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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate "For the merits in the development of science in the Republic of Kazakhstan", in 2007 and 2014 the state grant "The best university teacher", in 2016 the Order "Kurmet" (= "Honour").

R. Oinarov was born in the village Kul'Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis "Continuity and Lipschitzness of nonlinear integral operators of Uryson's type" at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis "Weighted estimates of integral and differential operators" at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov's kernels* or *Oinarov condition.* On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user=NzXYMS4AAAAJhl=ruoi=ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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UNCONDITIONAL BASES OF SUBSPACES RELATED TO NON-SELF-ADJOINT PERTURBATIONS OF SELF-ADJOINT OPERATORS

A.K. Motovilov, A.A. Shkalikov

Communicated by V.I. Burenkov

Dedicated to the 70^{th} birthday of Professor Ryskul Oinarov

Key words: Riesz basis, unconditional basis of subspaces, non-self-adjoint perturbations.

AMS Mathematics Subject Classification: 47A55, 47A15.

Abstract. Assume that T is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and that the spectrum of T is contained in the union $\bigcup_{j\in J}\Delta_j$, $J\subseteq\mathbb{Z}$, of the segments $\Delta_j=[\alpha_j,\beta_j]\subset\mathbb{R}$ such that $\alpha_{j+1}>\beta_j$ and

$$\inf_j\,(\alpha_{j+1}-\beta_j)=d>0.$$

If B is a bounded (in general non-self-adjoint) perturbation of T with $\|B\|=:b<d/2$, then the spectrum of the perturbed operator $A=T+B$ lies in the union $\bigcup_{j\in J}U_b(\Delta_j)$ of the mutually disjoint closed b-neighborhoods $U_b(\Delta_j)$ of the segments Δ_j in $\mathbb{C}$. Let Q_j be the Riesz projection onto the invariant subspace of A corresponding to the part of the spectrum of A lying in $U_b\,(\Delta_j)$, $j\in J$. Our main result is as follows: *The subspaces $\mathcal{L}_j=Q_j(\mathcal{H})$, $j\in J$ form an unconditional basis in the whole space $\mathcal{H}$.*

1 Introduction and main result

We begin with recalling some definitions (see [3, Chapter 6.5]). A sequence of nonzero subspaces $\{\mathcal{L}_k\}$ of a Hilbert spaces $\mathcal{H}$ is said to be *a basis* if any element $x\in\mathcal{H}$ is uniquely represented by the series

$$x=\sum_k x_k,\quad \text{where}\ \ x_k\in\mathcal{L}_k, \tag{1.1}$$

that converges in the norm of $\mathcal{H}$. A basis of subspaces $\{\mathcal{L}_k\}$ is said to be *unconditional* if the series (1.1) converges to x after any rearrangement of its elements. An unconditional basis of subspaces is also called a Riesz basis of subspaces. In the case in which all the subspaces $\mathcal{L}_j$ are one-dimensional (finite-dimensional) we can choose elements $y_j\in\mathcal{L}_j=Q_j(\mathcal{H})$ (a basis $\{y_{k_j}\}$ in $\mathcal{L}_j$). Then the sequence of the subspaces $\{\mathcal{L}_k\}$ is a basis if and only if the corresponding system is a basis (a basis with parentheses) in $\mathcal{H}$.

When one deals with bases of subspaces, it is convenient to work in terms of projections. We will use the definitions and results presented in [7, § 6]. Let J be a finite or infinite ordered set of indices, $J\subset\mathbb{Z}$. A system of projections $\{Q_j\}_{j\in J}$ is said to be *complete* if the equalities

$$(Q_jx,y)=0,\quad\text{for any}\quad x\in\mathcal{H}\quad\text{and any}\quad j\in J,$$

imply $y = 0$. It is easily seen that the system $\{Q_j\}_{j\in J}$ is complete if and only if any element $x \in \mathcal{H}$ can be approximated with an arbitrary accuracy by linear combinations of elements $x_k \in \mathcal{L}_k = Q_k(\mathcal{H})$, $k \in J$.

A system of projections $\{Q_j\}_{j\in J}$ is called *minimal* if

$$Q_j Q_k = \delta_{kj} Q_j \quad \text{for any} \quad j,k \in J.$$

It follows directly from the definitions that if $\{Q_j\}_{j\in J}$ is a minimal system of projections, then the sequence of the subspaces $\mathcal{L}_j = Q_j(\mathcal{H})$ forms a basis (an unconditional basis) if and only if the series $\sum_{j\in J} Q_j$ converges (converges after any rearrangement of the indices) in the strong operator topology to the identity operator.

Further we will make use of the following results (for the corresponding proofs see [3, Chapter 6] and [7, § 6]).

Theorem A. *Let $\{Q_j\}_{j\in J}$ be a system of projections in a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

1. *A sequence of subspaces $\mathcal{L}_j = Q_j(\mathcal{H})$, $j \in J$, is an unconditional basis of the Hilbert space $\mathcal{H}$.*
2. *There exists an equivalent inner product in $\mathcal{H}$ such that a sequence of subspaces $\mathcal{L}_j = Q_j(\mathcal{H})$, $j \in J$, is complete and mutually orthogonal ($\mathcal{L}_k \perp \mathcal{L}_j$ for $k \neq j$).*
3. *There exist a bounded and boundedly invertible operator K in $\mathcal{H}$, and a complete and minimal system of orthogonal projections $\{P_j\}$ such that*
$$Q_j = K^{-1} P_j K \qquad j \in J.$$
4. *The system of projections $\{Q_j\}_{j\in J}$ is complete, minimal, and the series $\sum_{j\in J} Q_j$ converges unconditionally.*
5. *The system of projections $\{Q_j\}_{j\in J}$ is complete, minimal, and*
$$\sum_{j\in J} |(Q_j x, x)| < \infty \qquad \text{for any} \quad x \in \mathcal{H}. \tag{1.2}$$

Now we are ready to formulate the main result of the paper. In what follows it is assumed that the set of indices coincides either with $\mathbb{N}$ or with $\mathbb{Z}$.

Theorem 1. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that the spectrum of T is contained in the union $\Delta := \bigcup_{j\in J} \Delta_j$ of the segments $\Delta_j = [\alpha_j, \beta_j] \subset \mathbb{R}$ such that $\alpha_{j+1} > \beta_j$ for all $j \in J$. Assume in addition that*

$$\inf_{j\in J} (\alpha_{j+1} - \beta_j) = d > 0. \tag{1.3}$$

Let B be a bounded (generally non-self-adjoint) operator on $\mathcal{H}$ with $\|B\| =: b < d/2$. Then the spectrum of the operator $A = T + B$ lies in the union $\bigcup_{j\in J} U_b(\Delta_j)$ of the mutually disjoint closed b-neighborhoods $U_b(\Delta_j)$ of the segments Δ_j in $\mathbb{C}$. If Q_j, $j \in J$, are the Riesz projections onto the invariant subspaces $\mathcal{L}_j$ of A corresponding to the isolated components of its spectrum lying in $U_b(\Delta_j)$, then the invariant subspaces $\mathcal{L}_j$, $\mathcal{L}_j = Q_j(\mathcal{H})$, $j \in J$, form an unconditional basis in $\mathcal{H}$.

There are many papers devoted to the Riesz basis property of the root vectors of non-self-adjoint operators, which are perturbations of self-adjoint ones. The corresponding results and references can be found in the book of Markus [6] and in the paper of Shkalikov [7]. First results related to Theorem 1 were obtained by Markus [5] and Kato (see [4, Chapter 5, Theorem 4.15a]): *Let T be a self-adjoint operator with discrete spectrum on a Hilbert space $\mathcal{H}$ such that its eigenvalues $\{\lambda_j\}$ are simple and subject the condition $\lambda_{j+1} - \lambda_j \to \infty$, as $j \to \infty$. Then for any bounded (generally non-self-adjoint) operator B the operator $T + B$ has discrete spectrum and its root vectors form a Riesz basis in the space $\mathcal{H}$.*

A more general result follows from the Markus-Matsaev theorem [6, Chapter 1, Theorem 6.12]: *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$, having a finite order (i.e. its eigenvalues, counting multiplicities, are subject to the condition $|\lambda_j| \geqslant Cj^p$ with some constants $C, p > 0$), and there exist gaps in the spectrum of T with the lengths $\geqslant d$. If $\|B\| \leqslant d/2$, then the the root subspaces of the perturbed operator $T + B$ form a Riesz basis with parentheses in $\mathcal{H}$ (or a Riesz basis consisting of finite-dimensional root subspaces).*

The condition for T to be of finite order was dropped in [7]. However, the compactness of the resolvent $(T - \lambda)^{-1}$ was essentially used in the proof. To the best of our knowledge, Theorem 1 is, apparently, the first result in this topic which deals with an unperturbed operator T possibly having a non-discrete spectrum.

We have an additional motivation to prove Theorem 1. We expect that this result might help to resolve some open problems concerning bounded perturbations of self-adjoint operators in spaces with indefinite metric (see [1, 2]).

2 Proof of Theorem 1

We divide the proof into two parts that are called below Step 1, Step 2 respectively. At Step 1 we prove that the sequence of subspaces $\{\mathcal{L}_j\}_{j\in J}$ forms a basis of $\mathcal{H}$. At Step 2 we prove that it is, actually, an unconditional basis.

Step 1. For the sake of definiteness, we assume that the index set J coincides with the set of all entire numbers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. This corresponds to the most general case when infinitely many segments Δ_j lie both on $\mathbb{R}^-$ and $\mathbb{R}^+$.

Under the hypothesis $\|B\| = b$ with $0 \leqslant b < d/2$ that we assume, the closed neighborhoods $U_b(\Delta_j)$ of the segments Δ_j, $j \in J$, are disjoint and, surely, $U_b(\Delta) = \bigcup_{j\in J} U_b(\Delta_j)$. The first assertion of the theorem on the inclusion of the spectrum $\sigma(A)$ of A in the union $\bigcup_{j\in J} U_b(\Delta_j)$ is a corollary of the well-known estimate

$$\left\|(T-\lambda)^{-1}\right\| \leqslant \frac{1}{\operatorname{dist}(\lambda, \sigma(T))} \leqslant \frac{1}{\operatorname{dist}(\lambda, \Delta)}, \qquad \Delta := \bigcup_{j\in J} \Delta_j, \tag{2.1}$$

where $\sigma(T)$, $\sigma(T) \subset \Delta$, is the spectrum of the self-adjoint operator T. For any λ lying outside $U_b\,(\Delta)$ we have

$$\delta := \operatorname{dist}(\lambda, \Delta) > b. \tag{2.2}$$

Then combining (2.1) and (2.2) with the bound

$$\left\|\left(\mathbf{1} + B(T-\lambda)^{-1}\right)^{-1}\right\| \leqslant \frac{1}{1 - b/\delta}, \tag{2.3}$$

for the resolvent $(A - \lambda)^{-1} = (T + B - \lambda)^{-1}$ one finds

$$\|(A-\lambda)^{-1}\| = \left\|(T-\lambda)^{-1}\left(\mathbf{1} + B(T-\lambda)^{-1}\right)^{-1}\right\|$$

$$\leqslant \frac{1}{\delta - b} < \infty, \tag{2.4}$$

where the quantity $\delta = \delta(\lambda)$ is defined in (2.2). Hence, any $\lambda \in \mathbb{C} \setminus U_b(\Delta)$ belongs to the resolvent set of the perturbed operator $A = T + B$ and then $\sigma(A) \subset U_b(\Delta) = \bigcup_{j \in J} U_b(\Delta_j)$. Since the neighborhoods $U_b(\Delta_j)$ for different $j \in J$ are disjoint, namely,

$$\operatorname{dist}(U_b(\Delta_j), U_b(\Delta_k)) \geqslant d - 2b, \quad j \neq k,$$

the spectral sets of A confined in $U_b(\Delta_j)$ are also disjoint. The Riesz projections Q_j for these sets are well defined. In particular, given an arbitrary $b' \in (b, d/2)$, one may write Q_j in the form

$$Q_j = -\frac{1}{2\pi i} \int_{\Gamma_j} (A - \lambda)^{-1} \, d\lambda, \qquad \Gamma_j = \partial U_{b'}(\Delta_j), \tag{2.5}$$

where the integration along contour Γ_j is performed in the anti-clockwise direction.

Below we will also use the representation

$$(A - \lambda)^{-1} = (T - \lambda)^{-1} - G(\lambda), \quad \lambda \notin U_b(\Delta), \tag{2.6}$$

where

$$G(\lambda) = (A - \lambda)^{-1} B (T - \lambda)^{-1} = (T - \lambda)^{-1} M(\lambda) B (T - \lambda)^{-1} \tag{2.7}$$

and

$$M(\lambda) = \left(\mathbf{1} + B(T - \lambda)^{-1}\right)^{-1}.$$

Now denote by R_n the rectangle in $\mathbb{C}$ whose vertical sides pass through the points

$$c_{-n} = (\beta_{-n-1} + \alpha_{-n})/2 \text{ and } c_n = (\beta_n + \alpha_{n+1})/2 \tag{2.8}$$

while the horizontal sides coincide with the segments $[c_{-n} \pm i\gamma_n, c_n \pm i\gamma_n]$, where

$$\gamma_n = \max\{|c_{-n}|, |c_n|\}. \tag{2.9}$$

Clearly, the sides of R_n do not intersect the set $U_b(\Delta)$. By virtue of (2.6) one obtains

$$\sum_{j=-n}^{n} Q_j x = -\frac{1}{2\pi i} \int_{\partial R_n} (A - \lambda)^{-1} x \, d\lambda = \sum_{-n}^{n} P_j x + I_n x, \tag{2.10}$$

where I_n are the respective contour integrals of the operator-valued function $G(\lambda)$ along ∂R_n,

$$I_n := \frac{1}{2\pi i} \int_{\partial R_n} G(\lambda) \, d\lambda, \tag{2.11}$$

and P_j are the spectral projections onto the spectral subspaces of the self-adjoint operator T associated with the parts of its spectrum inside the corresponding segments Δ_j.

Given an arbitrary $x \in \mathcal{H}$ we have

$$\sum_{j=-n}^{n} P_j x \to x \quad \text{as} \quad n \to \infty. \tag{2.12}$$

Thus, in order to prove that $\sum\limits_{j=-n}^{n} Q_j x \to x$ one only needs to show that the sequence of $I_n x$ in (2.10) converges to zero as $n \to \infty$.

First, let us show that the operators I_n are uniformly bounded. We have $I_n = I_n^1 + I_n^2$, where I_n^1 and I_n^2 are the integrals along the horizontal and vertical sides of the rectangles R_n, respectively. For λ varying on the horizontal sides of the rectangle ∂R_n we have the estimate

$$\|(T-\lambda)^{-1}\| \leqslant \frac{1}{\gamma_n}, \quad \lambda = \xi \pm i\gamma_n, \quad \xi \in [c_{-n}, c_n],$$

where γ_n is defined by (2.9). Hence, by virtue of (2.4) and (2.7), it follows

$$\|G(\lambda)\| \leqslant \frac{b}{(\gamma_n - b)\gamma_n}, \quad \lambda = \xi \pm i\gamma_n, \quad \xi \in [c_{-n}, c_n]. \tag{2.13}$$

Taking into account that the lengths of the horizontal sides do not exeed $2\gamma_n$ and $\gamma_n \to \infty$ as $n \to \infty$, we get $\|I_n^1\| \to 0$ as $n \to \infty$.

Let us estimate the norms of the operators I_n^2. For the sake of definiteness consider the right vertical side of the rectangle R_n and divide it in three parts

$$\omega_n \cup \omega_n^+ \cup \omega_n^-, \quad \text{where } \omega_n = (c_n - id, c_n + id), \quad \omega^\pm = [c_n \pm id, c_n \pm i\gamma_n].$$

The lengths of the intervals ω_n equal $2d$ and due to (2.4) and (2.7)

$$\left\| \int_{\omega_n} G(\lambda)\, d\lambda \right\| \leqslant \frac{2bd}{(d/2 - b)(d/2)}. \tag{2.14}$$

For $\lambda \in \omega_n^\pm$ we have

$$\delta(\lambda) - b \geqslant |\operatorname{Im} \lambda| - b. \tag{2.15}$$

Thus, again applying (2.4) and (2.7), one obtains

$$\left\| \int_{\omega_n^\pm} G(\lambda)\, d\lambda \right\| \leqslant \int_d^{\gamma_n} \frac{b}{\tau(\tau - b)}\, d\tau \leqslant \int_d^{\gamma_n} \frac{b}{(\tau - b)^2}\, d\tau \leqslant \frac{b}{d - b} < 1. \tag{2.16}$$

Therefore,

$$\|I_n\| \leqslant C, \tag{2.17}$$

where the constant C depends only on d and b.

Let us show that $\|I_n x\| \to 0$ as $n \to \infty$ for any fixed $x \in \mathcal{H}$. To this end, choose some $\varepsilon > 0$ and, first, find $N \in \mathbb{N}$ such that

$$\|x - x_N\| < \frac{\varepsilon}{2C}, \tag{2.18}$$

where

$$x_N = \sum_{j=-N}^{N} P_j x.$$

Obviously, from (2.17) and (2.18) it follows

$$\|I_n(x - x_N)\| < \frac{\varepsilon}{2} \quad \text{for any } n \in \mathbb{N}. \tag{2.19}$$

Let us estimate $\|I_n x_N\|$ as $n \to \infty$. Denote $e(t) := (E(t)x, x)$, where $E(t)$ is the spectral function of the self-adjoint operator T, and observe that, by the spectral theorem,

$$\left\|(T-\lambda)^{-1} x_N\right\|^2 = \int_{c_{-N}}^{c_N} \frac{de(t)}{(t-\xi)^2 + \tau^2}, \quad \lambda = \xi + i\tau \notin \sigma(T). \tag{2.20}$$

For N fixed the equality (2.20) implies

$$\left\|(T-\lambda)^{-1}x_N\right\| = O\left(|\lambda|^{-1}\right) \quad \text{as } |\lambda|\to\infty. \tag{2.21}$$

Now we can modify estimate (2.14) and get from (2.4) and (2.21)

$$\left\|\int_{\omega_n} G(\lambda)x_N\,d\lambda\right\| = O\left(|c_n|^{-1}\right) \to 0 \quad \text{as } n\to\infty.$$

Analogously, by taking into account (2.15), we can modify the estimate (2.16) and obtain

$$\left\|\int_{\omega_n^\pm} G(\lambda)x_N\,d\lambda\right\| \leqslant \int_d^{\gamma_n} \frac{d\tau}{(\tau-b)\sqrt{(c_n-c_N)^2+\tau^2}} = O\left(\frac{\ln c_n}{c_n}\right) = o(1) \quad \text{as } n\to\infty.$$

The last two estimates together give

$$I_n x_N \to 0 \quad \text{as } n\to\infty.$$

Hence, there is $N_1\in\mathbb{N}$ ($N_1\geqslant N$) such that $\|I_n x_N\|<\varepsilon/2$ whenever $n>N_1$. Together with (2.19) this yields $\|I_n x\|<\varepsilon$ for $n>N_1$. Therefore, we have proven that the sequence of I_n strongly converges to zero as $n\to\infty$ and then from (2.10) and (2.12) it follows that for any $x\in\mathcal{H}$ the two-sided series $\sum_{j=-n}^{n} Q_j x$ converges to x. Thus, the system of subspaces $\mathcal{L}_j = Q_j(\mathcal{H})$, $j\in\mathbb{Z}$ is complete. The fact that these subspaces are linearly independent follows from the mutual orthogonality of the Riez projections (2.5) in the sense that $Q_jQ_k=\delta_{jk}Q_j$ for any $j,k\in\mathbb{Z}$ (see, e.g. [3, Chapter 1, § 1.3]). Thus, the system $\{\mathcal{L}_j\}_{j=-\infty}^{\infty}$ represents a basis of subspaces in $\mathcal{H}$.

Step 2. By Theorem A, in order to prove that the above basis of subspaces $\{\mathcal{L}_j\}_{j=-\infty}^{\infty}$ is unconditional, it suffices to show that the series $\sum_{j=-\infty}^{\infty}|(Q_jx,x)|$ converges for any $x\in\mathcal{H}$. First, let us transform the contours Γ_j in integrals (2.5) into the contours $\widetilde{\Gamma}_j$, which surround the rectangles with the vertical sides $\lambda=c_{j-1}+i\tau$ and $\lambda=c_j+i\tau$ where τ is varying in $[-d,d]$, and the horizontal sides $\xi\pm id$, $\xi\in[c_{j-1},c_j]$. As previously, the numbers c_j are defined by (2.8) and coincide with the centers of the gaps between the neighboring intervals Δ_j and Δ_{j+1}.

Taking into account representation (2.6), we immediately conclude that

$$2\pi\sum_{j=-\infty}^{\infty}|(Q_jx,x)| \leqslant \sum_{j=-\infty}^{\infty}\left|\int_{\widetilde{\Gamma}_j}\left((T-\lambda)^{-1}x,x\right)\,d\lambda\right| + \sum_{j=-\infty}^{\infty}\left|\int_{\widetilde{\Gamma}_j}(G(\lambda)x,x)\,d\lambda\right|.$$

The first series converges since it simply coincides with the series $\sum_{j\in\mathbb{Z}}(P_jx,x)=\|x\|^2$; we recall that P_j are the spectral projections of the self-adjoint operator T associated with the segments Δ_j. By virtue of (2.7), the second series can be estimated as follows:

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\left|\int_{\widetilde{\Gamma}_j}(G(\lambda)x,x)\,d\lambda\right| &\leqslant C_1\sum_{j=-\infty}^{\infty}\int_{\widetilde{\Gamma}_j}\left\|(T-\lambda)^{-1}x\right\|^2|d\lambda| \\ &\leqslant C_1\left(\int_{\Gamma_+\cup\Gamma_-}\left\|(T-\lambda)^{-1}x\right\|^2|d\lambda| + 2\sum_{j\in\mathbb{Z}}\int_{\omega_j}\left\|(T-\lambda)^{-1}x\right\|^2|d\lambda|\right),\end{aligned} \tag{2.22}$$

where $\Gamma_\pm$ are the lines $\lambda=\xi\pm id$, $\xi\in\mathbb{R}$, ω_j are the vertical segments $\lambda=c_j+i\tau$, $-d\leqslant\tau\leqslant d$, and $C_1=const$. Convergence of the integrals and the series in (2.22) can be proven by using the spectral theorem. As before, denote $e(t) := (E(t)x, x)$, where $E(t)$ stands for the spectral function of T. Then

$$\begin{aligned}\int_{\Gamma_\pm} \left\|(T-\lambda)^{-1}x\right\|^2 |d\lambda| &= \int_{\Gamma_\pm} |d\lambda| \int_{\mathbb{R}} \frac{de(t)}{|t-\lambda|^2} \\ &= \int_{\mathbb{R}} de(t) \int_{\mathbb{R}} \frac{d\xi}{|t-\xi|^2+d^2} \\ &= \frac{\pi}{d}\int_{\mathbb{R}} de(t) \\ &= \frac{\pi}{d}\|x\|^2. \end{aligned} \tag{2.23}$$

Further, notice that for c_j given by (2.8), the lower bound (1.3) yields

$$|t-c_j| \geqslant \begin{cases} d/2 + d(j-k) & \text{for all} \quad t\in\Delta_k, \ \ k\leqslant j, \\ d/2+d(k-j-1) & \text{for all} \quad t\in\Delta_k, \ \ k\geqslant j+1. \end{cases}$$

Hence, for $\lambda = c_j + i\tau$, $\tau\in[-d,d]$, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\int_{\gamma_\pm} \left\|(T-\lambda)^{-1}x\right\|^2 |d\lambda| &= \sum_{j\in\mathbb{Z}}\int_{-d}^{d} d\tau \int_{\mathbb{R}} \frac{de(t)}{|t-c_j-i\tau|^2} \\ &= \sum_{j\in\mathbb{Z}}\int_{-d}^{d} d\tau \sum_{k\in\mathbb{Z}}\int_{\Delta_k} \frac{de(t)}{|t-c_j-i\tau|^2} \\ &\leqslant 2d\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_{\Delta_k}\frac{de(t)}{|t-c_j|^2} \\ &\leqslant \sum_{k\in\mathbb{Z}}\|P_kx\|^2\left(2\left(\frac{2}{d}\right)^2 + \sum_{j\in\mathbb{Z},\ j\neq k}\frac{1}{d^2|k-j|^2}\right) \\ &\leqslant \frac{2C_2}{d^2}\sum_{k\in\mathbb{Z}}\|P_kx\|^2 = \frac{2C_2}{d^2}\|x\|^2, \end{aligned} \tag{2.24}$$

where $C_2 = 4 + \sum\limits_{j=1}^{\infty}\frac{1}{j^2} = 4 + \frac{\pi^2}{6}$. The estimate (2.22) together with (2.23) and (2.24) entails bound (1.2). This completes the proof of Theorem 1. □

The following statement is a simple corollary of Theorem 1 (combined with Theorem A).

Corollary. *Assume the hypothesis of Theorem 1. Then there exists an inner product* $\langle\ ,\ \rangle$ *in* $\mathcal{H}$ *with the following properties.*

1. *The product* $\langle\ ,\ \rangle$ *is norm-equivalent to the original inner product* $(\ ,\ )$ *in* $\mathcal{H}$.
2. *The subspaces* $\mathcal{L}_j = Q_j\mathcal{H}$ *are mutually orthogonal with respect to the inner product* $\langle\ ,\ \rangle$ *and, with respect to* $\langle\ ,\ \rangle$, *the Hilbert space* $\mathcal{H}$ *admits the orthogonal decomposition*

$$\mathcal{H} = \bigoplus_{j\in J}\mathcal{L}_j. \tag{2.25}$$

(3) *The subspaces $\mathcal{L}_j$, $j \in J$, are reducing for the perturbed operator $A = T + B$ and, with respect to decomposition* (2.25), *this operator admits a block diagonal matrix representation*

$$A = \mathrm{diag}(\ldots, A_{-2}A_{-1}, A_0, A_1, A_2, \ldots),$$

where $A_j = A\big|_{\mathcal{L}_j}$, $j \in J$, denotes the part of A in the reducing subspace $\mathcal{L}_j$.

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