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HARDY AND RELLICH TYPE INEQUALITIES ON THE COMPLEX AFFINE GROUP

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Communicated by M. Ruzhansky

Dedicated to the 70^{th} birthday of Professor Ryskul Oinarov

Key words: complex affine group, fundamental solution, Rellich inequality, Hardy inequality.

AMS Mathematics Subject Classification: 35A23, 35H20.

Abstract. In this paper, by using properties of the fundamental solution of the canonical right-invariant Laplacian, versions of Hardy and Rellich type inequalities are proved on the complex affine group.

1 Introduction

In the modern analysis, the (L^2-)Hardy inequality is given by

$$\left\| \frac{f(x)}{|x|} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{n-2} \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad n \geq 3, \tag{1.1}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where $\frac{2}{n-2}$ is sharp.

Hardy type inequalities have also been widely studied on (homogeneous) stratified Lie groups. Let Q be the homogeneous dimension of the stratified Lie group $\mathbb{G}$, ∇_H be the horizontal gradient, and $d(x)$ be the so-called $\mathcal{L}$-gauge derived from the sub-Laplacian fundamental solution $d(x)^{2-Q}$ of Folland [6]. One has

$$\left\| \frac{f(x)}{d(x)} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-2} \|\nabla_H f\|_{L^2(\mathbb{G})}, \quad Q \geq 3, \tag{1.2}$$

for all $f \in C_0^\infty(\mathbb{G})$ where $\frac{2}{Q-2}$ is sharp. On the Heisenberg group inequality (1.2) was proved by Garofalo and Lanconelli [11] (see also [8] and [1] for the case $p \neq 2$). General weighted cases of (1.2) on stratified Lie groups are known, see Goldstein and Kombe [10] (see also [2]), and its further refinements, including boundary terms for bounded domains, were obtained in [15]. We refer to [16] for versions on more general Lie groups, namely, homogeneous groups.

Meanwhile, Rellich type inequalities also have a long history starting with Rellich's original work [14] and in the classical case it can be stated as

$$\left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)} \|\Delta f\|_{L^2(\mathbb{R}^n)}, \quad n \geq 5. \tag{1.3}$$

Different versions of this inequality have been also intensively investigated on stratified Lie groups. See, for example, [3], [4], [5], [7], [10], [13], and [17].

In the recent paper [18], Ruzhansky and the second author proved improved versions of Hardy's and Rellich's inequalities as well as of uncertainty principles for sums of squares of vector fields on bounded sets of smooth manifolds under certain assumptions on the vector fields, in particular, the obtained results were valid for sums of squares of vector fields on Euclidean spaces and for sub-Laplacians on stratified Lie groups. However, all this analysis was related to unimodular Lie groups. The aim of this paper is to show that some of those techniques are also applicable for non-unimodular Lie groups, namely, the complex affine group.

Let $\mathbb{G} = \mathbb{C} \rtimes \mathbb{C}^*$ be the complex affine group. Here and in the sequel $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. The group composition law of the complex affine group $\mathbb{G}$ is given by

$$(x, y) \circ (x', y') = (x + yx', yy')$$

for all $x, x' \in \mathbb{C}$ and $y, y' \in \mathbb{C}^*$ and the notations $x := t + is$ and $y := \tau + i\varsigma$ will be also useful in our analysis. The complex affine group is a Lie group and let us denote its Lie algebra by $\mathfrak{g}$.

We now fix a basis $\{X_1, X_2, X_3, X_4\}$ of $\mathfrak{g}$ such that

$$X_1 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial \tau} + \varsigma\frac{\partial}{\partial \varsigma},$$
$$X_2 = \frac{\partial}{\partial s}, \quad X_4 = -s\frac{\partial}{\partial t} + t\frac{\partial}{\partial s} - \varsigma\frac{\partial}{\partial \tau} + \tau\frac{\partial}{\partial \varsigma}.$$

These right invariant vector fields correspond to the canonical basis elements of $\mathfrak{g}$. Therefore, the (sub-)Laplacian

$$\Delta_X = -\sum_{j=1}^{4} X_j^2,$$

is called the right invariant canonical Laplacian of the complex affine group $\mathbb{G}$. The fundamental solution of the Laplacian Δ_X was computed explicitly by Gaudry and Sjögren [9] in the following form

$$\varepsilon = \frac{1}{4\pi^2}\frac{|y|^2}{|x|^2 + |1-y|^2}.$$

This explicit formula plays a key role in our proofs (see, e.g. (2.4)). We will also use the notation of the right invariant (canonical) gradient in the form

$$\nabla_X = (X_1, X_2, X_3, X_4).$$

The right-invariant and the left-invariant Haar measures on $\mathbb{G}$ are defined by

$$d\mu_r = dx\frac{dy}{|y|^2}, \quad d\mu_l = dx\frac{dy}{|y|^4}$$

with the modular function $m(x, y) = |y|^2$, respectively. In addition, one has the following integration rules with respect to the modular function

$$\int_{\mathbb{G}} f(\eta\zeta) d\mu_l(\eta) = m^{-1}(\zeta) \int_{\mathbb{G}} f(\eta) d\mu_l(\eta),$$
$$\int_{\mathbb{G}} f(\eta^{-1}) m^{-1}(\eta) d\mu_l(\eta) = \int_{\mathbb{G}} f(\eta) d\mu_l(\eta).$$

The plan of this short paper is as follows. In Section 2 we derive versions of Hardy inequality and uncertainty principle on the complex affine group $\mathbb{G}$. In Section 3 Rellich type inequality on $\mathbb{G}$ is studied.

2 Hardy type inequalities and uncertainty principles

We now present a Hardy type inequality on $\mathbb{G}$. The proof of Theorem 2.1 relies on properties of the fundamental solution of the right invariant canonical Laplacian Δ_X of the complex affine group $\mathbb{G}$.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $\alpha > 2-\beta$, $\beta > 2$. Then the following version of the Hardy inequality is valid:*

$$\int_{\mathbb{G}} \varepsilon^{\frac{\alpha}{2-\beta}} |\nabla_X u|^2 \, d\mu_l \geq \left(\frac{\beta+\alpha-2}{2}\right)^2 \int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\mu_l, \tag{2.1}$$

for any $u \in C_0^\infty(\mathbb{G})$, where $\nabla_X = (X_1, X_2, X_3, X_4)$.

Proof. Let $(\widetilde{\nabla}_X f)g := \sum_{k=1}^4 X_k f X_k g$ for any differentiable functions f and g. Setting $u = d^\gamma q$ for some (real-valued) functions $d > 0$, q, and a constant $\gamma \neq 0$ to be chosen later, we have

$$(\widetilde{\nabla}_X u)u = (\widetilde{\nabla}_X d^\gamma q) d^\gamma q = \sum_{k=1}^4 X_k(d^\gamma q) X_k(d^\gamma q)$$

$$= \gamma^2 d^{2\gamma-2} \sum_{k=1}^4 (X_k d)^2 q^2 + 2\gamma d^{2\gamma-1} q \sum_{k=1}^4 X_k d \, X_k q + d^{2\gamma} \sum_{k=1}^4 (X_k q)^2$$
$$= \gamma^2 d^{2\gamma-2} ((\widetilde{\nabla}_X d)d) q^2 + 2\gamma d^{2\gamma-1} q (\widetilde{\nabla}_X d) q + d^{2\gamma} (\widetilde{\nabla}_X q) q.$$

Integrating by parts we observe that

$$2\gamma \int_{\mathbb{G}} d^{\alpha+2\gamma-1} q (\widetilde{\nabla}_X d) q d\mu_l = \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} (\widetilde{\nabla}_X d^{\alpha+2\gamma}) q^2 d\mu_l$$
$$= \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} (\widetilde{\nabla}_X q^2) d^{\alpha+2\gamma} d\mu_l = -\frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} q^2 \Delta_X d^{\alpha+2\gamma} d\mu_l,$$

where we note that later on we will choose γ so that $d^{\alpha+2\gamma} = \varepsilon$. Consequently, we get

$$\int_{\mathbb{G}} d^\alpha (\widetilde{\nabla}_X u) u d\mu_l =$$
$$\gamma^2 \int_{\mathbb{G}} d^{\alpha+2\gamma-2} ((\widetilde{\nabla}_X d)d) \, q^2 d\mu_l + \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} (\widetilde{\nabla}_X d^{\alpha+2\gamma}) q^2 d\mu_l$$
$$+ \int_{\mathbb{G}} d^{\alpha+2\gamma} (\widetilde{\nabla}_X q) q d\mu_l = \gamma^2 \int_{\mathbb{G}} d^{\alpha+2\gamma-2} ((\widetilde{\nabla}_X d)d) \, q^2 d\mu_l$$
$$- \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} q^2 \Delta_X d^{\alpha+2\gamma} d\mu_l + \int_{\mathbb{G}} d^{\alpha+2\gamma} (\widetilde{\nabla}_X q) q d\mu_l$$
$$\geq \gamma^2 \int_{\mathbb{G}} d^{\alpha+2\gamma-2} ((\widetilde{\nabla}_X d)d) \, q^2 d\mu_l - \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{G}} q^2 \Delta_X d^{\alpha+2\gamma} d\mu_l, \tag{2.2}$$

since $d>0$ and $(\widetilde{\nabla}_X q)q = |\nabla_X q|^2 \geq 0$. On the other hand, it can be readily checked that for a vector field X we have

$$\begin{aligned}\frac{\gamma}{\alpha+2\gamma}X^2(d^{\alpha+2\gamma}) &= \gamma X(d^{\alpha+2\gamma-1}Xd) = \frac{\gamma}{2-\beta}X(d^{\alpha+2\gamma+\beta-2}X(d^{2-\beta}))\\ &= \frac{\gamma}{2-\beta}(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma+\beta-3}(Xd)X(d^{2-\beta})\\ &+ \frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}X^2(d^{2-\beta})\\ &= \gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(Xd)^2\\ &+ \frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}X^2(d^{2-\beta}).\end{aligned}$$

Consequently, we get the equality

$$-\frac{\gamma}{\alpha+2\gamma}\Delta_X d^{\alpha+2\gamma} = -\gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(\widetilde{\nabla}_X d)d - \frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}\Delta_X d^{2-\beta}. \quad (2.3)$$

Since $q^2 = d^{-2\gamma}u^2$, by substituting (2.3) into (2.2) we obtain

$$\begin{aligned}\int_{\mathbb{G}} d^{\alpha}(\widetilde{\nabla}_X u)u d\mu_l &\geq (-\gamma^2 - \gamma(\alpha+\beta-2))\int_{\mathbb{G}} d^{\alpha-2}((\widetilde{\nabla}_X d)d)u^2 d\mu_l\\ &-\frac{\gamma}{2-\beta}\int_{\mathbb{G}}(\Delta_X d^{2-\beta})d^{\alpha+\beta-2}u^2 dx.\end{aligned}$$

Taking $d=\varepsilon^{\frac{1}{2-\beta}}$, $\beta>2$, we get

$$\int_{\mathbb{G}}(\Delta_X\varepsilon)\varepsilon^{\frac{\alpha+\beta-2}{2-\beta}}u^2 dx = \left(\frac{1}{\varepsilon(e)}\right)^{\frac{\alpha+\beta-2}{\beta-2}} u^2(e) = 0,\ \alpha>2-\beta,\ \beta>2, \quad (2.4)$$

since ε is the fundamental solution to Δ_X. Here $e=(0,0,1,0)$ is the identity element of $\mathbb{G}$. Thus, with $d=\varepsilon^{\frac{1}{2-\beta}}$, $\beta>2$, we obtain

$$\int_{\mathbb{G}} \varepsilon^{\frac{\alpha}{2-\beta}}(\widetilde{\nabla}_X u)u\, d\mu_l \geq (-\gamma^2-\gamma(\alpha+\beta-2))\int_{\mathbb{G}}\varepsilon^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}_X\varepsilon^{\frac{1}{2-\beta}})\varepsilon^{\frac{1}{2-\beta}}u^2\, d\mu_l. \quad (2.5)$$

Now taking $\gamma=\frac{2-\beta-\alpha}{2}$, we obtain (2.1). □

Theorem 2.1 implies the following uncertainty principles:

Corollary 2.1 (Uncertainty principle on $\mathbb{G}$)**.** *Let $\beta>2$. Then for any $u\in C_0^\infty(\mathbb{G})$ we have*

$$\int_{\mathbb{G}}\varepsilon^{\frac{2}{2-\beta}}|\nabla_X\varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\nu\int_{\mathbb{G}}|\nabla_X u|^2 d\nu \geq \left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{G}}|\nabla_X\varepsilon^{\frac{1}{2-\beta}}|^2|u|^2 d\nu\right)^2, \quad (2.6)$$

and also

$$\int_{\mathbb{G}}\frac{\varepsilon^{\frac{2}{2-\beta}}}{|\nabla_X\varepsilon^{\frac{1}{2-\beta}}|^2}|u|^2 d\nu\int_{\mathbb{G}}|\nabla_X u|^2 d\nu \geq \left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{G}}|u|^2 d\nu\right)^2. \quad (2.7)$$

Proof. By taking $\alpha = 0$ in inequality (2.1) we get

$$\int_{\mathbb{G}} \varepsilon^{\frac{2}{2-\beta}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \int_{\mathbb{G}} |\nabla_X u|^2 d\nu$$

$$\geq \left(\frac{\beta-2}{2}\right)^2 \int_{\mathbb{G}} \varepsilon^{\frac{2}{2-\beta}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu \int_{\mathbb{G}} \frac{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}{\varepsilon^{\frac{2}{2-\beta}}} |u|^2 \, d\nu$$

$$\geq \left(\frac{\beta-2}{2}\right)^2 \left(\int_{\mathbb{G}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 d\nu\right)^2,$$

where we have used the Hölder inequality in the last line. This shows (2.6). The proof of (2.7) is similar. □

3 Rellich type inequalities

In this section, we present a version of the Rellich inequality.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $\beta > \alpha > 4 - \beta$ and $\beta > 2$. Then the following version of the Rellich inequality is valid:*

$$\int_{\mathbb{G}} \frac{\varepsilon^{\frac{\alpha}{2-\beta}}}{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2} |\Delta_X u|^2 d\mu_l \geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16} \int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 \, d\mu_l, \tag{3.1}$$

for any $u \in C_0^\infty(\mathbb{G})$, where ∇_X is the gradient and Δ_X is the Laplacian of $\mathbb{G}$ as defined in Introduction.

Proof. A direct calculation shows that

$$\Delta_X \varepsilon^{\frac{\alpha-2}{2-\beta}} = \sum_{k=1}^{4} X_k^2 \varepsilon^{\frac{\alpha-2}{2-\beta}} = (\alpha-2) \sum_{k=1}^{4} X_k \left(\varepsilon^{\frac{\alpha-3}{2-\beta}} X_k \varepsilon^{\frac{1}{2-\beta}}\right)$$

$$= (\alpha-2)(\alpha-3)\varepsilon^{\frac{\alpha-4}{2-\beta}} \sum_{k=1}^{4} \left|X_k \varepsilon^{\frac{1}{2-\beta}}\right|^2 + (\alpha-2)\varepsilon^{\frac{\alpha-3}{2-\beta}} \sum_{k=1}^{4} X_k \left(X_k \varepsilon^{\frac{1}{2-\beta}}\right)$$

$$= (\alpha-2)(\alpha-3)\varepsilon^{\frac{\alpha-4}{2-\beta}} \sum_{k=1}^{4} \left|X_k \varepsilon^{\frac{1}{2-\beta}}\right|^2 + \frac{\alpha-2}{2-\beta}\varepsilon^{\frac{\alpha-3}{2-\beta}} \sum_{k=1}^{4} X_k \left(\varepsilon^{\frac{\beta-1}{2-\beta}} X_k \varepsilon\right)$$

$$= (\alpha-2)(\alpha-3)\varepsilon^{\frac{\alpha-4}{2-\beta}} \sum_{k=1}^{4} \left|X_k \varepsilon^{\frac{1}{2-\beta}}\right|^2 + \frac{(\alpha-2)(\beta-1)}{2-\beta}\varepsilon^{\frac{\alpha-3}{2-\beta}}\varepsilon^{-1} \sum_{k=1}^{4} (X_k \varepsilon^{\frac{1}{2-\beta}})(X_k \varepsilon)$$

$$+ \frac{\alpha-2}{2-\beta}\varepsilon^{\frac{\beta+\alpha-4}{2-\beta}} \Delta_X \varepsilon = (\alpha-2)(\alpha-3)\varepsilon^{\frac{\alpha-4}{2-\beta}} \sum_{k=1}^{4} \left|X_k \varepsilon^{\frac{1}{2-\beta}}\right|^2$$

$$+ (\alpha-2)(\beta-1)\varepsilon^{\frac{\alpha-4}{2-\beta}} \sum_{k=1}^{4} (X_k \varepsilon^{\frac{1}{2-\beta}})(X_k \varepsilon^{\frac{1}{2-\beta}}) + \frac{\alpha-2}{2-\beta}\varepsilon^{\frac{\beta+\alpha-4}{2-\beta}} \Delta_X \varepsilon$$

$$= (\beta+\alpha-4)(\alpha-2)\varepsilon^{\frac{\alpha-4}{2-\beta}} |\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\varepsilon^{\frac{\beta+\alpha-4}{2-\beta}} \Delta_X \varepsilon,$$

that is,

$$\Delta_X \varepsilon^{\frac{\alpha-2}{2-\beta}} = (\beta+\alpha-4)(\alpha-2)\varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\varepsilon^{\frac{\beta+\alpha-4}{2-\beta}}\Delta_X \varepsilon. \tag{3.2}$$

As before we can assume that u is real-valued. Multiplying both sides of (3.2) by u^2 and integrating over $\mathbb{G}$, and taking into account that ε is the fundamental solution of Δ_X and $\beta+\alpha-4>0$, we get

$$\int_{\mathbb{G}} u^2 \Delta_X \varepsilon^{\frac{\alpha-2}{2-\beta}}\, d\mu_l = (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 u^2\, d\mu_l. \tag{3.3}$$

On the other hand, integrating by parts, we have

$$\int_{\mathbb{G}} u^2 \Delta_X \varepsilon^{\frac{\alpha-2}{2-\beta}}\, d\mu_l = \int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} \Delta_X u^2\, d\mu_l = \int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}}(2u\Delta_X u + 2|\nabla_X u|^2)\, d\mu_l, \tag{3.4}$$

Combining (3.3) and (3.4) we obtain

$$-2\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} u\Delta_X u d\mu_l + (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2\, u^2 d\mu_l = 2\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}}|\nabla_X u|^2 d\mu_l. \tag{3.5}$$

By using (2.1) we establish

$$-2\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} u\Delta_X u d\mu_l + (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2\, |u|^2 d\mu_l \geq 2\left(\frac{\beta+\alpha-4}{2}\right)^2 \int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2\, d\mu_l. \tag{3.6}$$

It follows that

$$-\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} u\Delta_X u d\mu_l \geq \left(\frac{\beta+\alpha-4}{2}\right)\left(\frac{\beta-\alpha}{2}\right)\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2\, d\mu_l. \tag{3.7}$$

On the other hand, for any $\nu>0$ Hölder's and Young's inequalities give

$$\begin{aligned} -\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-2}{2-\beta}} u\Delta_X u d\mu_l &\leq \left(\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 d\mu_l\right)^{\frac{1}{2}} \left(\int_{\mathbb{G}} \frac{\varepsilon^{\frac{\alpha}{2-\beta}}}{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}|\Delta_X u|^2 d\mu_l\right)^{\frac{1}{2}} \\ &\leq \nu\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2 d\mu_l + \frac{1}{4\nu}\int_{\mathbb{G}} \frac{\varepsilon^{\frac{\alpha}{2-\beta}}}{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}|\Delta_X u|^2 d\mu_l. \end{aligned} \tag{3.8}$$

Inequalities (3.8) and (3.7) imply that

$$\int_{\mathbb{G}} \frac{\varepsilon^{\frac{\alpha}{2-\beta}}}{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}|\Delta_X u|^2 d\mu_l \geq \left(-4\epsilon^2 + (\beta+\alpha-4)(\beta-\alpha)\epsilon\right)\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2\, d\mu_l.$$

Taking $\nu = \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$, we arrive at

$$\int_{\mathbb{G}} \frac{\varepsilon^{\frac{\alpha}{2-\beta}}}{|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2}|\Delta_X u|^2 d\mu_l \geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16}\int_{\mathbb{G}} \varepsilon^{\frac{\alpha-4}{2-\beta}}|\nabla_X \varepsilon^{\frac{1}{2-\beta}}|^2 |u|^2\, d\mu_l.$$

□

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References

[1] Adimurthi, A. Sekar, *Role of the fundamental solution in Hardy-Sobolev-type inequalities,* Proc. Roy. Soc. Edinburgh Sect. A, 136 (2003), no. 6, 1111–1130.

[2] P. Ciatti, M. G. Cowling, F. Ricci, *Hardy and uncertainty inequalities on stratified Lie groups,* Adv. Math. 277 (2015), 365–387.

[3] L. D'Ambrosio, *Hardy-type inequalities related to degenerate elliptic differential operators,* Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 (2005), no. 3, 451–486.

[4] D. Danielli, N. Garofalo, N.C. Phuc, *Hardy-Sobolev type inequalities with sharp constants in Carnot-Carathéodory spaces,* Potential Anal. 34 (2011), no. 3, 223–242.

[5] V. Fischer, M. Ruzhansky, *Quantization on nilpotent Lie groups,* Progress in Mathematics, 314, Birkhauser, 2016, xiii+557pp. ISBN: 978-3-319-29557-2.

[6] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups,* Ark. Mat. 13 (1975), no. 2, 161–207.

[7] Y. Jin, S. Shen, *Weighted Hardy and Rellich inequality on Carnot groups,* Arch. Math. (Basel) 96 (2011), no. 3, 263–271.

[8] P. Niu, H. Zhang, Y. Wang, *Hardy type and Rellich type inequalities on the Heisenberg group,* Proc. Amer. Math. Soc. 129 (2001), no. 12, 3623–3630.

[9] G. Gaudry, P. Sjögren, *Singular integrals on the complex affine group,* Colloq. Math. 75 (1998), no. 1, 133–148.

[10] J.A. Goldstein, I. Kombe, *The Hardy inequality and nonlinear parabolic equations on Carnot groups,* Nonlinear Anal. 69 (2008), no. 12, 4643–4653.

[11] N. Garofalo, E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation,* Ann. Inst. Fourier (Grenoble), 40 (1990), no. 2, 313–356.

[12] I. Kombe, *Sharp weighted Rellich and uncertainty principle inequalities on Carnot groups,* Commun. Appl. Anal. 14 (2010), no.2, 251–271.

[13] B. Lian, *Some sharp Rellich type inequalities on nilpotent groups and application,* Acta Math. Sci. Ser. B Engl. Ed. 33 (2013), no.1, 59–74.

[14] F. Rellich, *Halbbeschränkte differentialoperatoren höherer ordnung,* Proceedings of the International Congress of Mathematicians, 3 (1954), 243–250. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.

[15] M. Ruzhansky, D. Suragan, *Layer potentials, Green formulae, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups,* Adv. Math. 308 (2017), 483–528.

[16] M. Ruzhansky, D. Suragan, *Anisotropic L^2-weighted Hardy and L^2-Caffarelli-Kohn-Nirenberg inequalities,* Commun. Contemp. Math. 1750014 (2016), 1-12.

[17] M. Ruzhansky, D. Suragan, *On horizontal Hardy, Rellich, Caffarelli-Kohn-Nirenberg and p-sub-Laplacian inequalities on stratified groups,* J. Differential Equations, 262 (2017), 1799-1821.

[18] M. Ruzhansky, D. Suragan, *Local Hardy and Rellich inequalities for sums of squares of vector fields,* Adv. Diff. Equations, 22 (2017), 505-540.

[19] P. Sjögren, *An estimate for a first-order Riesz operator on the affine group,* Trans. Amer. Math. Soc. 351 (1999), no. 8, 3301–3314.

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