prove our findings:

Lemma 1.1. [3] *If p is analytic in $\mathbb{D}$, $p(0) = 1$ and $\Re\, p(z) > \frac{1}{2}$, $z \in \mathbb{D}$, then for any function F analytic in $\mathbb{D}$, the function $p * F$ takes its values in the convex hull of $F(\mathbb{D})$.*

A sequence $\{c_n\}_{n=0}^{\infty}$ of non-negative numbers is said to be a *convex null sequence* if $c_n \to 0$ as $n \to \infty$ and

$$c_0 - c_1 \geq c_1 - c_2 \geq ... \geq c_n - c_{n+1} \geq ... \geq 0.$$

Lemma 1.2. [2] *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function*

$$p(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k$$

is analytic in $\mathbb{D}$ and $\Re p(z) > 0$, $z \in \mathbb{D}$.

Motivated by [5] and [7], similarly according to their work and method, we derive some interesting results for the class $\mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$. These include the inclusion, convolution and the Polya-Schoenberg hypothesis.

2 Inclusion relations

Theorem 2.1. *If $\lambda = \beta$, then*

$$\mathcal{R}^{n+1}(\lambda, \alpha, \lambda, \gamma, \delta) \subset \mathcal{R}^n(\lambda, \alpha, \lambda, \gamma, \delta).$$

Proof. Let f belong to $\mathcal{R}^{n+1}(\lambda, \alpha, \lambda, \gamma, \delta)$. Then

$$\Re\{1 + \sum_{k=2}^{\infty} k\Psi(n+1, k, \lambda, \gamma) a_k z^{k-1}\} > \delta$$

or

$$\Re\{1 + \frac{1}{2(1-\delta)} \sum_{k=2}^{\infty} k\Psi(n+1, k, \lambda, \gamma) a_k z^{k-1}\} > \frac{1}{2}. \tag{2.1}$$

We now construct a function, say $q(z)$, with the property that given in Lemma 1.2 by setting $c_0 = 1$ and $c_k = \frac{1}{1+\lambda k}$ to be

$$q(z) = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{z^{k-1}}{1 + \lambda(k-1)},$$

which implies according to Lemma 1.2 that

$$\Re\left\{\frac{1}{2} + \sum_{k=2}^{\infty} \frac{z^{k-1}}{1 + \lambda(k-1)}\right\} > 0$$

or equivalently,

$$\Re\left\{1 + 2(1-\delta) \sum_{k=2}^{\infty} \frac{z^{k-1}}{1 + \lambda(k-1)}\right\} > \delta. \tag{2.2}$$

Since

$$\begin{aligned}\left(D^n_{\lambda,\alpha,\lambda,\gamma} f(z)\right)' &= 1 + \sum_{k=2}^{\infty} k\Psi(n, k, \lambda, \gamma) a_k z^{k-1} \\ &= 1 + \frac{1}{2(1-\delta)} \sum_{k=2}^{\infty} k\Psi(n+1, k, \lambda, \gamma) a_k z^{k-1} \\ &* 1 + 2(1-\delta) \sum_{k=2}^{\infty} \frac{z^{k-1}}{1 + \lambda(k-1)},\end{aligned}$$

it follows now from Lemma 1.1 and both conditions (2.1) and (2.2) that $\left(D^n_{\lambda,\alpha,\lambda,\gamma}f(z)\right)'$ takes its values in the convex hull of the range of $\mathbb{D}$ under

$$F(z) =: 1 + 2(1-\alpha)\sum_{k=2}^{\infty}\frac{z^{k-1}}{1+\lambda(k-1)}.$$

That is $f \in \mathcal{R}^n(\lambda,\alpha,\lambda,\gamma,\delta)$ as required. □

Theorem 2.1 is improved as follows.

Theorem 2.2. *Let* $f \in \mathcal{R}^{n+1}(\lambda,\alpha,\lambda,\gamma,\delta)$. *Then* $f \in \mathcal{R}^n(\lambda,\alpha,\lambda,\gamma,\mu)$, *where*

$$\mu = \frac{2\gamma^2 + (1+3\gamma)\delta}{(1+\gamma)(1+2\gamma)} \geq \delta.$$

Proof. Let f be in the class $\mathcal{R}^{n+1}(\lambda,\alpha,\lambda,\gamma,\delta)$. In [9] , it has been shown, that if $\gamma \geq 0$ and

$$g(z) = z + \sum_{k=2}^{\infty}\frac{z^k}{1+\lambda(k-1)},$$

then

$$\Re\left\{\frac{g(z)}{z}\right\} > \frac{4\gamma^2+3\gamma+1}{2(1+\gamma)(1+2\gamma)}.$$

Hence

$$\Re\left\{1 + 2(1-\alpha)\sum_{k=2}^{\infty}\frac{z^{k-1}}{1+\lambda(k-1)}\right\} > \frac{2\gamma^2+(1+3\gamma)\delta}{(1+\gamma)(1+2\gamma)}.$$

In view of Lemma 1.1 and Theorem 2.1, we obtain that

$$\Re\left(D^n_{\lambda,\alpha,\lambda,\gamma}f(z)\right)' > \frac{2\gamma^2+(1+3\gamma)\delta}{(1+\gamma)(1+2\gamma)} = \mu.$$

That is $f \in \mathcal{R}^n(\lambda,\alpha,\lambda,\gamma,\mu)$.

□

Remark 1. Since $D^n_{\lambda,\alpha,\beta,\gamma}$ is a linear function of γ, it is clear that

$$\mathcal{R}^n(\lambda,\alpha,\beta,\gamma,\delta) \subset \mathcal{R}^n(\lambda,\alpha,\beta,\gamma',\delta)$$

where $\gamma > \gamma'$.

Now we give the following:

Theorem 2.3. *The set* $\mathcal{R}^n(\lambda,\alpha,\beta,\gamma,\delta)$ *is convex.*

Proof. Let

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{ki}z^k \qquad (i = 1,2)$$

be the functions in the class $\mathcal{R}^n(\lambda,\alpha,\beta,\gamma,\delta)$. It is suffices to show that

$$h(z) = \eta_1 f_1(z) + \eta_2 f_2(z),$$

with η_1 and η_2 nonnegative and $\eta_1 + \eta_2 = 1$, belongs to the class $\mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$.
Since

$$h(z) = z + \sum_{k=2}^{\infty} (\eta_1 a_{k1} + \eta_2 a_{k2}) z^k,$$

then

$$(D^n_{\lambda,\alpha,\beta,\gamma} h(z))' = 1 + \sum_{k=2}^{\infty} k(\eta_1 a_{k1} + \eta_2 a_{k2}) \Psi(n+1, k, \lambda, \beta, \gamma) z^{k-1},$$

hence

$$\begin{aligned} \Re \left\{ (D^n_{\lambda,\alpha,\beta,\gamma} h(z))' \right\} &= \Re \left\{ 1 + \eta_1 \sum_{k=2}^{\infty} k\Psi(n, k, \lambda, \beta, \gamma) a_{k1} z^{k-1} \right\} \\ &\quad + \Re \left\{ 1 + \eta_2 \sum_{k=2}^{\infty} k\Psi(n, k, \lambda, \beta, \gamma) a_{k2} z^{k-1} \right\}. \end{aligned} \tag{2.3}$$

Since $f_1, f_2 \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$, this implies that

$$\Re \left\{ 1 + \eta_i \sum_{k=2}^{\infty} k\Psi(n, k, \lambda, \beta, \gamma) a_{ki} z^{k-1} \right\} > 1 + \eta_i(\delta - 1) \qquad (i = 1, 2). \tag{2.4}$$

Having used (2.4) in (2.3), we get

$$\Re \left\{ (D^n_{\lambda,\alpha,\beta,\gamma} h(z))' \right\} > 1 + \delta(\eta_1 + \eta_2) - (\eta_1 + \eta_2),$$

and since $\eta_1 + \eta_2 = 1$, the proof is complete. □

3 Convolution properties

It is known that the Polya-Schoenberg conjecture and its analogous results were verified by Ruscheweyh and Sheil-Small [7], that is, $\mathcal{C} * \mathcal{C} \subset \mathcal{C}, \mathcal{C} * \mathcal{S}^* \subset \mathcal{S}^*$, and $\mathcal{C} * \mathcal{K} \subset \mathcal{K}$. Note that the classes $\mathcal{C}, \mathcal{S}^*$, and $\mathcal{K}$ are denoted respectively as convex, starlike, and close-to-convex functions. Here, we prove an analogue of the Polya-Schoenberg hypothesis for the class $\mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$.

Theorem 3.1. *Let $f \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$ and $g \in \mathcal{C}$. Then $f * g \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$.*

Proof. If g is convex univalent in $\mathbb{D}$, then

$$\Re \left\{ \frac{g(z)}{z} \right\} > \frac{1}{2}.$$

By convolution, we obtain

$$\Re \left\{ \left(D^n_{\lambda,\alpha,\beta,\gamma} (f * g)(z) \right)' \right\} = \Re \left\{ (D^n_{\lambda,\alpha,\beta,\gamma} f(z))' * \frac{g(z)}{z} \right\}, \tag{3.1}$$

and by applying Lemma 1.1, the proof is complete.

□

Theorem 3.2. *Let $f, g \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$. Then $f * g \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \mu)$, where*

$$\mu = \frac{\gamma(2\delta + 1) + 4\delta - 1}{2(\gamma + 1)} \geq \delta.$$

Proof. Let $g = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$. Then

$$\Re\left\{1 + \sum_{k=2}^{\infty} k\Psi(n, k, \lambda, \beta, \gamma) b_k z^{k-1}\right\} > \delta. \tag{3.2}$$

Now we construct a function, say $q(z)$, with the property given in Lemma 1.2 by setting $c_0 = 1$ and

$$c_k = \frac{\gamma + 1}{(k+1)\Psi(n, k+1, \lambda, \beta, \gamma)} \qquad k \geq 1.$$

to be

$$q(z) = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{\gamma + 1}{(k+1)\Psi(n, k+1, \lambda, \beta, \gamma)} z^k,$$

which implies according to Lemma 1.2 that

$$\Re\left\{\frac{1}{2} + \sum_{k=2}^{\infty} \frac{\gamma + 1}{k\Psi(n, k, \lambda, \beta, \gamma)} z^{k-1}\right\} > 0,$$

or equivalently,

$$\Re\left\{1 + \sum_{k=2}^{\infty} \frac{\gamma + 1}{k\Psi(n, k, \lambda, \beta, \gamma)} z^{k-1}\right\} > \frac{1}{2}. \tag{3.3}$$

In view of Lemma 1.1 and conditions (3.2) and (3.3), the convolution

$$\left(1 + \sum_{k=2}^{\infty} k\Psi(n, k, \lambda, \beta, \gamma) b_k z^{k-1}\right) * \left(1 + \sum_{k=2}^{\infty} \frac{\gamma + 1}{k\Psi(n, k, \lambda, \beta, \gamma)} z^{k-1}\right)$$

$$= 1 + (\gamma + 1) \sum_{k=2}^{\infty} b_k z^{k-1},$$

implies that

$$\Re\left\{1 + (\gamma + 1) \sum_{k=2}^{\infty} b_k z^{k-1}\right\} > \delta \Rightarrow \Re\left\{1 + \sum_{k=2}^{\infty} b_k z^{k-1}\right\} > \frac{\gamma + \delta}{\gamma + 1}$$

$$\Rightarrow \Re\left\{\frac{g(z)}{z}\right\} > \frac{\gamma + \delta}{\gamma + 1} \Rightarrow \Re\left\{\frac{g(z)}{z} - \frac{2\delta + \gamma - 1}{2(\gamma + 1)}\right\} > \frac{1}{2}.$$

Again, by using Lemma 1.1 and since $f \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \delta)$, we obtain

$$\Re\left\{\left(D^n_{\lambda,\alpha,\beta,\gamma} f(z)\right)' * \left(\frac{g(z)}{z} - \frac{2\delta + \gamma - 1}{2(\gamma + 1)}\right)\right\} > \delta,$$

or

$$\Re\left\{\left(D^n_{\lambda,\alpha,\beta,\gamma} f(z)\right)' * \frac{g(z)}{z}\right\} > \frac{\gamma(2\delta + 1) + 4\delta - 1}{2(\gamma + 1)} = \mu.$$

From (3.1), we have that

$$\Re\left\{\left(D^n_{\lambda,\alpha,\beta,\gamma} (f * g)(z)\right)'\right\} > \mu.$$

That is $f * g \in \mathcal{R}^n(\lambda, \alpha, \beta, \gamma, \mu)$.

□

Acknowledgments

The authors thank the unknown referee for the informative remarks regarding the article. This work was supported by Universiti Kebangsaan Malaysia Grant: GUP-2017-064.

References

[1] M. Darus, K. Al-Shaqsi, *Differential sandwich theorems with generalised derivative operator,* International Journal of Computational and Mathematical Sciences. 2 (2008), no. 2, 75-78.

[2] L. Fejer, *Uber die positivitat von summen die nach trigonometrischen oder Legendreschen funktionen fortschreiten,* I. Acta Szeged, 2 (1925), 75-86.

[3] A.W. Goodman, *Univalent functions*, Mariner Comp. Tempa, Florida 2 (1983).

[4] M. Al-Kaseasbeh, M. Darus, *On an operator defined by the combination of both generalised operators of Salagean and Ruscheweyh,* Far East Journal of Mathematical Sciences, 97 (2015), no. 4, 443-455.

[5] F.M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator,* International Journal of Mathematics and Mathematical Sciences. 27 (2004), 1429-1436.

[6] S. Ruscheweyh, *New criteria for univalent functions,* Proceedings of the American Mathematical Society, 49 (1975), no. 1, 109-115.

[7] S. Ruscheweyh, T. Sheil-Small. *Hadamard products of Schlicht functions and the Polya-Schoenberg conjecture*, Commentarii Mathematici Helvetici, 48 (1973), no. 1, 119-135.

[8] G. S. Salagean, *Subclasses of univalent functions*, In Complex Analysis Fifth Romanian-Finnish Seminar . Springer Berlin Heidelberg, (1983), 362-372.

[9] Z. Zhongzhu, S. Owa. *Convolution properties of a class of bounded analytic functions,* Bulletin of the Australian Mathematical Society, 45 (1992), no. 1, 9-23.

Mohammad Al-Kaseasbeh
Department of Mathematics
Jerash University
26150 Jerash,
Jordan,
E-mail: zakariya.alkaseasbeh@gmail.com

Maslina Darus
School of Mathematical Sciences
University Kebangsaan Malaysia
43600 UKM Bangi,
Selangor, Malaysia,
E-mail:maslina@ukm.edu.my

Received: 15.07.2016
Revised version: 11.03.2017