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ON THE BOUNDEDNESS OF QUASILINEAR INTEGRAL OPERATORS OF ITERATED TYPE WITH OINAROV'S KERNELS ON THE CONE OF MONOTONE FUNCTIONS

V.D. Stepanov and G.E. Shambilova

Communicated by V. Kokilashvili

Dedicated to the 70^{th} birthday of Professor Ryskul Oinarov

Keywords: Hardy type inequality, weighted Lebesgue space, quasilinear integral operator, Oinarov's kernel, cone of monotone functions.

AMS Mathematics Subject Classification: 26D15

Abstract. We solve the characterization problem of $L_v^p - L_\rho^r$ weighted inequalities on Lebesgue cones of monotone functions on the half-axis for quasilinear integral operators of iterated type with Oinarov's kernels.

1 Introduction

Denote $\mathfrak{M}$ the set of all measurable functions on $\mathbb{R}_+ := [0, \infty)$, $\mathfrak{M}^+ \subset \mathfrak{M}$ the subset of all non-negative functions and $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+$ is the cone of all non-increasing functions.

If $0 < p \leq \infty$ and $v \in \mathfrak{M}^+$ we define

$$L_v^p := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^p} := \left(\int_0^\infty |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_v^\infty := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^\infty} := \operatorname*{ess\,sup}_{x \geq 0} v(x)|f(x)| < \infty \right\}.$$

The story was started since 90's of the last century, when in a process of characterization of the weighted Hardy inequality

$$\|Hf\|_{L_w^q} \leq C\|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{1.1}$$

with $1 \leq p \leq \infty, 0 < q \leq \infty$, where

$$Hf(x) := \int_0^x f(y)dy \quad \text{or} \quad Hf(x) := \int_x^\infty f(y)dy,$$

basically given in the papers [38] ($1 < p = q < \infty$), [6] ($1 \leq p = q \leq \infty$), [2] ($1 < p \leq q < \infty$), [5] ($1 \leq p, q \leq \infty$), [30] ($0 < q < 1 < p < \infty$), [31] ($0 < q < 1 = p$), further extensions were found for some convolution operators [34], [35], [4]. For a more general transformation a breakthrough was achieved by R. Oinarov [15] (with an earlier announcement in [16]) for the operator

$$Kf(x) := \int_0^x k(x,y)f(y)dy, \tag{1.2}$$

where a kernel $k(x,y) \geq 0$ satisfies

$$k(x,y) \approx k(x,z) + k(z,y),\ 0 \leq y \leq z \leq x. \tag{1.3}$$

The Oinarov techniques used to prove (1.1) (with K instead of H) and alternative proofs found in [1], [37] have opened up the study in many directions: for general Lebesgue spaces with measures, higher dimensions, general Banach function spaces, different classes of functions and so on (see [13], [14], [26] and references therein).

In particular, the problem appeared in connection with the necessity to consider weighted integral inequalities on the cones of monotone and quasiconcave functions which arise in the study of the classical operators in weighted Lebesgue and Lorentz spaces (see, for example, [27], [12], [8], [29], more recent papers [21], [19], [20], surveys [7], [9]). It was recently revealed that new quasilinear integral operators of iterated type are involved into study of the problems on the cones of monotone and quasi-concave functions (see, for instance, [3], [19], [20], [25]).

Let $u, v, w, \rho \in \mathfrak{M}^+,\ 0 < p, r, q \leq \infty$. In this paper we study the problem of characterizing the inequality

$$\|Rf\|_{L^r_\rho} \leq C\|f\|_{L^p_v},\ f \in \mathfrak{M}^\downarrow, \tag{1.4}$$

where a constant C does not depend on f and is assumed the least possible. Operator R is a quasilinear integral operators of the forms

$$\mathcal{T}f(x) := \left(\int_0^x k_1(x,y)w(y)\left(\int_y^\infty k_2(z,y)f(z)u(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{1.5}$$

$$\mathcal{S}f(x) := \left(\int_0^x k_1(x,y)w(y)\left(\int_0^y k_2(y,z)f(z)u(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{1.6}$$

where the kernels $k_i(x,y) \geq 0, (i = 1,2)$ satisfy Oinarov's condition (1.3). Symmetrical cases

$$Tf(x) := \left(\int_x^\infty k_1(y,x)w(y)\left(\int_0^y k_2(y,z)f(z)u(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{1.7}$$

$$Sf(x) := \left(\int_x^\infty k_1(y,x)w(y)\left(\int_y^\infty k_2(z,y)f(z)u(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{1.8}$$

were recently characterized in [33].

It have been solved for $k_1(x,y) = k_2(x,y) = 1, 0 \leq y \leq x$ (see [28]) by the reduction method [9] to weighted inequalities of an analogous form on cones of non-negative functions. These are actively studied [10], [11], [24], [25], [23] and found various applications [3], [17], [18], [19]. Observe, that characterization of (1.4) with two-kernel operators (1.5)–(1.6) required a more complicated technique . As in [28], the main results have a reduction form, i.e., for each of the operators (1.5)–(1.6), the inequality (1.4) is reduced to validity of analogous inequalities on the cone of nonnegative functions, criteria for which are known.

In Section 2, we obtain a criterion for the fulfillment of (1.4) with the operator $\mathcal{T}$, in Section 3, with the operator $\mathcal{S}$, and in Section 4, we formulate for completeness results for the operators (1.7) и (1.8).

Throughout the article, the products of the form $0 \cdot \infty$ are assumed equal to 0. The record $A \lesssim B$ means $A \le cB$ with c a constant depending only on p, q and r; $A \approx B$ is equivalent $A \lesssim B \lesssim A$. Also $\mathbb{Z}$ stands for the set of all integers,and χ_E – characteristic function (indicator) of a set $E \subset (0, \infty)$. We use the symbols := and =: for defining new quantities. If $1 \le p \le \infty$, then $p' := \frac{p}{p-1}$ for $1 < p < \infty$, $p' := \infty$ for $p = 1$ and $p' := 1$ for $p = \infty$.

2 The operator $\mathcal{T}$

Put $V(t) := \int_0^t v$, $U(t) := \int_0^t u$, $W(t) := \int_t^\infty w, \widetilde{u} := \frac{u}{V^{\frac{2}{p}}}$, $\widetilde{U}(t) := \int_t^\infty \widetilde{u}$, $0 < t < \infty$. Assume for simplicity that $0 < \int_x^\infty \rho < \infty, 0 < \int_x^\infty w < \infty$ for every $x > 0$ and $\int_0^\infty \rho = \infty, \int_0^\infty w = \infty$. Define the sequence $\{b_n\} \subset (0; \infty)$ from the equations

$$\int_{b_n}^\infty \rho = 2^{-n}, \ n \in \mathbb{Z}. \tag{2.1}$$

Let $\zeta : [0; \infty) \to [0; \infty)$ and $\zeta^{-1} : [0; \infty) \to [0; \infty)$ be define by the formulas (here $\sup \varnothing = 0$)

$$\zeta(x) := \sup\left\{y > 0 : \int_y^\infty \rho \ge \frac{1}{2}\int_x^\infty \rho\right\}, \zeta^{-1}(x) := \sup\left\{y > 0 : \int_y^\infty \rho \ge 2\int_x^\infty \rho\right\}. \tag{2.2}$$

Then ζ and ζ^{-1} are increasing functions such that $\int_{\zeta(x)}^\infty \rho = \frac{1}{2}\int_x^\infty \rho$, $\int_{\zeta^{-1}(x)}^\infty \rho = 2\int_x^\infty \rho$. In particular, $\zeta(b_n) = b_{n+1}$, $b_n = \zeta^{-1}(b_{n+1})$.

Given $0 < c < d \le \infty, 0 < t, p < \infty, h \in \mathfrak{M}^+$, put

$$\widetilde{T}_t h(x) := \chi_{(0,t]}(x) \left(\int_x^\infty k_2(s,x)\widetilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}} ds\right)^p, \tag{2.3}$$

$$\widetilde{T}_{[c,d]} h(x) := \chi_{[c,d]}(x) \left(\int_x^{\zeta(d)} k_2(s,x)\widetilde{u}(s)\left(\int_c^s hV\right)^{\frac{1}{p}} ds\right)^p. \tag{2.4}$$

$$\left\|\widetilde{T}_t\right\|_{L_v^p \to L_w^q} := \sup_{0 \ne h \in \mathfrak{M}^+} \frac{\left(\int_0^\infty [\widetilde{T}_t h]^q w\right)^{\frac{1}{q}}}{\left(\int_0^\infty [h]^p v\right)^{\frac{1}{p}}} \tag{2.5}$$

Theorem 2.1. *Let $0 < q < \infty, 0 < p < \infty, 0 < r < \infty$. Then the best constant $C_\mathcal{T}$ in*

$$\left(\int_0^\infty [\mathcal{T}f(x)]^r \rho(x)dx\right)^{\frac{1}{r}} \le C_\mathcal{T}\left(\int_0^\infty [f(x)]^p v(x)dx\right)^{\frac{1}{p}}, f \in \mathfrak{M}^\downarrow \tag{2.6}$$

satisfies

$$C_\mathcal{T} \approx \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + B,$$

where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ are the best constants in the inequalities

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)k_2(x,y)^q w(y)dy\right)^{\frac{r}{q}}\left(\int_x^\infty \left(\int_0^s hV\right)^{\frac{1}{p}} \tilde{u}(s)ds\right)^r dx\right)^{\frac{p}{r}} \le \mathcal{A}_1^p \int_0^\infty h, \tag{2.7}$$

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)dy\right)^{\frac{r}{q}}\left(\int_x^\infty k_2(s,x)\left(\int_0^s hV\right)^{\frac{1}{p}}\tilde{u}(s)ds\right)^r dx\right)^{\frac{p}{r}} \le \mathcal{A}_2^p\int_0^\infty h, \tag{2.8}$$

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\int_y^\infty k_2(s,y)\tilde{u}(s)ds\right)^q\left(\int_0^y hV\right)^{\frac{q}{p}}dy\right)^{\frac{r}{q}}dx\right)^{\frac{p}{r}} \le \mathcal{A}_3^p\int_0^\infty h, \tag{2.9}$$

$$\left(\int_0^\infty \rho(x)k_1(x,\zeta^{-2}(x))^{\frac{r}{q}}\left(\int_0^{\zeta^{-2}(x)} w(y)\left(\int_y^\infty k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}\right)^q dy\right)^{\frac{r}{q}}dx\right)^{\frac{p}{r}} \le \mathcal{A}_4^p\int_0^\infty h, \tag{2.10}$$

for $h \in \mathfrak{M}^+$, and the constant B has the form

$$B := \begin{cases} \sup_{t>0}\left(\int_t^\infty \rho\right)^{\frac{1}{r}}\left\|\widetilde{T}_t\right\|^{\frac{1}{p}}_{L^1\to L^{\frac{q}{p}}_{k_1(t,\cdot)w(\cdot)}}, & p\le r,\\ \left(\int_0^\infty \rho(x)\left[\left(\int_x^\infty \rho\right)\left\|\widetilde{T}_{[\zeta^{-1}(x),\zeta^2(x)]}\right\|_{L^1\to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}}\right]^{\frac{s}{p}}dx\right)^{\frac{1}{s}}, & r<p,\end{cases}$$

where $\frac{1}{s} := \frac{1}{r} - \frac{1}{p}$.

Proof. The change $f^p \to f$ in (2.6) leads to the inequality

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\int_y^\infty k_2(s,y)f^{\frac{1}{p}}(s)u(s)ds\right)^q dy\right)^{\frac{r}{q}}dx\right)^{\frac{p}{r}} \le C_{\mathcal{T}}^p\int_0^\infty fv.$$

Using Theorem 3.2 from [9], we obtain the equivalent inequality

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\int_y^\infty k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}}dx\right)^{\frac{p}{r}} \le C_{\mathcal{T}}^p\int_0^\infty h, \tag{2.11}$$

for $h \in \mathfrak{M}^+$.

The upper bound. Bellow we shall use well-known relation (see, for example, [12], Proposition 2.1)

$$\sum_{n\in\mathbb{Z}} 2^{-n}\left(\sum_{i\le n} a_i\right)^s \approx \sum_{n\in\mathbb{Z}} 2^{-n}a_n^s, \tag{2.12}$$

valid for all sequences of nonnegative numbers and every $s>0$. Put

$$T_ph(y) := \left(\int_y^\infty k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^p.$$

We write

$$J := \sum_n \int_{b_{n-1}}^{b_n} \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(T_ph(y)\right)^{\frac{q}{p}}dy\right)^{\frac{r}{q}}dx$$

$$\ll \sum_n 2^{-n}\left(\int_0^{b_n} k_1(b_n,y)w(y)\left(T_ph(y)\right)^{\frac{q}{p}}dy\right)^{\frac{r}{q}}$$

$$\approx \sum_n 2^{-n}\left(\int_0^{b_{n-2}} k_1(b_n,y)w(y)\left(T_ph(y)\right)^{\frac{q}{p}}dy\right)^{\frac{r}{q}}$$

$$+\sum_n 2^{-n}\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(T_ph(y)\right)^{\frac{q}{p}}dy\right)^{\frac{r}{q}} =: J_1+J_2.$$

$$J_2 \approx \sum_n 2^{-n}\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_y^{b_{n+1}} k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}}$$

$$+\sum_n 2^{-n}\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_{b_{n+1}}^{\infty} k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}} =: J_{2,1}+J_{2,2}.$$

By Oinarov's condition (1.3) we have $k_2(s,y)\approx k_2(x,y)+k_2(s,x)$ for $y\le x\le s$. Hence,

$$J_{2,2} \approx \sum_n \int_{b_n}^{b_{n+1}}\rho(x)dx\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_{b_{n+1}}^{\infty} k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}}$$

$$\approx \sum_n \int_{b_n}^{b_{n+1}}\rho(x)dx\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)k_2^q(x,y)w(y)\left(\int_{b_{n+1}}^{\infty}\left(\int_0^s hV\right)^{\frac{1}{p}}\tilde{u}(s)ds\right)^q dy\right)^{\frac{r}{q}}$$

$$+\sum_n \int_{b_n}^{b_{n+1}}\rho(x)dx\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_{b_{n+1}}^{\infty} k_2(s,x)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}}$$

$$\ll \int_0^{\infty}\rho(x)\left(\int_0^x k_1(x,y)k_2^q(x,y)w(y)dy\right)^{\frac{r}{q}}\left(\int_x^{\infty}\left(\int_0^s hV\right)^{\frac{1}{p}}\tilde{u}(s)ds\right)^r dx$$

$$+\int_0^{\infty}\rho(x)\left(\int_0^x k_1(x,y)w(y)dy\right)^{\frac{r}{q}}\left(\int_x^{\infty} k_2(s,x)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}}ds\right)^r dx$$

$$\ll (\mathcal{A}_1^r+\mathcal{A}_2^r)\|h\|_{L^1}^{\frac{r}{p}}. \tag{2.13}$$

Write

$$J_{2,1} \approx \sum_n 2^{-n}\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_y^{b_{n+1}} k_2(s,y)\tilde{u}(s)\left(\int_0^{b_{n-2}} hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}}$$

$$+\sum_n 2^{-n}\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_y^{b_{n+1}} k_2(s,y)\tilde{u}(s)\left(\int_{b_{n-2}}^s hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}} =: J_{2,1,1}+J_{2,1,2}$$

We have

$$J_{2,1,1} \approx \sum_n \int_{b_n}^{b_{n+1}}\rho(x)\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y)\left(\int_y^{b_{n+1}} k_2(s,y)\tilde{u}(s)\left(\int_0^{b_{n-2}} hV\right)^{\frac{1}{p}}ds\right)^q dy\right)^{\frac{r}{q}}$$

$$\leq \int_0^\infty \rho(x) \left(\int_0^x k_1(x,y) w(y) \left(\int_y^\infty k_2(s,y) \tilde{u}(s) ds \right)^q \left(\int_0^y hV \right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} dx \leq \mathcal{A}_3^r \|h\|_{L^1}^{\frac{r}{p}}. \quad (2.14)$$

Next, using (2.3) and (2.4), we obtain

$$J_{2,1,2} = \sum_n 2^{-n} \left[\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y) w(y) \left(\widetilde{T}_{[b_{n-2},b_n]} h(y) \right)^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \right]^{\frac{r}{p}}$$

$$\leq \sum_n 2^{-n} \left\| \widetilde{T}_{[b_{n-2},b_n]} \right\|^{\frac{r}{p}}_{L^1[b_{n-2},b_{n+1}] \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}[b_{n-2},b_n]} \left(\int_{b_{n-2}}^{b_{n+1}} h \right)^{\frac{r}{p}}.$$

For $p \leq r$ by Jensen's inequality

$$J_{2,1,2} \leq \sup_n \left(\int_{b_{n+1}}^\infty \rho \right) \left\| \widetilde{T}_{[b_{n-2},b_n]} \right\|^{\frac{r}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}} \left(\int_0^\infty h \right)^{\frac{r}{p}}$$

$$\leq \sup_{t>0} \left(\int_t^\infty \rho \right) \left\| \widetilde{T}_t \right\|^{\frac{r}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(t,\cdot)w(\cdot)}} \left(\int_0^\infty h \right)^{\frac{r}{p}}.$$

Hence,

$$J_{2,1,2}^{\frac{p}{r}} \leq B^p \int_0^\infty h. \quad (2.15)$$

For $r < p$, by Hölder's inequality, we find with $\frac{1}{s} = \frac{1}{r} - \frac{1}{p}$ that

$$J_{2,1,2} \leq \left(\sum_n 2^{\frac{-ns}{r}} \left\| \widetilde{T}_{[b_{n-2},b_n]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}} \right)^{\frac{r}{s}} \left(\int_0^\infty h \right)^{\frac{r}{p}}.$$

because

$$\sum_n 2^{\frac{-ns}{r}} \left\| \widetilde{T}_{[b_{n-2},b_n]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}}$$

$$\approx \sum_n \left(\int_{b_{n-2}}^{b_{n-1}} \rho \right) \left(\int_{b_{n-1}}^\infty \rho \right)^{\frac{s}{p}} \left\| \widetilde{T}_{[\zeta^{-1}(b_{n-1}),\zeta^2(b_{n-2})]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(b_{n-2}),\cdot)w(\cdot)}}$$

$$\leq \sum_n \int_{b_{n-2}}^{b_{n-1}} \rho(x) \left[\left(\int_x^\infty \rho \right) \left\| \widetilde{T}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx$$

$$= \int_0^\infty \rho(x) \left[\left(\int_x^\infty \rho \right) \left\| \widetilde{T}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx = B^s.$$

Therefore,

$$J_{2,1,2}^{\frac{p}{r}} \leq B^p \int_0^\infty h. \quad (2.16)$$

Consider

$$J_1 = \sum_n 2^{-n} \left(\int_0^{b_{n-2}} k_1(b_n,y) w(y) \left(T_p h(y) \right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$= \sum_n 2^{-n} \left(\sum_{i \le n} \int_{b_{i-3}}^{b_{i-2}} k_1(b_n, y) w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\approx \sum_n 2^{-n} \left(\sum_{i \le n} \int_{b_{i-3}}^{b_{i-2}} k_1(b_n, b_{i-1}) w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$+ \sum_n 2^{-n} \left(\sum_{i \le n} \int_{b_{i-3}}^{b_{i-2}} k_1(b_{i-1}, y) w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} =: J_{1,1} + J_{1,2}.$$

Applying (2.12), we get

$$J_{1,2} \le \sum_n 2^{-n} \left(\int_{b_{n-3}}^{b_{n-1}} k_1(b_{n-1}, y) w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} \approx J_2$$

$$\le (\mathcal{A}_1^r + \mathcal{A}_2^r + \mathcal{A}_3^r + B^r) \|h\|_{L^1}^{\frac{r}{p}}. \tag{2.17}$$

To estimate $J_{1,1}$, we use inequality (see [9], Lemma 3.1)

$$k_1(b_n, b_{i-1}) \ll \left(\sum_{j=i}^{n} k_1(b_j, b_{j-1})^{\alpha} \right)^{\frac{1}{\alpha}}, \ \alpha \in (0,1) \tag{2.18}$$

and Minkowski's inequality to derive that

$$J_{1,1} = \sum_n 2^{-n} \left(\sum_{i \le n} k_1(b_n, b_{i-1}) \int_{b_{i-3}}^{b_{i-2}} w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\ll \sum_n 2^{-n} \left(\sum_{i \le n} \left(\sum_{j=i}^{n} k_1(b_j, b_{j-1})^{\alpha} \right)^{\frac{1}{\alpha}} \int_{b_{i-3}}^{b_{i-2}} w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\le \sum_n 2^{-n} \left(\sum_{j \le n} k_1(b_j, b_{j-1})^{\alpha} \left(\sum_{i \le j} \int_{b_{i-3}}^{b_{i-2}} w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\alpha} \right)^{\frac{r}{\alpha q}}$$

$$= \sum_n 2^{-n} \left(\sum_{j \le n} k_1(b_j, b_{j-1})^{\alpha} \left(\int_0^{b_{j-2}} w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right)^{\alpha} \right)^{\frac{r}{\alpha q}}$$

$$\overset{(2.12)}{\approx} \sum_n 2^{-n} \left(k_1(b_n, b_{n-1}) \left(\int_0^{b_{n-2}} w(y) \left(T_p h(y)\right)^{\frac{q}{p}} dy \right) \right)^{\frac{r}{q}}$$

$$\approx \sum_n \int_{b_n}^{b_{n+1}} \rho(x) k_1(b_n, b_{n-1})^{\frac{r}{q}} \left(\int_0^{b_{n-2}} w(y) \left(T_p h(y)^{\frac{q}{p}} dy \right) \right)^{\frac{r}{q}} dx$$

$$\le \sum_n \int_{b_n}^{b_{n+1}} \rho(x) k_1(x, \zeta^{-2}(x))^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} w(y) \left(T_p h(y)^{\frac{q}{p}} dy \right) \right)^{\frac{r}{q}} dx$$

$$= \int_0^\infty \rho(x) k_1(x,\zeta^{-2}(x))^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} w(y) \left(T_p h(y)^{\frac{q}{p}} dy \right) \right)^{\frac{r}{q}} dx \le \mathcal{A}_4^r \|h\|_{L^1}^{\frac{r}{p}} \tag{2.19}$$

It follows from (2.13)-(2.19) that the upper estimate $C_{\mathcal{T}} \ll \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + B$ is proved.

The lower bound. Diminish the domains of integration in (2.11):

(1) $[y,\infty] \to [x,\infty]$ and obtain $C_{\mathcal{T}} \ge \mathcal{A}_1 + \mathcal{A}_2$ (since $k_2(s,y) \approx k_2(x,y) + k_2(s,x)$ for $0 < y \le x \le s$).

(2) $[0,s] \to [0,y]$ and obtain $C_{\mathcal{T}} \ge \mathcal{A}_3$.

(3) $[0,x] \to [0,\zeta^{-2}(x)]$ and obtain $C_{\mathcal{T}} \ge \mathcal{A}_4$ (since $k_1(x,y) \gtrsim k_1(x,\zeta^{-2}(x))$ for $y \le \zeta^{-2}(x) < x$).

Inequality (2.11) for $h \in \mathfrak{M}^+$ implies that

$$\left(\int_t^\infty \rho\right)^{\frac{p}{r}} \left(\int_0^t k_1(t,y) w(y) \left(\int_y^\infty k_2(s,y) \tilde{u}(s) \left(\int_0^s hV \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{p}{q}} \le C_{\mathcal{T}}^p \int_0^{\zeta(t)} h,$$

therefore, $C_{\mathcal{T}} \gg B$, and the theorem is proved for $p \le r$.

It remains to find the lower bound for B when $r < p$. We have

$$B^s = \int_0^\infty \rho(x) \left[\left(\int_x^\infty \rho \right) \left\| \widetilde{T}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx$$

$$= \sum_n \int_{b_{n-1}}^{b_n} \rho(x) \left[\left(\int_x^\infty \rho \right) \left\| \widetilde{T}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx$$

$$\le \sum_n \int_{b_{n-1}}^{b_n} \rho(x) dx \left(\int_{b_{n-1}}^\infty \rho \right)^{\frac{s}{p}} \left\| \widetilde{T}_{[\zeta^{-1}(b_{n-1}),\zeta^2(b_n)]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}}$$

$$\approx \sum_n 2^{-\frac{ns}{r}} \left\| \widetilde{T}_{[b_{n-2},b_{n+2}]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(b_{n+2},\cdot)w(\cdot)}}.$$

Hence

$$B^s \ll \sum_n 2^{-\frac{ns}{r}} \left\| \widetilde{T}_{[b_{n-2},b_{n+2}]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(b_{n+2},\cdot)w(\cdot)}} =: \mathcal{B}^s.$$

Let $\theta \in (0,1)$ be some fixed number. Then for every $n \in \mathbb{Z}$ there exists $h_n \in L^1[b_{n-2}, b_{n+3}]$ such that $\|h_n\|_{L^1[b_{n-2},b_{n+3}]} = 1$ and

$$\left\| \widetilde{T}_{[b_{n-2},b_{n+2}]} h_n \right\|_{L^{\frac{q}{p}}_{k_1(b_{n+2},\cdot)w(\cdot)}} \ge \theta \left\| \widetilde{T}_{[b_{n-2},b_{n+2}]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(b_{n+2},\cdot)w(\cdot)}}.$$

Put

$$g_n := 2^{\frac{-ns}{r}} \left\| \widetilde{T}_{[b_{n-2},b_{n+2}]} \right\|^{\frac{s}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(b_{n+2},\cdot)w(\cdot)}} h_n;\ \mathbf{T}_n := \left\| \widetilde{T}_{[b_{n-2},b_{n+2}]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(b_{n+2},\cdot)w(\cdot)}};\ g := \sum g_n.$$

Then

$$\|g\|_{L^1} \ll \sum_n \int_{b_{n-2}}^{b_{n+3}} g_n = \sum_n 2^{\frac{-ns}{r}} \mathbf{T}_n^{\frac{s}{p}} = \mathcal{B}^s.$$

On the other hand

$$D := \int_0^\infty \rho(x) \left(\int_0^x k_1(x,y) w(y) (T_p g(y))^{\frac{q}{p}} \right)^{\frac{r}{q}} dx$$

$$\gg \sum_n \int_{b_{n+2}}^{b_{n+3}} \rho(x) dx \left[\left(\int_{b_{n-2}}^{b_{n+2}} k_1(b_{n+2}, y) w(y) (T_p g(y))^{\frac{q}{p}} \right)^{\frac{p}{q}} \right]^{\frac{r}{p}}$$

$$\gg \sum_n 2^{-n} \left[\left(\int_{b_{n-2}}^{b_{n+2}} k_1(b_{n+2}, y) w(y) \left[\left(\int_y^{b_{n+3}} k_2(s,y) \tilde{u}(s) \left(\int_{b_{n-2}}^s g_n V \right)^{\frac{1}{p}} ds \right)^p \right]^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \right]^{\frac{r}{p}}$$

$$= \sum_n 2^{-n} \left[\left(\int_{b_{n-2}}^{b_{n+2}} k_1(b_{n+2}, y) w(y) \left(\widetilde{T}_{[b_{n-2}, b_{n+2}]} g_n \right)^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \right]^{\frac{r}{p}}$$

$$= \sum_n 2^{-n} 2^{\frac{-ns}{p}} \mathbf{T}_n^{\frac{sr}{p^2}} \left\| \widetilde{T}_{[b_{n-2}, b_{n+2}]} h_n \right\|^{\frac{r}{p}}_{L^{\frac{q}{p}}_{k_1(b_{n+2}, \cdot) w(\cdot)}} \geq \theta^{\frac{r}{p}} \sum_n 2^{\frac{-ns}{r}} \mathbf{T}_n^{\frac{s}{p}} = \theta^{\frac{r}{p}} \mathcal{B}^s.$$

Inequality (2.11) yields

$$\mathcal{B}^s C_{\mathcal{T}}^p \gg C_{\mathcal{T}}^p \|g\|_{L^1} \overset{(2.11)}{\geq} D^{\frac{p}{r}} \gg \theta \mathcal{B}^{\frac{sp}{r}}.$$

Consequently,

$$C_{\mathcal{T}} \gg \theta^{\frac{1}{p}} \mathcal{B} \geq \theta^{\frac{1}{p}} B.$$

Since $\theta \in (0,1)$ is arbitrary, $C_{\mathcal{T}} \gg B$. □

Remark 1. (1) *If* $q = \infty$ *then* $C_{\mathcal{T}} \approx \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}$, *where* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ *and* $\mathcal{A}_4$ *are the best constants in*

$$\left(\int_0^\infty \rho(x) [\operatorname*{ess\,sup}_{y \in (0,x)} k_1(x,y) k_2(x,y) w(y)]^r \left(\int_x^\infty \left(\int_0^s hV \right)^{\frac{1}{p}} \tilde{u}(s) ds \right)^r dx \right)^{\frac{p}{r}} \leq \mathcal{A}_1^p \int_0^\infty h, \tag{2.20}$$

$$\left(\int_0^\infty \rho(x) [\operatorname*{ess\,sup}_{y \in (0,x)} k_1(x,y) w(y)]^r \left(\int_x^\infty k_2(s,x) \left(\int_0^s hV \right)^{\frac{1}{p}} \tilde{u}(s) ds \right)^r dx \right)^{\frac{p}{r}} \leq \mathcal{A}_2^p \int_0^\infty h, \tag{2.21}$$

$$\left(\int_0^\infty \rho(x) \left[\operatorname*{ess\,sup}_{y \in (0,x)} k_1(x,y) w(y) \left(\int_y^\infty k_2(s,y) \tilde{u}(s) ds \right) \left(\int_0^y hV \right)^{\frac{1}{p}} \right]^r dx \right)^{\frac{p}{r}} \leq \mathcal{A}_3^p \int_0^\infty h, \tag{2.22}$$

$$\left(\int_0^\infty \rho(x) k_1(x, \zeta^{-2}(x))^r \left[\operatorname*{ess\,sup}_{y \in (0, \zeta^{-2}(x))} w(y) \left(\int_y^\infty k_2(s,y) \tilde{u}(s) \left(\int_0^s hV \right)^{\frac{1}{p}} ds \right) \right]^r dx \right)^{\frac{p}{r}} \leq \mathcal{A}_4^p \int_0^\infty h, \tag{2.23}$$

for $h \in \mathfrak{M}^+$ *and the constant* B *has the form*

$$B := \begin{cases} \sup_{t>0} \left(\int\limits_t^\infty \rho \right)^{\frac{1}{r}} \left\| \tilde{T}_t \right\|^{\frac{1}{p}}_{L^1 \to L^\infty_{k_1(t,\cdot)w(\cdot)}}, & p \le r, \\ \left(\int\limits_0^\infty \rho(x) \left[\left(\int\limits_x^\infty \rho \right) \left\| \tilde{T}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L^1 \to L^\infty_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx \right)^{\frac{1}{s}}, & r < p. \end{cases}$$

(2) *if* $p = \infty$ *or* $r = \infty$, *then*

$$C_{\mathcal{T}} = \|\mathcal{T}(\frac{1}{v})\|_{L^r_\rho}, \; p = \infty;$$

$$C_{\mathcal{T}} \approx \sup_{t>0} \mathcal{R}(t) \|\widetilde{T}_t\|^{\frac{1}{p}}_{L^1_V \to L^{\frac{q}{p}}_{k_1(t,\cdot)w(\cdot)}}, \; r = \infty,$$

where

$$\mathcal{R}(t) := \operatorname*{ess\,sup}_{z \ge t} \rho(z).$$

Proof. (1) For $q = \infty$, we have

$$J := \sum_n \int_{b_{n-1}}^{b_n} \rho(x) \left[\operatorname*{ess\,sup}_{y \le x} \; k_1(x,y) w(y) \left(T_p h(y)\right)^{\frac{1}{p}} \right]^r dx$$

$$\ll \sum_n 2^{-n} \left[\operatorname*{ess\,sup}_{y \in (0, b_{n-2}]} \; k_1(b_n,y) w(y) \left(T_p h(y)\right)^{\frac{1}{p}} \right]^r$$

$$+ \sum_n 2^n \left[\operatorname*{ess\,sup}_{y \in (b_{n-2}, b_n]} \; k_1(b_n,y) w(y) \left(T_p h(y)\right)^{\frac{1}{p}} \right]^r =: J_1 + J_2.$$

By analogy to Theorem 2.1, we prove the inequality

$$J_2 \ll (\mathcal{A}_1^r + \mathcal{A}_2^r + \mathcal{A}_3^r + B^r) \|h\|^{\frac{r}{p}}_{L^1}. \tag{2.24}$$

To estimate J_1 use relation

$$\sum_{n \in \mathbb{Z}} 2^{-n} \left(\sup_{i \ge n} a_i \right)^s \approx \sum_{n \in \mathbb{Z}} 2^{-n} a_n^s. \tag{2.25}$$

instead of (2.12). We have

$$J_1 = \sum_n 2^{-n} \left[\operatorname*{ess\,sup}_{y \in (0, b_{n-2}]} \; k_1(b_n,y) w(y) \left(T_p h(y)\right)^{\frac{1}{p}} \right]^r =$$

$$= \sum_n 2^{-n} \left[\sup_{i \le n} \; \operatorname*{ess\,sup}_{y \in (b_{i-3}, b_{i-2}]} \; k_1(b_n,y) w(y) \left(T_p h(y)\right)^{\frac{1}{p}} \right]^r$$

$$\approx \sum_n 2^{-n} \left[\sup_{i \le n} \; \operatorname*{ess\,sup}_{y \in (b_{i-3}, b_{i-2}]} \; k_1(b_n,b_{i-1}) w(y) \left(T_p h(y)\right)^{\frac{1}{p}} \right]^r$$

$$+\sum_n 2^n \left[\sup_{i\geq n} \operatorname*{ess\,sup}_{y\in(b_{i-3},b_{i-2}]} k_1(b_{i-1},y)w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r =: J_{1,1}+J_{1,2}.$$

Applying (2.25) as in Theorem 2.1, we obtain

$$J_{1,2} \leq (\mathcal{A}_1^r+\mathcal{A}_2^r+\mathcal{A}_3^r+B^r)\|h\|_{L^1}^{\frac{r}{p}}. \tag{2.26}$$

To estimate $J_{1,1}$ we use Minkowski's inequality and (2.18) to see that

$$\begin{aligned}
J_{1,1} &= \sum_n 2^{-n}\left[\sup_{i\leq n} \operatorname*{ess\,sup}_{y\in(b_{i-3},b_{i-2}]} k_1(b_n,b_{i-1})w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r \\
&\ll \sum_n 2^{-n}\left[\sup_{i\leq n}\left(\sum_{j=i}^n k_1(b_j,b_{j-1})^\alpha\right)^{\frac{1}{\alpha}} \operatorname*{ess\,sup}_{y\in(b_{i-3},b_{i-2}]} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r \\
&\leq \sum_n 2^{-n}\left[\sum_{j\leq n} k_1(b_j,b_{j-1})^\alpha\left(\sup_{i\leq j}\operatorname*{ess\,sup}_{y\in(b_{i-3},b_{i-2}]} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right)^\alpha\right]^{\frac{r}{\alpha}} \\
&= \sum_n 2^{-n}\left[\sum_{j\leq n} k_1(b_j,b_{j-1})\operatorname*{ess\,sup}_{y\leq b_{j-2}} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r \\
&\overset{(2.25)}{\approx} \sum_n 2^{-n}k_1(b_n,b_{n-1})^r\left[\operatorname*{ess\,sup}_{y\leq b_{n-2}} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r \\
&\approx \sum_n \int_{b_n}^{b_{n+1}}\rho(x)dx k_1(b_n,b_{n-1})^r\left[\operatorname*{ess\,sup}_{y\leq b_{n-2}} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r \\
&\leq \sum_n \int_{b_n}^{b_{n+1}}\rho(x)k_1(x,\zeta^{-2}(x))^r\left[\operatorname*{ess\,sup}_{y\leq\zeta^{-2}(x)} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r dx \\
&= \int_0^\infty \rho(x)k_1(x,\zeta^{-2}(x))^r\left[\operatorname*{ess\,sup}_{y\leq\zeta^{-2}(x)} w(y)\left(T_ph(y)\right)^{\frac{1}{p}}\right]^r dx \leq \mathcal{A}_4^r\|h\|_{L_1}^{\frac{r}{p}}.
\end{aligned} \tag{2.27}$$

Relations (2.24)-(2.27) imply the upper estimate $C_{\mathcal{T}} \ll \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+B$. The lower estimate is proved by analogy to Theorem 2.1. □

Remark 2. Sharp two-sided estimates of the best constants in (2.7)–(2.9), (2.20)–(2.22) and the constants B by explicit integral functionals are found by reduction theorems in [24], [25] and criteria for the boundedness of Hardy-type integral operators [15], [31], [32], [22]. However, (2.10) and (2.23) contain an additional iteration on the left-hand sides, therefore, we give the separate reduction for this case (see Lemma 2.1).

Supposed that $\lambda,\mu,\nu,\eta \in \mathfrak{M}^+$ and the kernel $k(x,y)$ satisfies Oinarov's condition (1.3), while the sequence b_n and the function $\zeta(x)$ are defined by (2.1) and (2.2) with λ in place of ρ. Given $0<c<d\leq\infty, 0<t,q<\infty, h\in\mathfrak{M}^+$ put

$$\widetilde{T}_th(x) := \chi_{(0,t]}(x)\left(\int_x^\infty k(s,x)\nu(s)\left(\int_0^s h\right)^{\frac{1}{q}}ds\right)^q, \tag{2.28}$$

$$\widetilde{T}_{[c,d]}h(x) := \chi_{[c,d]}(x)\left(\int\limits_x^{\zeta(d)} k(s,x)\nu(s)\left(\int\limits_c^s h\right)^{\frac{1}{q}} ds\right)^q. \tag{2.29}$$

Lemma 2.1. *Let $0 < p, q, r < \infty$. Then for the best constant C in the inequality*

$$\left(\int\limits_0^\infty \lambda(x)\left(\int\limits_0^x \mu(y)\left(\int\limits_y^\infty k(s,y)\nu(s)\left(\int\limits_0^s h\right)^q ds\right)^{\frac{r}{q}} dy\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} \le C\int\limits_0^\infty h\eta, h \in \mathfrak{M}^+$$

satisfies

$$C \approx G_1 + G_2 + G_3 + G,$$

where G_1, G_2, G_3 are the best constants in the inequalities

$$\left(\int\limits_0^\infty \lambda(x)\left(\int\limits_0^x \mu(y)[k(x,y)]^{\frac{r}{q}} dy\right)^{\frac{p}{r}}\left(\int\limits_x^\infty \nu(s)\left(\int\limits_0^s h\right)^q ds\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \le G_1\int\limits_0^\infty h\eta,$$

$$\left(\int\limits_0^\infty \lambda(x)\left(\int\limits_0^x \mu\right)^{\frac{p}{r}}\left(\int\limits_x^\infty k(s,x)\nu(s)\left(\int\limits_0^s h\right)^q ds\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \le G_2\int\limits_0^\infty h\eta,$$

$$\left(\int\limits_0^\infty \lambda(x)\left(\int\limits_0^x \mu(y)\left(\int\limits_y^\infty k(s,y)\nu(s)ds\right)^{\frac{r}{q}}\left(\int\limits_0^y h\right)^r dy\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} \le G_3\int\limits_0^\infty h\eta,$$

for $h \in \mathfrak{M}^+$ and the constant G has the form

$$G := \begin{cases} \sup_{t>0}\left(\int\limits_t^\infty \lambda\right)^{\frac{1}{p}} \left\|\widetilde{T}_t\right\|_{L^1_\eta \to L^r_\mu}, & p \ge 1, \\ \left(\int\limits_0^\infty \lambda(x)\left[\left(\int\limits_t^\infty \lambda\right)\left\|\widetilde{T}_{[\zeta^{-1}(x),\zeta(x)]}\right\|_{L^1_\eta \to L^r_\mu}\right]^{\frac{p}{1-p}} dx\right)^{\frac{1-p}{p}}, & 0 < p < 1. \end{cases}$$

Proof. The upper bound. Write

$$J^p := \int_0^\infty \lambda(x)\left(\int_0^x \mu(y)\left(\int_y^\infty k(s,y)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{r}{q}} dy\right)^{\frac{p}{r}} dx$$

$$\approx \sum_n 2^{-n}\left(\int_{b_{n-1}}^{b_n} \mu(y)\left(\int_y^\infty k(s,y)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{r}{q}} dy\right)^{\frac{p}{r}}$$

$$\approx \sum_n 2^{-n}\left(\int_{b_{n-1}}^{b_n} \mu(y)\left(\int_y^{b_{n+1}} k(s,y)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{r}{q}} dy\right)^{\frac{p}{r}}$$

$$+\sum_n 2^{-n}\left(\int_{b_{n-1}}^{b_n}\mu(y)\left(\int_{b_{n+1}}^{\infty}k(s,y)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{r}{q}}dy\right)^{\frac{p}{r}} =: J_1^p + J_2^p.$$

Estimate of J_2.

$$J_2^p \approx \sum_n \int_{b_n}^{b_{n+1}}\lambda(x)dx\left(\int_{b_{n-1}}^{b_n}\mu(y)\left(\int_{b_{n+1}}^{\infty}k(s,y)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{r}{q}}dy\right)^{\frac{p}{r}} =: J_1^p + J_2^p.$$

Since $k(s,y)\approx k(x,y)+k(s,x)$ for $y\le b_n\le x\le b_{n+1}\le s$, we get

$$J_2^p \approx \sum_n \int_{b_n}^{b_{n+1}}\lambda(x)\left(\int_{b_{n-1}}^{b_n}\mu(y)[k(x,y)]^{\frac{r}{q}}dy\right)^{\frac{p}{r}}dx\left(\int_{b_{n+1}}^{\infty}\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{p}{q}}$$

$$+\sum_n \int_{b_n}^{b_{n+1}}\lambda(x)\left(\int_{b_{n+1}}^{\infty}k(s,x)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{p}{q}}dx\left(\int_{b_{n-1}}^{b_n}\mu\right)^{\frac{p}{r}}$$

$$\ll \sum_n \int_{b_n}^{b_{n+1}}\lambda(x)\left(\int_0^x\mu(y)[k(x,y)]^{\frac{r}{q}}dy\right)^{\frac{p}{r}}\left(\int_x^{\infty}\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{p}{q}}dx$$

$$+\sum_n \int_{b_n}^{b_{n+1}}\lambda(x)\left(\int_0^x\mu\right)^{\frac{p}{r}}\left(\int_x^{\infty}k(s,x)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{p}{q}}dx$$

$$=\int_0^{\infty}\lambda(x)\left(\int_0^x\mu(y)[k(x,y)]^{\frac{r}{q}}dy\right)^{\frac{p}{r}}\left(\int_x^{\infty}\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{p}{q}}dx$$

$$+\int_0^{\infty}\lambda(x)\left(\int_0^x\mu\right)^{\frac{p}{r}}\left(\int_x^{\infty}k(s,x)\nu(s)\left(\int_0^s h\right)^q ds\right)^{\frac{p}{q}}dx$$

$$\le (G_1^p+G_2^p)\left(\int_0^{\infty}h\eta\right)^p.$$

Estimate of J_1.

$$J_1^p \approx \sum_n 2^{-n}\left(\int_{b_{n-1}}^{b_n}\mu(y)\left(\int_y^{b_{n+1}}k(s,y)\nu(s)\left(\int_0^{b_{n-1}}h\right)^q ds\right)^{\frac{r}{q}}dy\right)^{\frac{p}{r}}$$

$$+\sum_n 2^{-n}\left(\int_{b_{n-1}}^{b_n}\mu(y)\left(\int_y^{b_{n+1}}k(s,y)\nu(s)\left(\int_{b_{n-1}}^{s}h\right)^q ds\right)^{\frac{r}{q}}dy\right)^{\frac{p}{r}} =: J_{1,1}^p + J_{1,2}^p.$$

Estimate of $J_{1,1}$.

$$J_{1,1}^p \approx \sum_n \int_{b_n}^{b_{n+1}}\lambda(x)dx\left(\int_{b_{n-1}}^{b_n}\mu(y)\left(\int_y^{b_{n+1}}k(s,y)\nu(s)ds\right)^{\frac{r}{q}}dy\right)^{\frac{p}{r}}\left(\int_0^{b_{n-1}}h\right)^p$$

$$\ll \int_0^{\infty}\lambda(x)\left(\int_0^x\mu(y)\left(\int_y^{\infty}k(s,y)\nu(s)ds\right)^{\frac{r}{q}}\left(\int_0^y h\right)^r dy\right)^{\frac{p}{r}} \le G_3^p\left(\int_0^{\infty}h\eta\right)^p.$$

Now we estimate $J_{1,2}$. Using notations (2.28) and (2.29), write

$$J_{1,2}^p = \sum_n 2^{-n} \left(\int_{b_{n-1}}^{b_n} \mu(y) \left[\widetilde{T}_{[b_{n-1},b_n]} h(y) \right]^r dy \right)^{\frac{p}{r}}$$

$$\leq \sum_n 2^{-n} \left\| \widetilde{T}_{[b_{n-1},b_n]} \right\|^p_{L^1_\eta \to L^r_\mu} \left(\int_{b_{n-1}}^{b_{n+1}} h\eta \right)^p .$$

For $p \geq 1$, applying Jensen's inequality, we have

$$J_{1,2}^p \ll \sup_n \left(\int_{b_n}^\infty \lambda \right) \left\| \widetilde{T}_{[b_{n-1},b_n]} \right\|^p_{L^1_\eta \to L^r_\mu} \left(\int_0^\infty h\eta \right)^p \ll G^p \left(\int_0^\infty h\eta \right)^p .$$

For $0 < p < 1$, by Hölder's inequality, we obtain

$$J_{1,2}^p \ll \left(\sum_n \left(\int_{b_{n-1}}^{b_n} \lambda(x) dx \right)^{\frac{1}{1-p}} \left\| \widetilde{T}_{[b_{n-1},b_n]} \right\|^{\frac{p}{1-p}}_{L^1_\eta \to L^r_\mu} \right)^{1-p} \left(\int_0^\infty h\eta \right)^p \ll G^p \left(\int_0^\infty h\eta \right)^p .$$

Therefore, the upper estimate $C \ll G_1 + G_2 + G_3 + G$ is proved. For the lower estimate it suffices to repeat the corresponding arguments in the proof of Theorem 2.1. □

3 The main results for $\mathcal{S}$

Given $0 < c < d \leq \infty, 0 < t < \infty, h \in \mathfrak{M}^+$ put

$$\widetilde{\mathcal{T}}_t h(x) := \chi_{(0,t]}(x) \left(\int_0^x k_2(x,s) u(s) \left(\int_s^\infty h \right)^{\frac{1}{p}} ds \right)^p , \tag{3.1}$$

$$\widetilde{\mathcal{T}}_{[c,d]} h(x) := \chi_{[c,d]}(x) \left(\int_{\zeta^{-1}(c)}^x k_2(x,s) u(s) \left(\int_s^d h \right)^{\frac{1}{p}} ds \right)^p . \tag{3.2}$$

$$\left\| \widetilde{\mathcal{T}}_t \right\|_{L^p_v \to L^q_w} := \sup_{0 \neq h \in \mathfrak{M}^+} \frac{\left(\int_0^\infty [\widetilde{\mathcal{T}}_t h]^q w \right)^{\frac{1}{q}}}{\left(\int_0^\infty [h]^p v \right)^{\frac{1}{p}}} \tag{3.3}$$

Theorem 3.1. *Let $0 < q < \infty, 0 < p < \infty, 0 < r < \infty$. Then the best constant $C_{\mathcal{S}}$ in*

$$\left(\int_0^\infty [\mathcal{S} f(x)]^r \rho(x) dx \right)^{\frac{1}{r}} \leq C_{\mathcal{S}} \left(\int_0^\infty [f(x)]^p v(x) dx \right)^{\frac{1}{p}} , f \in \mathfrak{M}^\downarrow \tag{3.4}$$

satisfies $C_{\mathcal{S}} \approx \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 + \mathbf{B}$, where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are the best constants in inequalities

$$\left(\int_0^\infty \rho(x) \left(\int_{\zeta^{-3}(x)}^x k_1(x,y) k_2^q(y, \zeta^{-3}(x)) w(y) dy \right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-3}(x)} \left(\int_s^\infty h \right)^{\frac{1}{p}} u(s) ds \right)^r dx \right)^{\frac{p}{r}}$$

$$\leq \mathbf{A}_1^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)\left(\int_{\zeta^{-3}(x)}^x k_1(x,y)w(y)dy\right)^{\frac{r}{q}}\left(\int_0^{\zeta^{-3}(x)} k_2(\zeta^{-3}(x),s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}} ds\right)^r dx\right)^{\frac{p}{r}}$$

$$\leq \mathbf{A}_2^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\int_0^y k_2(y,s)u(s)ds\right)^q\left(\int_y^\infty h\right)^{\frac{q}{p}} dy\right)^{\frac{r}{q}} dx\right)^{\frac{p}{r}} \leq \mathbf{A}_3^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)k_1(x,\zeta^{-2}(x))^{\frac{r}{q}}\left[\int_0^{\zeta^{-2}(x)} w(y)\left(\int_0^y k_2(y,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}}\right)^q dy\right]^{\frac{r}{q}} dx\right)^{\frac{p}{r}}$$

$$\leq \mathbf{A}_4^p \int_0^\infty hV,$$

for $h \in \mathfrak{M}^+$ and the constant $\mathbf{B}$ has the form

$$\mathbf{B} := \begin{cases} \sup_{t>0}\left(\int_t^\infty \rho\right)^{\frac{1}{r}} \left\|\widetilde{\mathcal{T}}_t\right\|^{\frac{1}{p}}_{L^1_V \to L^{\frac{q}{p}}_{k_1(t,\cdot)w(\cdot)}}, & p \leq r, \\ \left(\int_0^\infty \rho(x)\left[\left(\int_x^\infty \rho\right)\left\|\widetilde{\mathcal{T}}_{[\zeta^{-1}(x),\zeta^2(x)]}\right\|_{L^1_V \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}}\right]^{\frac{s}{p}} dx\right)^{\frac{1}{s}}, & r < p, \end{cases}$$

where $\frac{1}{s} := \frac{1}{r} - \frac{1}{p}$.

Proof. The change $f^p \to f$ in (3.4) leads to the inequality

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\int_0^y k_2(y,s)f^{\frac{1}{p}}(s)u(s)ds\right)^q dy\right)^{\frac{r}{q}} dx\right)^{\frac{p}{r}} \leq C_{\mathcal{S}}^p \int_0^\infty fv.$$

Using Proposition 2.1 [9] and the Monotone Convergence Theorem, we obtain the equivalent inequality

$$\left(\int_0^\infty \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\int_0^y k_2(y,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}} ds\right)^q dy\right)^{\frac{r}{q}} dx\right)^{\frac{p}{r}} \leq C_{\mathcal{S}}^p \int_0^\infty hV, \tag{3.5}$$

for $h \in \mathfrak{M}^+$.

The upper bound. Put

$$\mathcal{T}_p h(y) := \left(\int_0^y k_2(y,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}} ds\right)^p.$$

Plainly,

$$I := \sum_n \int_{b_{n-1}}^{b_n} \rho(x)\left(\int_0^x k_1(x,y)w(y)\left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy\right)^{\frac{r}{q}} dx$$

$$\ll \sum_n 2^{-n} \left(\int_0^{b_n} k_1(b_n, y) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\approx \sum_n 2^{-n} \left(\int_0^{b_{n-2}} k_1(b_n, y) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$+ \sum_n 2^{-n} \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} =: I_1 + I_2.$$

$$I_2 \approx \sum_n 2^{-n} \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_0^{b_{n-3}} k_2(y, s) u(s) \left(\int_s^\infty h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}}$$

$$+ \sum_n 2^{-n} \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_{b_{n-3}}^{y} k_2(y, s) u(s) \left(\int_s^\infty h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}} =: I_{2,1} + I_{2,2}.$$

Next, by Oinarov's condition (1.3) $k_2(y, s) \approx k_2(y, \zeta^{-3}(x)) + k_2(\zeta^{-3}(x), s)$ for $0 < s \le \zeta^{-3}(x) \le y$, Hence,

$$I_{2,1} \approx \sum_n \int_{b_n}^{b_{n+1}} \rho(x) dx \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_0^{b_{n-3}} k_2(y, s) u(s) \left(\int_s^\infty h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}}$$

$$\approx \sum_n \int_{b_n}^{b_{n+1}} \rho(x) dx \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) k_2^q(y, \zeta^{-3}(x)) w(y) \left(\int_0^{b_{n-3}} \left(\int_s^\infty h \right)^{\frac{1}{p}} u(s) ds \right)^q dy \right)^{\frac{r}{q}}$$

$$+ \sum_n \int_{b_n}^{b_{n+1}} \rho(x) dx \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_0^{b_{n-3}} k_2(\zeta^{-3}(x), s) u(s) \left(\int_s^\infty h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}}$$

$$\ll \int_0^\infty \rho(x) \left(\int_{\zeta^{-3}(x)}^{x} k_1(x, y) k_2^q(y, \zeta^{-3}(x)) w(y) dy \right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-3}(x)} \left(\int_s^\infty h \right)^{\frac{1}{p}} u(s) ds \right)^r dx$$

$$+ \int_0^\infty \rho(x) \left(\int_{\zeta^{-3}(x)}^{x} k_1(x, y) w(y) dy \right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-3}(x)} k_2(\zeta^{-3}(x), s) u(s) \left(\int_s^\infty h \right)^{\frac{1}{p}} ds \right)^r dx$$

$$\ll (\mathbf{A}_1^r + \mathbf{A}_2^r) \|h\|_{L_V^1}^{\frac{r}{p}}. \tag{3.6}$$

Write

$$I_{2,2} \approx \sum_n 2^{-n} \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_{b_{n-3}}^{y} k_2(y, s) u(s) \left(\int_s^{b_n} h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}}$$

$$+ \sum_n 2^{-n} \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_{b_{n-3}}^{y} k_2(y, s) u(s) \left(\int_{b_n}^\infty h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}} =: I_{2,2,1} + I_{2,2,2}$$

Then

$$I_{2,2,2} \approx \sum_n \int_{b_n}^{b_{n+1}} \rho(x) \left(\int_{b_{n-2}}^{b_n} k_1(b_n, y) w(y) \left(\int_{b_{n-3}}^{y} k_2(y, s) u(s) \left(\int_{b_n}^\infty h \right)^{\frac{1}{p}} ds \right)^q dy \right)^{\frac{r}{q}}$$

$$\leq \int_0^\infty \rho(x) \left(\int_0^x k_1(x,y)w(y) \left(\int_0^y k_2(y,s)u(s)ds \right)^q \left(\int_y^\infty h \right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} dx \leq \mathbf{A}_3^r \|h\|_{L_V^1}^{\frac{r}{p}}. \quad (3.7)$$

Next, using (3.1) and (3.2), we have

$$I_{2,2,1} = \sum_n 2^{-n} \left[\left(\int_{b_{n-2}}^{b_n} k_1(b_n,y)w(y) \left(\widetilde{\mathcal{T}}_{[b_{n-2},b_n]} h(y) \right)^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \right]^{\frac{r}{p}}$$

$$\leq \sum_n 2^{-n} \left\| \widetilde{\mathcal{T}}_{[b_{n-2},b_n]} \right\|^{\frac{r}{p}}_{L_V^1[b_{n-2},b_{n+1}] \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}[b_{n-2},b_n]} \left(\int_{b_{n-2}}^{b_{n+1}} hV \right)^{\frac{r}{p}}.$$

For $p \leq r$, applying Jensen's inequality, we infer

$$I_{2,2,1} \leq \sup_n \left(\int_{b_{n+1}}^\infty \rho \right) \left\| \widetilde{\mathcal{T}}_{[b_{n-2},b_n]} \right\|^{\frac{r}{p}}_{L_V^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}} \left(\int_0^\infty hV \right)^{\frac{r}{p}}$$

$$\leq \sup_{t>0} \left(\int_t^\infty \rho \right) \left\| \widetilde{\mathcal{T}}_t \right\|^{\frac{r}{p}}_{L_V^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}} \left(\int_0^\infty hV \right)^{\frac{r}{p}}.$$

Hence,

$$I_{2,2,1}^{\frac{p}{r}} \leq \mathbf{B}^p \int_0^\infty hV. \quad (3.8)$$

For $r < p$, by Hölder's inequality, we find that

$$I_{2,2,1} \leq \left(\sum_n 2^{\frac{-ns}{r}} \left\| \widetilde{\mathcal{T}}_{[b_{n-2},b_n]} \right\|^{\frac{s}{p}}_{L_V^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}} \right)^{\frac{r}{s}} \left(\int_0^\infty hV \right)^{\frac{r}{p}}.$$

because

$$\sum_n 2^{\frac{-ns}{r}} \left\| \widetilde{\mathcal{T}}_{[b_{n-2},b_n]} \right\|^{\frac{s}{p}}_{L_V^1 \to L^{\frac{q}{p}}_{k_1(b_n,\cdot)w(\cdot)}}$$

$$\approx \sum_n \left(\int_{b_{n-2}}^{b_{n-1}} \rho \right) \left(\int_{b_{n-1}}^\infty \rho \right)^{\frac{s}{p}} \left\| \widetilde{\mathcal{T}}_{[\zeta^{-1}(b_{n-1}),\zeta^2(b_{n-2})]} \right\|^{\frac{s}{p}}_{L_V^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(b_{n-2}),\cdot)w(\cdot)}}$$

$$\leq \sum_n \int_{b_{n-2}}^{b_{n-1}} \rho(x) \left[\left(\int_x^\infty \rho \right) \left\| \widetilde{\mathcal{T}}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L_V^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx$$

$$= \int_0^\infty \rho(x) \left[\left(\int_x^\infty \rho \right) \left\| \widetilde{\mathcal{T}}_{[\zeta^{-1}(x),\zeta^2(x)]} \right\|_{L_V^1 \to L^{\frac{q}{p}}_{k_1(\zeta^2(x),\cdot)w(\cdot)}} \right]^{\frac{s}{p}} dx = \mathbf{B}^s.$$

Therefore,

$$I_{2,2,1}^{\frac{p}{r}} \leq \mathbf{B}^p \int_0^\infty hV. \quad (3.9)$$

Consider

$$I_1 = \sum_n 2^{-n} \left(\int_0^{b_{n-2}} k_1(b_n,y)w(y) \left(\mathcal{T}_p h(y) \right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$= \sum_n 2^{-n} \left(\sum_{i \le n} \int_{b_{i-3}}^{b_{i-2}} k_1(b_n, y) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\approx \sum_n 2^{-n} \left(\sum_{i \le n} \int_{b_{i-3}}^{b_{i-2}} k_1(b_n, b_{i-1}) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$+ \sum_n 2^{-n} \left(\sum_{i \le n} \int_{b_{i-3}}^{b_{i-2}} k_1(b_{i-1}, y) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} =: I_{1,1} + I_{1,2}.$$

Applying (2.12), we get

$$I_{1,2} \le \sum_n 2^{-n} \left(\int_{b_{n-3}}^{b_{n-1}} k_1(b_{n-1}, y) w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} \approx I_2$$

$$\le (\mathbf{A}_1^r + \mathbf{A}_2^r + \mathbf{A}_3^r + \mathbf{B}^r) \|h\|_{L_V^1}^{\frac{r}{p}}. \tag{3.10}$$

To estimate $I_{1,1}$, we use (2.18) and Minkowski's inequality to find that

$$I_{1,1} = \sum_n 2^{-n} \left(\sum_{i \le n} k_1(b_n, b_{i-1}) \int_{b_{i-3}}^{b_{i-2}} w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\ll \sum_n 2^{-n} \left(\sum_{i \le n} \left(\sum_{j=i}^{n} k_1(b_j, b_{j-1})^{\alpha} \right)^{\frac{1}{\alpha}} \int_{b_{i-3}}^{b_{i-2}} w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}}$$

$$\le \sum_n 2^{-n} \left(\sum_{j \le n} k_1(b_j, b_{j-1})^{\alpha} \left(\sum_{i \le j} \int_{b_{i-3}}^{b_{i-2}} w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\alpha} \right)^{\frac{r}{\alpha q}}$$

$$= \sum_n 2^{-n} \left(\sum_{j \le n} k_1(b_j, b_{j-1})^{\alpha} \left(\int_0^{b_{j-2}} w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right)^{\alpha} \right)^{\frac{r}{\alpha q}}$$

$$\overset{(2.12)}{\approx} \sum_n 2^{-n} \left(k_1(b_n, b_{n-1}) \left(\int_0^{b_{n-2}} w(y) \left(\mathcal{T}_p h(y)\right)^{\frac{q}{p}} dy \right) \right)^{\frac{r}{q}}$$

$$\approx \sum_n \int_{b_n}^{b_{n+1}} \rho(x) k_1(b_n, b_{n-1})^{\frac{r}{q}} \left(\int_0^{b_{n-2}} w(y) \left(\mathcal{T}_p h(y)^{\frac{q}{p}} dy\right) \right)^{\frac{r}{q}} dx$$

$$\le \sum_n \int_{b_n}^{b_{n+1}} \rho(x) k_1(x, \zeta^{-2}(x))^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} w(y) \left(\mathcal{T}_p h(y)^{\frac{q}{p}} dy\right) \right)^{\frac{r}{q}} dx$$

$$= \int_0^{\infty} \rho(x) k_1(x, \zeta^{-2}(x))^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} w(y) \left(\mathcal{T}_p h(y)^{\frac{q}{p}} dy\right) \right)^{\frac{r}{q}} dx \le A_4^r \|h\|_{L_V^1}^{\frac{r}{p}} \tag{3.11}$$

It follows from (3.6)-(3.11) that the upper estimate $C_{\mathcal{S}} \ll \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + A_4 + \mathbf{B}$ is proved.

The lower bound. Diminish the domains of integration in (3.5):

(1) $[0,x] \to [\zeta^{-3}(x), x]$ and obtain $C_{\mathcal{S}} \geq \mathbf{A}_1 + \mathbf{A}_2$ (since $k_2(y,s) \approx k_2(y, \zeta^{-3}(x)) + k_2(\zeta^{-3}(x), s)$ for $0 < s \leq \zeta^{-3}(x) \leq y$).

(2) $[s,\infty]) \to [y,\infty)$ and obtain $C_{\mathcal{S}} \geq \mathbf{A}_3$.

(3) $[0,x] \to [0,\zeta^{-2}(x)]$ and obtain $C_{\mathcal{S}} \geq \mathbf{A}_4$ (since $k_1(x,y) \gtrsim k_1(x,\zeta^{-2}(x))$ for $y \leq \zeta^{-2}(x) < x$).

$C_{\mathcal{S}} \gg \mathbf{B}$ is proved analogously to Theorem 2.1.

□

For the limit values of parameters, we have

Remark 3. (1) $C_{\mathcal{S}} \approx \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 + \mathbf{B}$ *for* $q = \infty$, *where* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ *and* $\mathbf{A}_4$ *are the best constants in the inequalities*

$$\left(\int_0^\infty \rho(x)[\operatorname*{ess\,sup}_{y\in(\zeta^{-3}(x),x)} k_1(x,y)k_2(y,\zeta^{-3}(x))w(y)]^r \left(\int_0^{\zeta^{-3}(x)} \left(\int_s^\infty h\right)^{\frac{1}{p}} u(s)ds\right)^r dx\right)^{\frac{p}{r}}$$

$$\leq \mathbf{A}_1^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)[\operatorname*{ess\,sup}_{y\in(\zeta^{-3}(x),x)} k_1(x,y)w(y)]^r \left(\int_0^{\zeta^{-3}(x)} k_2(\zeta^{-3}(x),s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}} ds\right)^r dx\right)^{\frac{p}{r}}$$

$$\leq \mathbf{A}_2^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)\left[\operatorname*{ess\,sup}_{y\in(0,x)} k_1(x,y)w(y)\left(\int_0^y k_2(y,s)u(s)ds\right)\left(\int_y^\infty h\right)^{\frac{1}{p}}\right]^r dx\right)^{\frac{p}{r}} \leq \mathbf{A}_3^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)k_1(x,\zeta^{-2}(x))^r \left[\operatorname*{ess\,sup}_{y\in(0,\zeta^{-2}(x))} w(y)\left(\int_0^y k_2(y,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}} ds\right)\right]^r dx\right)^{\frac{p}{r}}$$

$$\leq \mathbf{A}_4^p \int_0^\infty hV, \tag{3.12}$$

for $h \in \mathfrak{M}^+$, *and the constant* $\mathbf{B}$ *has the form*

$$\mathbf{B} := \begin{cases} \sup_{t>0}\left(\int_t^\infty \rho\right)^{\frac{1}{r}} \left\|\tilde{\mathcal{T}}_t\right\|^{\frac{1}{p}}_{L^1_V \to L^\infty_{k_1(t,\cdot)w(\cdot)}}, & p \leq r, \\ \left(\int_0^\infty \rho(x)\left[\left(\int_x^\infty \rho\right)\left\|\tilde{\mathcal{T}}_{[\zeta^{-1}(x),\zeta^2(x)]}\right\|_{L^1_V \to L^\infty_{k_1(\zeta^2(x),\cdot)w(\cdot)}}\right]^{\frac{s}{p}} dx\right)^{\frac{1}{s}}, & r < p. \end{cases}$$

(2) *For* $p = \infty$ *and for* $r = \infty$, *we have*

$$C_{\mathcal{S}} = \|\mathcal{S}(\frac{1}{v})\|_{L^r_\rho},\ p = \infty;$$

$$C_{\mathcal{S}} \approx \sup_{t>0} \mathcal{R}(t)\|\widetilde{\mathcal{T}}_t\|^{\frac{1}{p}}_{L^1_V \to L^{\frac{q}{p}}_{k_1(t,\cdot)w(\cdot)}},\ r = \infty,$$

where

$$\mathcal{R}(t) := \operatorname*{ess\,sup}_{z\geq t}\rho(z).$$

Suppose $\lambda,\mu,\nu,\eta \in \mathfrak{M}^+$, the kernel $k(x,y)$ satisfies Oinarov's condition (1.3), and the sequence b_n and the function $\zeta(x)$ are defined by (2.1) and (2.2) with λ instead of ρ. Given $0 < c < d \leq \infty, 0 < t, q < \infty, h \in \mathfrak{M}^+$ put

$$\widetilde{\mathcal{T}}_t h(x) := \chi_{(0,t]}(x)\left(\int_0^x k(x,s)\nu(s)\left(\int_s^\infty h\right)^{\frac{1}{q}} ds\right)^q,$$

$$\widetilde{\mathcal{T}}_{[c,d]} h(x) := \chi_{[c,d]}(x)\left(\int_{\zeta^{-1}(c)}^x k(x,s)\nu(s)\left(\int_s^d h\right)^{\frac{1}{q}} ds\right)^q.$$

Similar to Remark 2, we prove the following lemma that enables us to reduce the inequalities with the constants $\mathbf{A}_4$.

Lemma 3.1. *Suppose $0 < p,q,r < \infty$. Then the best constant C^* in*

$$\left(\int_0^\infty \lambda(x)\left(\int_0^x \mu(y)\left(\int_0^y k(y,s)\nu(s)\left(\int_s^\infty h\right)^q ds\right)^{\frac{r}{q}} dy\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}}$$

$$\leq C^*\int_0^\infty h\eta, h\in\mathfrak{M}^+$$

satisfies

$$C^* \approx G_1^* + G_2^* + G_3^* + G^*,$$

where G_1^*, G_2^*, G_3^* are the best constants in

$$\left(\int_0^\infty \lambda(x)\left(\int_{\zeta^{-2}(x)}^x \mu(y)[k(y,\zeta^{-2}(x))]^{\frac{r}{q}} dy\right)^{\frac{p}{r}}\left(\int_0^{\zeta^{-2}(x)} \nu(s)\left(\int_s^\infty h\right)^q ds\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$\leq G_1^*\int_0^\infty h\eta,$$

$$\left(\int_0^\infty \lambda(x)\left(\int_{\zeta^{-2}(x)}^x \mu\right)^{\frac{p}{r}}\left(\int_0^{\zeta^{-2}(x)} k(\zeta^{-2}(x),s)\nu(s)\left(\int_s^\infty h\right)^q ds\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \leq G_2^*\int_0^\infty h\eta,$$

$$\left(\int_0^\infty \lambda(x)\left(\int_0^x \mu(y)\left(\int_0^y k(y,s)\nu(s)ds\right)^{\frac{r}{q}}\left(\int_y^\infty h\right)^r dy\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} \leq G_3^*\int_0^\infty h\eta$$

for $h \in \mathfrak{M}^+$ *and the constant* G^* *has the form*

$$G^* := \begin{cases} \sup_{t>0} \left(\int_t^\infty \lambda\right)^{\frac{1}{p}} \left\|\widetilde{\mathcal{T}}_t\right\|_{L^1_\eta \to L^r_\mu}, & p \geq 1, \\ \left(\int_0^\infty \lambda(x) \left[\left(\int_x^\infty \lambda\right) \left\|\widetilde{\mathcal{T}}_{[\zeta^{-1}(x),\zeta(x)]}\right\|_{L^1_\eta \to L^r_\mu}\right]^{\frac{p}{1-p}} dx\right)^{\frac{1-p}{p}}, & 0 < p < 1. \end{cases}$$

Inequality (3.12) is reduced similarly.

4 The main results for T и S

The characterization of (1) for the operators T и S is proved in [33]. For completeness below we state the results. We will assume that $0 < \int_0^x \rho < \infty$ for every $x > 0$ and $\int_0^\infty \rho = \infty, \int_0^\infty w = \infty$. Define the sequence $\{a_n\} \subset (0;\infty)$ from the equations

$$\int_0^{a_n} \rho = 2^n, \ n \in \mathbb{Z}.$$

Let $\sigma : [0;\infty) \to [0;\infty)$ и $\sigma^{-1} : [0;\infty) \to [0;\infty)$ be defined by the formulas (here $\inf \varnothing = \infty$)

$$\sigma(x) := \inf\left\{y > 0 : \int_0^y \rho \geq 2\int_0^x \rho\right\}, \sigma^{-1}(x) := \inf\left\{y > 0 : \int_0^y \rho \geq \frac{1}{2}\int_0^x \rho\right\}, \ x \geq 0.$$

For $0 < c < d \leq \infty, 0 < t < \infty, h \in \mathfrak{M}^+$ put

$$\mathcal{T}_t h(x) := \chi_{[t,\infty)}(x) \left(\int_0^x k_2(x,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}} ds\right)^p,$$

$$\mathcal{T}_{[c,d]} h(x) := \chi_{[c,d]}(x) \left(\int_{\sigma^{-1}(c)}^x k_2(x,s)u(s)\left(\int_s^d h\right)^{\frac{1}{p}} ds\right)^p,$$

$$\|\mathcal{T}_t\|_{L^p_v \to L^q_w} := \sup_{0 \neq h \in \mathfrak{M}^+} \frac{\left(\int_0^\infty [\mathcal{T}_t h]^q w\right)^{\frac{1}{q}}}{\left(\int_0^\infty [h]^p v\right)^{\frac{1}{p}}}$$

Theorem 4.1. *Suppose that* $0 < q < \infty, 0 < p < \infty, 0 < r < \infty$. *Then the best constant* C_T *in*

$$\left(\int_0^\infty [Tf(x)]^r \rho(x) dx\right)^{\frac{1}{r}} \leq C_T \left(\int_0^\infty [f(x)]^p v(x) dx\right)^{\frac{1}{p}}, f \in \mathfrak{M}^\downarrow$$

satisfies

$$C_T \approx A_1 + A_2 + A_3 + A_4 + B,$$

where A_1, A_2, A_3, A_4 *are the best constants in*

$$\left(\int_0^\infty \rho(x)\left(\int_x^\infty k_1(y,x)k_2(y,x)^q w(y)dy\right)^{\frac{r}{q}} \left(\int_0^x \left(\int_s^\infty h\right)^{\frac{1}{p}} u(s)ds\right)^r dx\right)^{\frac{p}{r}} \leq A_1^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)\left(\int_x^\infty k_1(y,x)w(y)dy\right)^{\frac{r}{q}} \left(\int_0^x k_2(x,s)\left(\int_s^\infty h\right)^{\frac{1}{p}} u(s)ds\right)^r dx\right)^{\frac{p}{r}} \leq A_2^p \int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)\left(\int_x^\infty k_1(y,x)w(y)\left(\int_0^y k_2(y,s)u(s)ds\right)^q\left(\int_y^\infty h\right)^{\frac{q}{p}}dy\right)^{\frac{r}{q}}dx\right)^{\frac{p}{r}} \le A_3^p\int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)k_1(\sigma^2(x),x)^{\frac{r}{q}}\left[\int_{\sigma^2(x)}^\infty w(y)\left(\int_0^y k_2(y,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}}\right)^q dy\right]^{\frac{r}{q}}dx\right)^{\frac{p}{r}} \le A_4^p\int_0^\infty hV,$$

for $h\in\mathfrak{M}^+$ *and the constant* B *has the form*

$$B:=\begin{cases}\sup_{t>0}\left(\int_0^t\rho\right)^{\frac{1}{r}}\|\mathcal{T}_{\sqcup t}\|^{\frac{1}{p}}_{L^1_V\to L^{\frac{q}{p}}_{k_1(\cdot,t)w(\cdot)}}, & p\le r;\\ \left(\int_0^\infty \rho(x)\left[\left(\int_0^x\rho\right)\left\|\mathcal{T}_{[\sigma^{-1}(x),\sigma^2(x)]}\right\|_{L^1_V\to L^{\frac{q}{p}}_{k_1(\cdot,\sigma^{-1}(x))w(\cdot)}}\right]^{\frac{s}{p}}dx\right)^{\frac{1}{s}}, & r<p,\end{cases}$$

where $\frac{1}{s}:=\frac{1}{r}-\frac{1}{p}$.

Remark 4. (1) $C_T\approx A_1+A_2+A_3+A_4+B$ *for* $q=\infty$ *where* A_1,A_2,A_3,A_4 *are the best constants in*

$$\left(\int_0^\infty \rho(x)[\operatorname*{ess\,sup}_{y\ge x} k_1(y,x)k_2(y,x)w(y)]^r\left(\int_0^x\left(\int_s^\infty h\right)^{\frac{1}{p}}u(s)ds\right)^r dx\right)^{\frac{p}{r}} \le A_1^p\int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)[\operatorname*{ess\,sup}_{y\ge x} k_1(y,x)w(y)]^r\left(\int_0^x k_2(x,s)\left(\int_s^\infty h\right)^{\frac{1}{p}}u(s)ds\right)^r dx\right)^{\frac{p}{r}} \le A_2^p\int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)\left[\operatorname*{ess\,sup}_{y\ge x} k_1(y,x)w(y)\left(\int_0^y k_2(y,s)u(s)ds\right)\left(\int_y^\infty h\right)^{\frac{1}{p}}\right]^r dx\right)^{\frac{p}{r}} \le A_3^p\int_0^\infty hV,$$

$$\left(\int_0^\infty \rho(x)k_1(\sigma^2(x),x)^r\left[\operatorname*{ess\,sup}_{y\ge\sigma^2(x)} w(y)\left(\int_0^y k_2(y,s)u(s)\left(\int_s^\infty h\right)^{\frac{1}{p}}ds\right)\right]^r dx\right)^{\frac{p}{r}} \le A_4^p\int_0^\infty hV,$$

for $h\in\mathfrak{M}^+$ *and the constant* B *has the form*

$$B:=\begin{cases}\sup_{t>0}\left(\int_0^t\rho\right)^{\frac{1}{r}}\|\mathcal{T}_{\sqcup t}\|^{\frac{1}{p}}_{L^1_V\to L^\infty_{k_1(\cdot,t)w(\cdot)}}, & p\le r,\\ \left(\int_0^\infty \rho(x)\left[\left(\int_0^x\rho\right)\left\|\mathcal{T}_{[\sigma^{-1}(x),\sigma^2(x)]}\right\|_{L^1_V\to L^\infty_{k_1(\cdot,\sigma^{-1}(x))w(\cdot)}}\right]^{\frac{s}{p}}dx\right)^{\frac{1}{s}}, & r<p.\end{cases}$$

(2) *For* $p=\infty$ *or* $r=\infty$, *we have*

$$C_T=\|T(\frac{1}{v})\|_{L^r_\rho},\ p=\infty;$$

$$C_T\approx\sup_{t>0}R(t)\|\mathcal{T}_t\|^{\frac{1}{p}}_{L^1_V\to L^{\frac{q}{p}}_w},\ r=\infty,$$

where $R(t) := \operatorname{ess\,sup}_{0<z<t} \rho(z)$.

For $0 < c < d \le \infty,\ 0 < t < \infty,\ h \in \mathfrak{M}^+$ put

$$T_t h(x) := \chi_{[t,\infty)}(x) \left(\int_x^\infty k_2(s,x) \left(\int_0^s hV \right)^{\frac{1}{p}} \widetilde{u}(s) ds \right)^p,$$

$$T_{[c,d]} h(x) := \chi_{[c,d]}(x) \left(\int_x^{\sigma(d)} k_2(s,x) \left(\int_c^s hV \right)^{\frac{1}{p}} \widetilde{u}(s) ds \right)^p.$$

$$\|T_t\|_{L_v^p \to L_w^q} := \sup_{0 \ne h \in \mathfrak{M}^+} \frac{\left(\int_0^\infty [T_t h]^q w \right)^{\frac{1}{q}}}{\left(\int_0^\infty [h]^p v \right)^{\frac{1}{p}}}$$

Theorem 4.2. *Suppose that* $0 < q < \infty, 0 < p < \infty, 0 < r < \infty$. *Then the best constant* C_S *in*

$$\left(\int_0^\infty [Sf(x)]^r \rho(x) dx \right)^{\frac{1}{r}} \le C_S \left(\int_0^\infty [f(x)]^p v(x) dx \right)^{\frac{1}{p}}, f \in \mathfrak{M}^{\downarrow}$$

satisfies

$$C_S \approx \mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3 + \mathbb{A}_4 + \mathbb{B},$$

where $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4$ *are the best constants in*

$$\left(\int_0^\infty \rho(x) \left(\int_x^{\sigma^3(x)} k_1(y,x) k_2(\sigma^3(x),x)^q w(y) dy \right)^{\frac{r}{q}} \left(\int_{\sigma^3(x)}^\infty \left(\int_0^s hV \right)^{\frac{1}{p}} \tilde{u}(s) ds \right)^r dx \right)^{\frac{p}{r}}$$
$$\le \mathbb{A}_1^p \int_0^\infty h,$$

$$\left(\int_0^\infty \rho(x) \left(\int_x^{\sigma^3(x)} k_1(y,x) w(y) dy \right)^{\frac{r}{q}} \left(\int_{\sigma^3(x)}^\infty k_2(s,\sigma^3(x)) \left(\int_0^s hV \right)^{\frac{1}{p}} \tilde{u}(s) ds \right)^r dx \right)^{\frac{p}{r}}$$
$$\le \mathbb{A}_2^p \int_0^\infty h,$$

$$\left(\int_0^\infty \rho(x) \left(\int_x^\infty k_1(y,x) w(y) \left(\int_y^\infty k_2(s,y) \tilde{u}(s) ds \right)^q \left(\int_0^y hV \right)^{\frac{q}{p}} dy \right)^{\frac{r}{q}} dx \right)^{\frac{p}{r}} \le \mathbb{A}_3^p \int_0^\infty h,$$

$$\left(\int_0^\infty \rho(x) k_1(\sigma^2(x),x)^{\frac{r}{q}} \left[\int_{\sigma^2(x)}^\infty w(y) \left(\int_y^\infty k_2(s,y) \tilde{u}(s) \left(\int_0^s hV \right)^{\frac{1}{p}} \right)^q dy \right]^{\frac{r}{q}} dx \right)^{\frac{p}{r}} \le \mathbb{A}_4^p \int_0^\infty h,$$

for $h \in \mathfrak{M}^+$ *and the constant* $\mathbb{B}$ *has the form*

$$\mathbb{B} := \begin{cases} \sup_{t>0} \left(\int_0^t \rho \right)^{\frac{1}{r}} \|T_t\|^{\frac{1}{p}}_{L^1 \to L^{\frac{q}{p}}_{k_1(\cdot,t)w(\cdot)}}, & p \le r, \\ \left(\int_0^\infty \rho(x) \left[\left(\int_0^x \rho \right) \left\| T_{[\sigma^{-1}(x),\sigma^2(x)]} \right\|_{L^1 \to L^{\frac{q}{p}}_{k_1(\cdot,\sigma^{-1}(x))w(\cdot)}} \right]^{\frac{s}{p}} dx \right)^{\frac{1}{s}}, & r < p. \end{cases}$$

Remark 5. (1) *$C_S \approx \mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3 + \mathbb{A}_4 + \mathbb{B}$ for $q = \infty$, where $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4$ are the best constants in*

$$\left(\int_0^\infty \rho(x)[\operatorname*{ess\,sup}_{x<y<\sigma^3(x)} k_1(y,x)k_2(\sigma^3(x),x)w(y)]^r \left(\int_{\sigma^3(x)}^\infty \left(\int_0^s hV\right)^{\frac{1}{p}} \tilde{u}(s)ds\right)^r dx\right)^{\frac{p}{r}}$$

$$\leq \mathbb{A}_1^p \int_0^\infty h,$$

$$\left(\int_0^\infty \rho(x)[\operatorname*{ess\,sup}_{x<y<\sigma^3(x)} k_1(y,x)w(y)]^r \left(\int_{\sigma^3(x)}^\infty k_2(s,\sigma^3(x)) \left(\int_0^s hV\right)^{\frac{1}{p}} \tilde{u}(s)ds\right)^r dx\right)^{\frac{p}{r}}$$

$$\leq \mathbb{A}_2^p \int_0^\infty h,$$

$$\left(\int_0^\infty \rho(x)\left[\operatorname*{ess\,sup}_{y\geq x} k_1(y,x)w(y)\left(\int_y^\infty k_2(s,y)\tilde{u}(s)ds\right)\left(\int_0^y hV\right)^{\frac{1}{p}}\right]^r dx\right)^{\frac{p}{r}} \leq \mathbb{A}_3^p \int_0^\infty h,$$

$$\left(\int_0^\infty \rho(x)k_1(\sigma^2(x),x)^r \left[\operatorname*{ess\,sup}_{y\geq\sigma^2(x)} w(y)\left(\int_y^\infty k_2(s,y)\tilde{u}(s)\left(\int_0^s hV\right)^{\frac{1}{p}} ds\right)\right]^r dx\right)^{\frac{p}{r}} \leq \mathbb{A}_4^p \int_0^\infty h,$$

for $h \in \mathfrak{M}^+$ and the constant $\mathbb{B}$ has the form

$$\mathbb{B} := \begin{cases} \sup_{t>0}\left(\int_0^t \rho\right)^{\frac{1}{r}} \|T_t\|^{\frac{1}{p}}_{L^1\to L^\infty_{k_1(\cdot,t)w(\cdot)}}, & p \leq r; \\ \left(\int_0^\infty \rho(x)\left[\left(\int_0^x \rho\right)\left\|T_{[\sigma^{-1}(x),\sigma^2(x)]}\right\|_{L^1\to L^\infty_{k_1(\cdot,\sigma^{-1}(x))w(\cdot)}}\right]^{\frac{s}{p}} dx\right)^{\frac{1}{s}}, & r < p. \end{cases}$$

(2) *For $p = \infty$ and $r = \infty$, we have*

$$C_S = \|S(\frac{1}{v})\|_{L^r_\rho},\ p = \infty;$$

$$C_S \approx \sup_{t>0} R(t)\|T_t\|^{\frac{1}{p}}_{L^1_V\to L^{\frac{q}{p}}_{k_1(\cdot,t)w(\cdot)}},\ r = \infty,$$

where $R(t) := \operatorname*{ess\,sup}_{0<z<t}\rho(z)$.

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