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ON A WEIGHTED SOBOLEV SPACE ON REAL LINE

D.V. Prokhorov

Communicated by M.L. Goldman

Dedicated to the 70th birthday of Professor Ryskul Oinarov

Key words: Sobolev space, density of smooth functions

AMS Mathematics Subject Classification: 46E35

Abstract. A criterion of density of smooth functions in a weighted Sobolev space on real line is obtained. In one partial case an alternative description of the space associated with the weighted Sobolev space are given.

1 Introduction

Let $I := (a,b) \subset \mathbb{R}$. For $1 \le p < \infty$ we denote $L^p(I)$ the Lebesgue space with the norm $\|f\|_{L^p(I)} := \left(\int_I |f|^p\right)^{\frac{1}{p}}$. Let $\mathcal{V}_p(I) := \{v \in L^p_{\text{loc}}(I) : v \ge 0, \|v\|_{L^1(I)} \ne 0\}$ be the set of weight functions (weights). Denote $W^1_{1,\text{loc}}(I)$ the space of all functions $u \in L^1_{\text{loc}}(I)$, whose distributional derivatives Du belong to $L^1_{\text{loc}}(I)$. In the papers [3, 4, 5] are studied some properties of the weighted Sobolev space

$$W^1_p(I) := \{u \in W^1_{1,\text{loc}}(I) : \|u\|_{W^1_p(I)} < \infty\},$$

where

$$\|u\|_{W^1_p(I)} := \|vu\|_{L^p(I)} + \|\rho Du\|_{L^p(I)}, \quad v, \rho \in \mathcal{V}_p(I),\ \frac{1}{\rho} \in L^{p'}_{\text{loc}}(I), \tag{1.1}$$

and its subspaces

$$\overset{\circ\circ}{W}{}^1_p(I) := \{f \in AC(I) : \operatorname{supp} f \text{ compact in } I, \|vf\|_{L^p(I)} + \|\rho f'\|_{L^p(I)} < \infty\}$$

and $\overset{\circ}{W}{}^1_p(I) = \overline{\overset{\circ\circ}{W}{}^1_p(I)}^{W^1_p(I)}$ — the closure of the space $\overset{\circ\circ}{W}{}^1_p(I)$ in $W^1_p(I)$. In particular, for $1 < p < \infty$ the description of elements of spaces $\overset{\circ}{W}{}^1_p(I), W^1_p(I)$, criterion of equality $\overset{\circ}{W}{}^1_p(I) = W^1_p(I)$ and two-sided estimates on supremums

$$\sup_{f \in X} \frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}} \quad \text{and} \quad \sup_{f \in X} \frac{\int_I |fg|}{\|f\|_{W^1_p(I)}},$$

are proved, where $g \in L^1_{\text{loc}}(I)$, $X \in \{W^1_p(I), \overset{\circ}{W}{}^1_p(I), \overset{\circ\circ}{W}{}^1_p(I)\}$. Usually in theory of Sobolev spaces by $\overset{\circ}{W}{}^1_p(I)$ denote the closure of space $C_0^\infty(I)$ in $W^1_p(I)$. In Section 2 we prove, that for the weights, which satisfy the conditions (1.1), there are the equalities $\overline{\overset{\circ\circ}{W}{}^1_p(I)}^{W^1_p(I)} = \overline{C_0^\infty(I)}^{W^1_p(I)}$ and

$$\sup_{f\in\overset{\circ\circ}{W}{}^1_p(I)} \frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}} = \sup_{f\in C_0^\infty(I)} \frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}} \tag{1.2}$$

for any $g \in L^1_{\mathrm{loc}}(I)$. In Section 3 we prove a criterion of finiteness of the supremum

$$\sup_{f\in\overset{\circ}{W}{}^1_p(I)} \frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}}, \quad g \in L^1_{\mathrm{loc}}(I), \tag{1.3}$$

in case, when $\overset{\circ}{W}{}^1_p(I) \neq W^1_p(I)$ and each $u \in \overset{\circ}{W}{}^1_p(I)$ has a representative $f \in AC_{\mathrm{loc}}(I)$ with $f(a+0) = f(b-0) = 0$. The criterion gives the answer on the question: under what conditions on the function $g \in L^1_{\mathrm{loc}}(I)$ the map $f \mapsto \int_I fg$ is defined a bounded linear functional on the weighted Sobolev space $\overset{\circ}{W}{}^1_p(I)$; and complements the results of papers [3, 4, 5, 1].

2 Density of smooth functions

Theorem 2.1. *Let $I = (a,b) \subset \mathbb{R}$, $1 \le p < \infty$, $\rho, v \in \mathcal{V}_p(I)$, $\frac{1}{\rho} \in L^{p'}_{\mathrm{loc}}(I)$, $f \in \overset{\circ\circ}{W}{}^1_p(I)$ and $\operatorname{supp} f \subset (a^*,b^*) \subset\subset (a,b)$. Then for arbitrary $\varepsilon > 0$ there exists $h \in C_0^\infty(I)$ such that $\operatorname{supp} h \subset (a^*,b^*)$, $\|f-h\|_{C(I)} < \varepsilon$ and $\|f-h\|_{W^1_p(I)} < \varepsilon$.*

Proof. Since $v, \rho \in L^p_{\mathrm{loc}}(I)$ then $C_0^1(I) \subset W^1_p(I)$. Fix any $f \in \overset{\circ\circ}{W}{}^1_p(I)$. Let $\operatorname{supp} f \subset (a_0,b_0) \subset\subset (a_3,b_3) \subset\subset (a^*,b^*)$. Fix an arbitrary $\varepsilon > 0$. Let

$$0 < \varepsilon_0 < \frac{\varepsilon}{2} \min\left\{ \left(\int_{a_3}^{b_3} v^p\right)^{-\frac{1}{p}} \left(\int_{a_3}^{b_3} \frac{1}{\rho^{p'}}\right)^{-\frac{1}{p'}}, \left(\int_{a_3}^{b_3} \frac{1}{\rho^{p'}}\right)^{-\frac{1}{p'}}, 1 \right\}.$$

We take $b_2 \in (b_0,b_3)$ such that $\left(\int_{b_2}^{b_3} \rho^p\right)^{\frac{1}{p}} < \frac{\varepsilon_0}{4}$. Let $0 < \varepsilon_1 < \min\{\frac{\varepsilon_0}{12}, \frac{1}{3}(b_3-b_2)\}$. Since $\rho \in L^p([a_0,b_0])$ then there exists (see [6, Theorem 3.14]) $h_1 \in C([a_0,b_0])$ such that

$$\left(\int_{a_0}^{b_0} |f' - h_1|^p \rho^p\right)^{\frac{1}{p}} < \varepsilon_1 \min\left\{ \left(\int_{a_0}^{b_0} \frac{1}{\rho^{p'}}\right)^{-\frac{1}{p'}}, 1 \right\}.$$

Now we take $a_1 \in (a_3,a_0)$, $b_1 \in (b_0,b_2)$ such that

$$|h_1(a_0)| \max\left\{ \left[\int_{a_1}^{a_0} \rho^p\right]^{\frac{1}{p}}, (a_0-a_1) \right\} < \varepsilon_1, \quad |h_1(b_0)| \max\left\{ \left[\int_{b_0}^{b_1} \rho^p\right]^{\frac{1}{p}}, (b_1-b_0) \right\} < \varepsilon_1.$$

We extend the function h_1 on $(a,b_1]$ such that $h_1 = 0$ on $(a,a_1]$, h_1 on $[a_1,a_0]$ is the function whose graph is the segment connecting the points $(a_1,0)$ and $(a_0,h_1(a_0))$, h_1 on $[b_0,b_1]$ is the function whose graph is the segment connecting the points $(b_0,h_1(b_0))$ and $(b_1,0)$.

We have

$$\int_a^{b_1} h_1 = \int_{a_1}^{a_0} h_1 + \int_{a_0}^{b_0} h_1 + \int_{b_0}^{b_1} h_1.$$

By construction,

$$\left|\int_{a_1}^{a_0} h_1\right| \le |h_1(a_0)|(a_0-a_1) < \varepsilon_1, \quad \left|\int_{b_0}^{b_1} h_1\right| \le |h_1(b_0)|(b_1-b_0) < \varepsilon_1,$$

$$\left|\int_{a_0}^{b_0} h_1\right| = \left|\int_{a_0}^{b_0} (h_1 - f')\right| \le \left(\int_{a_0}^{b_0} |h_1 - f'|^p \rho^p\right)^{\frac{1}{p}} \left(\int_{a_0}^{b_0} \frac{1}{\rho^{p'}}\right)^{\frac{1}{p'}} < \varepsilon_1.$$

Thus $\left|\int_a^{b_1} h_1\right| < 3\varepsilon_1$. We define $h_1 = 0$ on $(b_1, b_2]$. If $\int_a^{b_1} h_1 = 0$, then we define $h_1 = 0$ on (b_2, b). Now let $\alpha := \operatorname{sign}\left(\int_a^{b_1} h_1\right) \ne 0$. We define $h_1 = 0$ on $[b_3, b)$. Since

$$0 < \left|\int_a^{b_1} h_1\right| < 3\varepsilon_1 < (b_3 - b_2),$$

then there exist $d \in (0,1)$ and $c \in (0, \frac{1}{2}(b_3-b_2))$ such that $d(b_3 - b_2 - c) = |\int_a^{b_1} h_1|$. We extend h_1 on $[b_2, b_3]$ such that its graph is the polygonal line with vertices $(b_2, 0)$, $(b_2 + c, -\alpha d)$, $(b_3 - c, -\alpha d)$, $(b_3, 0)$. In both cases $h_1 \in C_0(I)$ and $\int_I h_1 = 0$.

We put $h_0(x) := \int_a^x h_1$. Then $h_0 \in C_0^1(I)$ and $h_0'(x) = h_1(x)$, $x \in I$. Hence

$$\|(f' - h_0')\rho\|_{L^p(I)} = \left[\int_{a_1}^{a_0} |h_1|^p \rho^p + \int_{a_0}^{b_0} |f' - h_1|^p \rho^p + \int_{b_0}^{b_1} |h_1|^p \rho^p + \int_{b_2}^{b_3} |h_1|^p \rho^p\right]^{\frac{1}{p}} < \frac{\varepsilon_0}{2}.$$

Besides that,

$$\begin{aligned} \sup_{x\in I} |f(x) - h_0(x)| &= \sup_{x \in I} \left|\int_a^x f' - \int_a^x h_1\right| \le \int_{a_3}^{b_3} |f' - h_1| \\ &\le \left(\int_{a_3}^{b_3} |f' - h_1|^p \rho^p\right)^{\frac{1}{p}} \left(\int_{a_3}^{b_3} \frac{1}{\rho^{p'}}\right)^{\frac{1}{p'}} < \frac{\varepsilon_0}{2} \left(\int_{a_3}^{b_3} \frac{1}{\rho^{p'}}\right)^{\frac{1}{p'}} < \frac{\varepsilon}{4}, \end{aligned}$$

$$\|(f - h_0)v\|_{L^p(I)} \le \left(\int_{a_3}^{b_3} v^p\right)^{\frac{1}{p}} \sup_{x \in I} |f(x) - h_0(x)| < \frac{\varepsilon_0}{2} \left(\int_{a_3}^{b_3} v^p\right)^{\frac{1}{p}} \left(\int_{a_3}^{b_3} \frac{1}{\rho^{p'}}\right)^{\frac{1}{p'}} < \frac{\varepsilon}{4}.$$

Consequently, $\|f - h_0\|_{C(I)} < \frac{\varepsilon}{4}$ and $\|f - h_0\|_{W_p^1(I)} < \frac{\varepsilon}{2}$.

By g_τ we denote a mollification of g with radius τ. Since $\operatorname{supp} h_0 \subset [a_1, b_3]$, then there exists $\tau^* > 0$ such that $\operatorname{supp}\{(h_0)_\tau\} \subset (a^*, b^*)$ holds for any $\tau \in (0, \tau^*)$. Besides that, $(h_0)_\tau \in C^\infty(I)$ and $(h_0')_\tau(x) = ((h_0)_\tau)'(x)$, $x \in I$. Since the functions h_0 and h_0' are continuous and have compact supports, then (see [2, Theorem C.19 (i)])

$$\|h_0 - (h_0)_\tau\|_{C(I)} \to 0, \quad \|h_0' - ((h_0)_\tau)'\|_{C(I)} \to 0$$

as $\tau \to 0+0$. We take $\tau' \in (0, \tau^*)$ such that

$$\|h_0 - h\|_{C(I)} < \frac{\varepsilon}{4} \min\left\{\left(\int_{a^*}^{b^*} v^p\right)^{-\frac{1}{p}}, 1\right\}, \quad \|h_0' - h'\|_{C(I)} < \frac{\varepsilon}{4}\left(\int_{a^*}^{b^*} \rho^p\right)^{-\frac{1}{p}}$$

for $h := (h_0)_{\tau'}$. Therefore $\|f - h\|_{C(I)} < \varepsilon$ and $\|f - h\|_{W_p^1(I)} < \varepsilon$. □

Corollary 2.1. *Let* $I=(a,b)\subset\mathbb{R}$, $1\le p<\infty$, $\rho,v\in\mathcal{V}_p(I)$, $\frac{1}{\rho}\in L^{p'}_{\mathrm{loc}}(I)$, $g\in L^1_{\mathrm{loc}}(I)$. *Then* $\overline{\overset{\circ\circ}{W}{}^1_p(I)}^{W^1_p(I)}=\overline{C_0^\infty(I)}^{W^1_p(I)}$ *and* (1.2) *holds.*

Proof. We have $C_0^\infty(I)\subset\overset{\circ\circ}{W}{}^1_p(I)$ and, by Theorem 2.1, $\overset{\circ\circ}{W}{}^1_p(I)\subset\overline{C_0^\infty(I)}^{W^1_p(I)}$. Hence $\overline{\overset{\circ\circ}{W}{}^1_p(I)}^{W^1_p(I)}=\overline{C_0^\infty(I)}^{W^1_p(I)}$.

It is clear that right side of (1.2) is not greater than the left side of (1.2). Fix an arbitrary $f\in\overset{\circ\circ}{W}{}^1_p(I)$. By Theorem 2.1 there exist $(a^*,b^*)\subset\subset I$ and a sequence $\{h_n\}\subset C_0^\infty(I)$ such that $\operatorname{supp}h_n\subset(a^*,b^*)$, $\|f-h_n\|_{C(I)}\to 0$ and $\|f-h_n\|_{W^1_p(I)}\to 0$ as $n\to\infty$. Since $g\in L^1([a^*,b^*])$ then

$$\frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}}=\lim_{n\to\infty}\frac{\left|\int_I h_n g\right|}{\|h_n\|_{W^1_p(I)}}\le\sup_{f\in C_0^\infty(I)}\frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}}.$$

□

Corollary 2.2. *Let* $I=(a,b)\subset\mathbb{R}$, $1<p<\infty$, $\rho,v\in\mathcal{V}_p(I)$, $\frac{1}{\rho}\in L^{p'}_{\mathrm{loc}}(I)$. *Then* $C_0^\infty(I)$ *is dense in* $W^1_p(I)$ *if and only if* $\|v\|_{L^p((a,c))}\|\frac{1}{\rho}\|_{L^{p'}((a,c))}=\|v\|_{L^p((c,b))}\|\frac{1}{\rho}\|_{L^{p'}((c,b))}=\infty$, *where the point* $c\in I$ *is taken such that* $\|v\|_{L^1((a,c))}>0$ *and* $\|v\|_{L^1((c,b))}>0$.

Proof. Statement follows from [3, Lemma 1.6] and Corollary 2.1. □

3 Finiteness of the supremum

Let $I=(a,b)$, $1<p<\infty$ and weight functions be satisfy the following set of conditions

$$\rho,v\in\mathcal{V}_p(I),\ \frac{1}{\rho}\in L^{p'}_{\mathrm{loc}}(I),\ \|\tfrac{1}{\rho}\|_{L^{p'}((a,c))}\|v\|_{L^p((a,c))}<\infty,\ \|\tfrac{1}{\rho}\|_{L^{p'}((c,b))}\|v\|_{L^p((c,b))}<\infty, \tag{3.1}$$

where the point $c\in I$ is taken such that $\|v\|_{L^1((a,c))}>0$ and $\|v\|_{L^1((c,b))}>0$. By [3, Lemma 1.6], when the condition (3.1) holds $f\in\overset{\circ}{W}{}^1_p(I)$ if and only if $f\in W^1_p(I)$ and $\bar f(a+0)=\bar f(b-0)=0$, where $\bar f$ is the representative of f, which existence is proved in [5, Corollary 2.2]. By using [3, Theorem 3.1] and [5, Theorem 2.6], a criterion of finiteness of the supremum (1.3) is formulated in terms of special functions constructed with the Oinarov and Otelbaev scheme [3]. In this paper we prove a criterion that does not use Oinarov-Otelbaev functions.

We first prove a result for the vector space $\overset{\circ}{AC}_p(\rho,I):=\{f\in AC_{\mathrm{loc}}(I):f(a+0)=f(b-0)=0,\|f'\rho\|_{L^p(I)}<\infty\}$, equipped with the norm

$$\|f\|:=\|f'\rho\|_{L^p(I)}+|f(c)|,\quad f\in\overset{\circ}{AC}_p(\rho,I),$$

where $c\in I$ is a fixed point. This space was considered in the paper [1] (see, also, the references to the article).

Theorem 3.1. *Let* $I:=(a,b)\subset\mathbb{R}$, $1<p<\infty$, $g\in L^1_{\mathrm{loc}}(I)$, $\frac{1}{\rho}\in L^{p'}(I)$, $c\in I$. *Then*

$$A_0:=\sup_{f\in\overset{\circ}{AC}_p(\rho,I)}\frac{\left|\int_I fg\right|}{\|f'\rho\|_{L^p(I)}}<\infty\quad\Leftrightarrow\quad B_1<\infty,$$

where

$$B_1 := \left(\int_I \left| \int_c^x |g| \right|^{p'} |\rho(x)|^{-p'} dx \right)^{\frac{1}{p'}} .$$

In this case

$$\int_I fg = \int_I G \cdot f', \quad f \in \overset{\circ}{AC}_p(\rho, I),$$

where $G(x) := -\int_c^x g$, $x \in I$, *and*

$$A_0 \le B_2 := \inf_{\gamma \in \mathbb{R}} \left(\int_I \left| \gamma + \int_c^x g \right|^{p'} |\rho(x)|^{-p'} dx \right)^{\frac{1}{p'}} ;$$

if, besides that, $\rho \in L^p_{\mathrm{loc}}(I)$, *then* $A_0 = B_2$.

Proof. Necessity. Let $A_0 < \infty$. Then for any $f \in \overset{\circ}{AC}_p(\rho, I)$ there exist the integral $\int_I fg$, and definition of Lebesgue integral implies $\int_I |fg| < \infty$.

Fix a point $a_1 \in (a, c)$ an arbitrary Lebesgue measurable function h such that $\rho h \in L^p((c, b))$. Since $\int_c^b |\rho|^{-p'} < \infty$ then $\int_c^b |h| < \infty$. Let h_0 be such that $\rho h_0 \in L^p((a_1, c))$ and $\|\rho h_0\|_{L^p((a_1,c))} > 0$. Since $\frac{1}{\rho} \in L^{p'}_{\mathrm{loc}}(I)$ we have $\int_{a_1}^c |h_0| < \infty$. We put $h_1 := h_0 \frac{\int_c^b h}{\int_{a_1}^c h_0}$. Then $\int_{a_1}^c h_1 = \int_c^b h$. We define $f(x) := \int_a^x (h_1 \chi_{(a_1,c)} - h \chi_{(c,b)})$. Then $f \in AC(I)$, $\rho f' \in L^p(I)$ and

$$f(x) = \int_{a_1}^c h_1 - \int_c^x h = \int_x^b h$$

holds for $x \in (c, b)$. In particular, $f \in \overset{\circ}{AC}_p(\rho, I)$. Therefore

$$\int_c^b \left| \int_x^b h \right| |g(x)| \, dx < \infty$$

for any Lebesgue measurable function h with $\|\rho h\|_{L^p((c,b))} < \infty$.

By [5, Lemma 2.4] (where $X := \{f : \|\rho f\|_{L^p(I)} < \infty\}$, $Y := L^1(I)$, $(Th)(x) = g(x) \int_x^b h$) we have the inequality

$$\int_c^b \left| \int_x^b h \right| |g(x)| \, dx \le C \left(\int_c^b |h\rho|^p \right)^{\frac{1}{p}} . \tag{3.2}$$

Using the result [7, Theorem 2.4], we find that

$$B_{11} := \left(\int_c^b \left(\int_x^b |\rho|^{-p'} \right) \left(\int_c^x |g| \right)^{p'-1} |g(x)| \, dx \right)^{\frac{1}{p'}} < \infty.$$

Integrating by parts, we get the estimate

$$B_{11}^{p'} \ge \int_c^\beta \left(\int_x^b |\rho|^{-p'} \right) \left(\int_c^x |g| \right)^{p'-1} |g(x)| \, dx \ge \frac{1}{p'} \int_c^\beta \left(\int_c^x |g| \right)^{p'} |\rho(x)|^{-p'} dx$$

for any point $\beta \in (c, b)$. Analogously we prove the finiteness of

$$B_{12} := \left(\int_a^c \left(\int_a^x |\rho|^{-p'} \right) \left(\int_x^c |g| \right)^{p'-1} |g(x)| \, dx \right)^{\frac{1}{p'}}$$

and the estimate

$$B_{12}^{p'} \geq \frac{1}{p'}\int_\alpha^c \left(\int_x^c |g|\right)^{p'} |\rho(x)|^{-p'}dx, \quad \alpha \in (a,c).$$

Using the monotone convergence theorem, we obtain the estimate $p'(B_{11}^{p'} + B_{12}^{p'}) \geq B_1^{p'}$.

Sufficiency. Let $B_1 < \infty$. We have

$$0 = \lim_{\beta\to b-0}\left(\int_\beta^b \left(\int_c^x |g|\right)^{p'} |\rho(x)|^{-p'}dx\right)^{\frac{1}{p'}} \geq \lim_{\beta\to b-0}\left(\int_\beta^b |\rho|^{-p'}\right)^{\frac{1}{p'}} \int_c^\beta |g|$$

and, analogously,

$$\lim_{\alpha\to a+0}\left(\int_a^\alpha |\rho|^{-p'}\right)^{\frac{1}{p'}} \int_\alpha^c |g| = 0.$$

Define $\bar{G}(x) := -\int_c^x |g|$, $x \in (a,b)$,

$$\bar{L}(f) := \int_I \bar{G}\cdot f', \quad f \in \overset{\circ}{AC}_p(\rho, I).$$

Then $\bar{L} \in (\overset{\circ}{AC}_p(\rho, I))^*$. Integrating by parts, we get

$$\begin{aligned}\bar{L}(f) &= \int_a^c \bar{G}f' + \int_c^b \bar{G}f' = \lim_{\alpha\to a+0}\int_\alpha^c \bar{G}f' + \lim_{\beta\to b-0}\int_c^\beta \bar{G}f' \\ &= \lim_{\alpha\to a+0}\left(\bar{G}(\alpha)f(\alpha) + \int_\alpha^c f|g|\right) + \lim_{\beta\to b-0}\left(\bar{G}(\beta)f(\beta) + \int_c^\beta f|g|\right).\end{aligned}$$

Now fix an arbitrary $f \in \overset{\circ}{AC}_p(\rho, I)$. Since $f \in AC_{\mathrm{loc}}(I)$ then

$$f(x) = f(\alpha) + \int_\alpha^x f'.$$

From $f(a+0) = 0$ and from the estimate

$$\int_a^x |f'| \leq \left(\int_a^x |\rho f'|^p\right)^{\frac{1}{p}} \left(\int_a^x |\rho|^{-p'}\right)^{\frac{1}{p'}} < \infty,$$

we have $f(x) = \int_a^x f'$. Therefore

$$|\bar{G}(\alpha)f(\alpha)| = \left|\int_\alpha^c |g| \cdot \int_a^\alpha f'\right| \leq \int_\alpha^c |g| \cdot \left(\int_a^\alpha |\rho|^{-p'}\right)^{\frac{1}{p'}} \left(\int_a^\alpha |f'\rho|^p\right)^{\frac{1}{p}}$$

and

$$\limsup_{\alpha\to a+0} |\bar{G}(\alpha)f(\alpha)| \leq \lim_{\alpha\to a+0}\int_\alpha^c |g| \cdot \left(\int_a^\alpha |\rho|^{-p'}\right)^{\frac{1}{p'}} \left(\int_a^\alpha |f'\rho|^p\right)^{\frac{1}{p}} = 0.$$

Analogously, $\lim_{\beta\to b-0} |\bar{G}(\beta)f(\beta)| = 0$. Thus,

$$\bar{L}(f) = \lim_{\alpha\to a+0}\int_\alpha^c f|g| + \lim_{\beta\to b-0}\int_c^\beta f|g|.$$

Since $|f| \in \overset{\circ}{AC}_p(\rho, I)$ then there exists the integral

$$\int_I |fg| = \lim_{\alpha \to a+0} \int_\alpha^c |fg| + \lim_{\beta \to b-0} \int_c^\beta |fg| = \bar{L}(|f|).$$

Now we put $G(x) := -\int_c^x g$, $x \in (a,b)$, and

$$L(f) := \int_I G \cdot f', \quad f \in \overset{\circ}{AC}_p(\rho, I).$$

Then $L \in (\overset{\circ}{AC}_p(\rho, I))^*$. Using similar arguments, we obtain the equality

$$L(f) = \lim_{\alpha \to a+0} \int_\alpha^c fg + \lim_{\beta \to b-0} \int_c^\beta fg.$$

And the existence of the integral $\int_I |fg|$ implies

$$L(f) = \int_I fg.$$

This proves the finiteness of A_0.

Remark that

$$\int_I f' = \lim_{\beta \to b-0} \int_a^\beta f' = \lim_{\beta \to b-0} f(\beta) = 0$$

for any $f \in \overset{\circ}{AC}_p(\rho, I)$. Then for arbitrary $\gamma \in \mathbb{R}$ we have

$$A_0 = \sup_{f \in \overset{\circ}{AC}_p(\rho, I)} \frac{\left|\int_I (\gamma + G) f'\right|}{\|f'\rho\|_{L^p(I)}} \le \left(\int_I \left| \gamma + \int_c^x g \right|^{p'} |\rho(x)|^{-p'} dx \right)^{\frac{1}{p'}},$$

that is $A_0 \le B_2$.

Now let $\rho \in L^p_{\text{loc}}(I)$. Then $C_0^1(I) \subset \overset{\circ}{AC}_p(\rho, I)$. Denote

$$A_1 := \sup_{\phi \in C_0^1(I)} \frac{\left|\int_I G \phi'\right|}{\|\phi' \rho\|_{L^p(I)}}.$$

Let $A_1 < \infty$. The set $Y := \{\phi' : \phi \in C_0^1(I)\}$ is a subspace of the weighted Lebesgue space $L^p_\rho(I) := \{f : \|f\|_{L^p_\rho(I)} := \|f\rho\|_{L^p(I)} < \infty\}$. Since $A_1 < \infty$ then $\Lambda : f \mapsto \int_I Gf$ is a linear functional on Y and $|\Lambda(f)| \le A_1 \|f\|_{L^p_\rho(I)}$ for any $f \in Y$. Denote by $\tilde{\Lambda}$ the extension by Hahn-Banach theorem of the functional Λ on all $L^p_\rho(I)$. Then there exists the function $F \in L^{p'}_{\frac{1}{\rho}}(I)$ such that $\tilde{\Lambda}(f) = \int_I Ff$, $f \in L^p_\rho(I)$, and

$$A_1 = \sup_{f \in L^p_\rho(I)} \frac{\left|\int_I Ff\right|}{\|f\|_{L^p_\rho(I)}} = \left(\int_I |F|^{p'} |\rho|^{-p'} \right)^{\frac{1}{p'}}.$$

Since $\tilde{\Lambda}$ coincides with Λ on Y, then there exists the constant $\gamma \in \mathbb{R}$ such that $F = G + \gamma$ a.e. on I. Consequently,

$$A_0 \ge A_1 = \left(\int_I |G + \gamma|^{p'} |\rho|^{-p'} \right)^{\frac{1}{p'}} \ge B_2.$$

□

Now we formulate a criterion of finiteness of the supremum (1.3).

Corollary 3.1. *Let $I := (a,b) \subset \mathbb{R}$, $1 < p < \infty$, $g \in L^1_{\mathrm{loc}}(I)$, the point $c \in I$ be taken such that $\|v\|_{L^1((a,c))} > 0$ and $\|v\|_{L^1((c,b))} > 0$, weight functions ρ, v be satisfy the set of conditions* (3.1). *Then*

$$A_2 := \sup_{f \in \overset{\circ}{W}{}^1_p(I)} \frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}} < \infty \quad \Leftrightarrow \quad B_1 < \infty.$$

In this case

$$\int_I fg = \int_I G \cdot Df, \quad f \in \overset{\circ}{W}{}^1_p(I),$$

where $G(x) := -\int_c^x g$, $x \in I$. In particular, $\left(1 + \|v\|_{L_p(I)} \|\frac{1}{\rho}\|_{L_{p'}(I)}\right)^{-1} B_2 \le A_2 \le B_2$.

Proof. The set of conditions (3.1) is equivalent to the following set of conditions

$$\rho, v \in \mathcal{V}_p(I), \ \tfrac{1}{\rho} \in L^{p'}(I), \ v \in L^p(I). \tag{3.3}$$

Required only to show that (3.1) implies (3.3). Since $\rho \in L^p_{\mathrm{loc}}(I)$ then $\|\frac{1}{\rho}\|_{L^{p'}((a,c))} > 0$ and $\|\frac{1}{\rho}\|_{L^{p'}((c,b))} > 0$. Since $\|\frac{1}{\rho}\|_{L^{p'}((a,c))} \|v\|_{L^p((a,c))} < \infty$ and $\|\frac{1}{\rho}\|_{L^{p'}((c,b))} \|v\|_{L^p((c,b))} < \infty$ then $\|v\|_{L^p(I)} < \infty$. Given a choice of the point $c \in I$, we get $\frac{1}{\rho} \in L^{p'}(I)$.

Therefore

$$\frac{1}{1 + \|v\|_{L^p(I)} \|\frac{1}{\rho}\|_{L^{p'}(I)}} \sup_{f \in \overset{\circ}{AC}_p(\rho, I)} \frac{\left|\int_I fg\right|}{\|f'\rho\|_{L^p(I)}} \le \sup_{f \in \overset{\circ}{W}{}^1_p(I)} \frac{\left|\int_I fg\right|}{\|f\|_{W^1_p(I)}} \le \sup_{f \in \overset{\circ}{AC}_p(\rho, I)} \frac{\left|\int_I fg\right|}{\|f'\rho\|_{L^p(I)}}.$$

Applying Theorem 3.1, we obtain the required result. □

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