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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate “For the merits in the development of science in the Republic of Kazakhstan”, in 2007 and 2014 the state grant “The best university teacher”, in 2016 the Order “Kurmet” (= “Honour”).

R. Oinarov was born in the village Kul’Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis “Continuity and Lipschitzness of nonlinear integral operators of Uryson’s type” at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis “Weighted estimates of integral and differential operators” at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov’s kernels* or *Oinarov condition.* On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user=NzXYMS4AAAAJhl=ruoi=ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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TIME DEPENDENT BOUNDARY NORMS FOR KERNELS AND REGULARIZING PROPERTIES OF THE DOUBLE LAYER HEAT POTENTIAL

M. Lanza de Cristoforis, P. Luzzini

Communicated by T.V. Tararykova

Dedicated to the 70^{th} *birthday of Professor Ryskul Oinarov*

Key words: integral operators on Lipschitz parabolic cylinders, double layer heat potential.

2000 AMS Mathematics Subject Classification: 31B10.

Abstract. We introduce a class of norms for time dependent kernels on the boundary of Lipschitz parabolic cylinders and we prove theorems of joint continuity of integral operators upon variation of both the kernel and the density function. As an application, we prove that the integral operator associated to the double layer heat potential has a regularizing property on the boundary.

1 Introduction

This paper is mainly devoted to continuity and regularizing properties of boundary integral operators defined on the boundary of parabolic cylinders upon variation of both the kernel and the density (or moment) function and to their applications to integral operators, the double layer heat potential in particular. Throughout the paper, we assume that

$$n \in \mathbb{N} \setminus \{0, 1\},$$

where $\mathbb{N}$ denotes the set of all natural numbers including 0. Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. The analysis of the properties of the integral operator associated to the double layer heat potential is a classical topic. Indeed, the double layer heat potential has been systematically exploited in the analysis of boundary value problems for the heat equation.

A first systematic treatment of the properties of heat layer potentials can be found in the works of Gevrey [5], [6], where the author has studied the properties of heat potentials in the case $n = 1$.

Then Van Tun [20], [21], [22] has developed in a series of papers the work of Gevrey and has obtained some results on the Schauder regularity of heat potentials. In particular, Van Tun has proved that the integral operator associated to the double layer heat potential defined on the boundary of a parabolic cylinder improves by 1/2 the Hölder exponent of the density.

In the case $m \in \mathbb{N}$ and Ω is of class $C^{m,\alpha}$, it has long been known that if the density μ is of class $C^{(m+\alpha)/2;m+\alpha}(\overline{]-\infty,T[} \times \partial\Omega)$, then the restriction of the double layer potential to the set $\overline{]-\infty, T[} \times \Omega$ can be extended to a function of $C^{(m+\alpha)/2;m+\alpha}(\overline{]-\infty, T[} \times \mathrm{cl}\Omega)$ (cf. *e.g.* Ladyzhenskaja, Solonnikov and Ural'ceva [15].)

In the case $m \in \mathbb{N}$ and Ω is of class $C^{m+2,\alpha}$, Kamynin [9], [10], [11], [12] has proved that the integral operator associated to the double layer heat potential is bounded from the Schauder space $C^{(m+\alpha)/2;m+\alpha}([0,T] \times \partial\Omega)$ to $C^{(m+1+\alpha')/2;m+1+\alpha'}([0,T] \times \partial\Omega)$ for $\alpha' \in]0, \alpha[$, $T < +\infty$.

Then Costabel [3] has considered the case of anisotropic Sobolev spaces and has proved some mapping property of heat potentials in Sobolev spaces on Lipschitz domains. We also mention the works of Lewis and Murray [16] and Hofmann and Lewis [8] for time dependent Lipschitz domains.

Our interest is two-fold. On one hand we want to prove that the integral operator associated to the double layer heat potential improves the regularity of the density in Schauder spaces and thus extend the above mentioned work of Kamynin, and on the other hand we are interested in the dependence of an integral operator upon variation of the density and of the kernel. We plan to apply both types of results in the analysis of singularly perturbed boundary value problems.

In this paper we plan to consider the cases in which Ω is a bounded open Lipschitz subset of $\mathbb{R}^n$ and a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then in a forthcoming paper, we plan to exploit the results of the present paper and to prove a formula for the tangential derivatives of the double layer heat potential and correspoinding regularizing properties of the integral operator associated to the double layer heat potential in spaces of fuction with high order derivatives in Hölder spaces in a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ with $m \geqslant 1$.

Thus we plan to prove in a parabolic setting, the corresponding results of [2] for integral operators defined on the boundary of Ω and for layer potentials corresponding to the fundamental solution of an arbitrary second order elliptic operator with constant coefficients. For references to previous contributions on the double layer potential for second order elliptic operators, we refer to [2].

In Sections 2 and 3, we introduse some notation and preliminaries. In Section 4 we collect some inequalities for the kernel of the double layer heat potential in the case Ω is of class $C^{1,\alpha}$. Then in Section 6 we introduce a class of function spaces and norms for kernels of integral operators defined on the boundary of parabolic cylinders, and we verify that the kernel associated to the fundamental solution of the heat equation and its first order derivatives and the kernel of the double layer heat potential belong to such classes.

In Section 7, we estimate the norm of an integral operator with kernel K applied to a density μ in terms of the norm of K in the above classes and of the L^∞-norm of μ. In Section 8, we apply the results of Section 7 to the double layer heat potential and deduce corresponding inequalities.

In Section 9, we estimate the norm of an integral operator with kernel K applied to a density μ in terms of the norm of K in the above classes and of the Hölder norm of μ. In Section 10, we apply the results of Section 9 to the double layer heat potential and deduce corresponding inequalities.

The authors believe that the introduction of the above mentioned norms could be applied to simplify some of the classical proofs of the mapping properties of layer heat potentials.

2 Notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Deimling [1]. Let $s \in \mathbb{N} \setminus \{0\}$. Let $\mathbb{D} \subseteq \mathbb{R}^s$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, and $\mathrm{diam}(\mathbb{D})$ denotes the diameter of $\mathbb{D}$. The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^s$ or in $\mathbb{C}$. For all $R \in]0,+\infty[$, $x \in \mathbb{R}^s$, x_j denotes the j-th coordinate of x for all $j \in \{1,\dots,s\}$, and $\mathbb{B}_s(x,R)$ denotes the ball $\{y \in \mathbb{R}^s : |x-y| < R\}$. $B(\mathbb{D},\mathcal{X})$ and $C^0(\mathbb{D},\mathcal{X})$ denote the space of bounded and continuous functions from $\mathbb{D}$ to $\mathcal{X}$, respectively. We endow $B(\mathbb{D},\mathcal{X})$ with the sup-norm and we set $C^0_b(\mathbb{D},\mathcal{X}) \equiv C^0(\mathbb{D},\mathcal{X}) \cap B(\mathbb{D},\mathcal{X})$. Let Ω be an open subset of $\mathbb{R}^s$. The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{C})$, or more simply by $C^m(\Omega)$. Let $f \in C^m(\Omega)$. Then Df denotes the Jacobian matrix of f. Let $\eta \equiv (\eta_1,\dots,\eta_s) \in \mathbb{N}^s$, $|\eta| \equiv \eta_1 + \cdots + \eta_s$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_s^{\eta_s}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leqslant m$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$.

The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose derivatives up to order m are bounded is denoted $C^m_b(\mathrm{cl}\,\Omega)$. Then $C^m_b(\mathrm{cl}\,\Omega)$ endowed with the norm $\|f\|_{C^m_b(\mathrm{cl}\,\Omega)} \equiv \sum_{|\eta|\leqslant m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. If Ω is bounded, then $C^m_b(\mathrm{cl}\,\Omega) = C^m(\mathrm{cl}\,\Omega)$. Now let ω be a function of $]0,+\infty[$ to itself such that

$$\omega \text{ is increasing and } \lim_{r\to 0^+} \omega(r) = 0\,. \tag{2.1}$$

If f is a function from a subset $\mathbb{D}$ of $\mathbb{R}^s$ to a normed space $\mathcal{X}$, we set

$$|f : \mathbb{D}|_{\omega(\cdot)} \equiv \sup\left\{ \frac{\|f(x)-f(y)\|_{\mathcal{X}}}{\omega(|x-y|)} : x,y \in \mathbb{D}, x \neq y \right\}.$$

If $|f : \mathbb{D}|_{\omega(\cdot)} < \infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f : \mathbb{D}|_{\omega(\cdot)}$. The subspace of $C^0(\mathbb{D},\mathcal{X})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted $C^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X})$. The space $C^{0,\omega(\cdot)}_b(\mathbb{D},\mathcal{X}) \equiv C^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X}) \cap B(\mathbb{D},\mathcal{X})$ endowed with the norm $\|f\|_{C^{0,\omega(\cdot)}_b(\mathbb{D},\mathcal{X})} \equiv \sup_{\mathbb{D}} \|f\|_{\mathcal{X}} + |f : \mathbb{D}|_{\omega(\cdot)}$ is a Banach space. If $\mathcal{X} = \mathbb{C}$, we simply write $C^0(\mathbb{D})$, $C^{0,\omega(\cdot)}(\mathbb{D})$, $C^{0,\omega(\cdot)}_b(\mathbb{D})$ instead of $C^0(\mathbb{D},\mathbb{C})$, $C^{0,\omega(\cdot)}(\mathbb{D},\mathbb{C})$, $C^{0,\omega(\cdot)}_b(\mathbb{D},\mathbb{C})$, respectively.

Particularly important is the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha \in]0,1]$. In this case, we simply write $C^{0,\alpha}(\mathbb{D})$, $C^{0,\alpha}_b(\mathbb{D})$, $|\cdot : \mathbb{D}|_\alpha$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C^{0,r^\alpha}_b(\mathbb{D})$, $|\cdot : \mathbb{D}|_{r^\alpha}$, respectively.

Remark 1. Let $s \in \mathbb{N} \setminus \{0\}$. Let ω be as in (2.1). Let $\mathbb{D}$ be a subset of $\mathbb{R}^s$. Let $\mathcal{X}$ be a normed space. Let $f \in C^0_b(\mathbb{D},\mathcal{X})$, $a \in]0,+\infty[$. Then,

$$\sup_{x,y\in\mathbb{D},\ |x-y|\geqslant a} \frac{\|f(x)-f(y)\|_{\mathcal{X}}}{\omega(|x-y|)} \leqslant \frac{2}{\omega(a)} \sup_{\mathbb{D}} \|f\|_{\mathcal{X}}\,.$$

Thus the difficulty of estimating the Hölder quotient $\frac{\|f(x)-f(y)\|_{\mathcal{X}}}{\omega(|x-y|)}$ of a bounded function f lies entirely in case $0 < |x-y| < a$. Then we have the following.

Lemma 2.1. *Let $s \in \mathbb{N} \setminus \{0\}$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^s$. Let ψ_1, ψ_2, ψ_3 be as in (2.1). Let conditions $\sup_{j=1,2} \sup_{r\in]0,1[} \psi_j(r)\psi_3^{-1}(r) < \infty$ hold. Then the pointwise product is bilinear and continuous from $C^{0,\psi_1(\cdot)}_b(\mathbb{D}) \times C^{0,\psi_2(\cdot)}_b(\mathbb{D})$ to $C^{0,\psi_3(\cdot)}_b(\mathbb{D})$.*

For the definition of open subsets of $\mathbb{R}^n$ of class C^1 or $C^{1,\alpha}$ for some $\alpha \in]0,1]$, we refer to Gilbarg and Trudinger [7]. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. We denote by $\nu \equiv (\nu_l)_{l=1,\dots,n}$ the external unit normal to $\partial\Omega$.

Next we introduce the Hölder spaces on cylindrical domains. If $T \in]-\infty,+\infty]$ and if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then we set

$$\mathbb{D}_T \equiv \overline{]-\infty,T[} \times \mathbb{D}\,, \qquad \partial_T\mathbb{D} \equiv (\partial\mathbb{D})_T = \overline{]-\infty,T[} \times \partial\mathbb{D}\,.$$

Clearly $\overline{]-\infty,T[} =]-\infty,T]$ if $T \in \mathbb{R}$ and $\overline{]-\infty,T[} =]-\infty,+\infty[$ if $T=+\infty$. We also note that

$$(\mathrm{cl}\,\mathbb{D})_T = \mathrm{cl}\,\mathbb{D}_T\,.$$

Remark 2. As is well known, the map Ξ from the vector space $\mathbb{C}^{\mathbb{D}_T}$ of functions from $\mathbb{D}_T$ to $\mathbb{C}$ to the vector space $(\mathbb{C}^{\mathbb{D}})^{\overline{]-\infty,T[}}$ of functions from $\overline{]-\infty,T[}$ to $\mathbb{C}^{\mathbb{D}}$, which takes a function f to the function Ξf from $\overline{]-\infty,T[}$ to $\mathbb{C}^{\mathbb{D}}$ which takes t to $f(t,\cdot)$ is an isomorphism. As a rule, we omit to write the canonical identification map Ξ.

Then we have the following.

Definition 1. Let α', $\alpha'' \in]0,1]$, $T \in]-\infty,+\infty]$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$. Then $C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)$ denotes the space of bounded functions u from $\mathbb{D}_T$ to $\mathbb{C}$ such that

$$\|u\|_{C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)} \equiv \sup_{\mathbb{D}_T}|u| + \sup_{t_1,t_2\in\overline{]-\infty,T[},t_1\neq t_2} \frac{\|u(t_1,\cdot)-u(t_2,\cdot)\|_{C^0_b(\mathbb{D})}}{|t_1-t_2|^{\alpha'}} + \sup_{t\in\overline{]-\infty,T[}} |u(t,\cdot) : \mathbb{D}|_{\alpha''} < +\infty\,.$$

It is well known that $(C^{0,\alpha';0,\alpha''}(\mathbb{D}_T), \|\cdot\|_{C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)})$ is a Banach space. By Remark 2, $u \in C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)$ if and only if the canonically identified map Ξu belongs to $C^{0,\alpha'}(\overline{]-\infty,T[}, C^0_b(\mathbb{D})) \cap B(\overline{]-\infty,T[}, C^{0,\alpha''}_b(\mathbb{D}))$. Then we have the following.

Definition 2. Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$. Then $C^{\alpha/2;\alpha}(\mathbb{D}_T)$ denotes the space of bounded functions u from $\mathbb{D}_T$ to $\mathbb{C}$ such that

$$\|u\|_{C^{\alpha/2;\alpha}(\mathbb{D}_T)} \equiv \sup_{(t,x)\in\mathbb{D}_T}|u(t,x)| + \sup_{(t,x),(\tau,y)\in\mathbb{D}_T,(t,x)\neq(\tau,y)} \frac{|u(t,x)-u(\tau,y)|}{(|t-\tau|^{1/2}+|x-y|)^{\alpha}} < +\infty\,.$$

It is well known that $(C^{\alpha/2;\alpha}(\mathbb{D}_T), \|\cdot\|_{C^{\alpha/2;\alpha}(\mathbb{D}_T)})$ is a Banach space. Then we have the following (for a proof cf. *e.g.*, Krylov [14, p. 120].)

Proposition 2.1. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let $\mathbb{D}$ is a subset of $\mathbb{R}^n$. Then $C^{\alpha/2;\alpha}(\mathbb{D}_T)$ coincides with $C^{0,\alpha/2;0,\alpha}(\mathbb{D}_T)$ both algebraically and topologically.*

3 Preliminary inequalities

We start with the following elementary lemma, which collects either known inequalities or variants of known inequalities, which we need in the sequel.

Lemma 3.1. *The following statements hold.*

(i)

$$\frac{1}{2}|x'-y| \leqslant |x''-y| \leqslant 2|x'-y| \qquad \forall y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x'-x''|)\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$.

(ii) Let $h \in]-\infty, 0]$. *Then*

$$\left| e^{h|x'-y|^2} - e^{h|x''-y|^2} \right| \leqslant 2|h| \rho_{2,y}(x',x'') e^{h\rho^2_{1,y}(x',x'')} |x'-x''|\,,$$

for all x', x'', $y \in \mathbb{R}^n$, *where*

$$\rho_{1,y}(x',x'') \equiv \min\{|x'-y|, |x''-y|\}\,, \quad \rho_{2,y}(x',x'') \equiv \max\{|x'-y|, |x''-y|\}\,.$$

(iii)

$$\frac{1}{2}|x'-y| \leqslant \rho_{1,y}(x',x'') \leqslant \rho_{2,y}(x',x'') \leqslant 2|x'-y| \qquad \forall y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x'-x''|)\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$.

Proof. Statement (i) follows by the triangular inequality. Statement (ii) follows by applying the mean value inequalty to the function e^{hs^2} of $s \in [0, +\infty[$. Statement (iii) is an immediate consequence of statement (i). □

Then we have the following well-known statement.

Lemma 3.2. *Let* $\alpha \in]0,1]$. *Let* Ω *be a bounded open connected subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Then there exists* $c_{\Omega,\alpha} > 0$ *such that*

$$|\nu(y)^t(x-y)| \leqslant c_{\Omega,\alpha}|x-y|^{1+\alpha} \qquad \forall x, y \in \partial\Omega\,.$$

Next we introduce a list of classical inequalities which can be verified by exploiting the local parametrizations of $\partial\Omega$.

Lemma 3.3. *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$. *Then the following statements hold.*

(i) Let $\gamma \in]-\infty, n-1[$. *Then*

$$c'_{\Omega,\gamma} \equiv \sup_{x \in \partial\Omega} \int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^\gamma} < +\infty\,.$$

(ii) Let $\gamma \in]-\infty, n-1[$. *Then*

$$c''_{\Omega,\gamma} \equiv \sup_{x',x'' \in \partial\Omega,\ x' \neq x''} |x'-x''|^{-(n-1)+\gamma} \int_{\mathbb{B}_n(x',3|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|x'-y|^\gamma} < +\infty\,.$$

(iii) Let $\gamma \in]n-1, +\infty[$. *Then*

$$c'''_{\Omega,\gamma} \equiv \sup_{x',x'' \in \partial\Omega,\ x' \neq x''} |x'-x''|^{-(n-1)+\gamma} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x'-y|^\gamma} < +\infty\,.$$

(iv)

$$c_{\Omega}^{iv} \equiv \sup_{x',x''\in\partial\Omega,\ 0<|x'-x''|<1/e} |\ln|x'-x''||^{-1} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x'-y|^{n-1}} < +\infty\,.$$

Next we have the following technical elementary lemma, which collects either known inequalities or variants of known inequalities, which we need in the sequel.

Lemma 3.4.

(i) $e^{-v} \leqslant \delta^{\delta} e^{-\delta} v^{-\delta}$ *for all* $v, \delta \in]0,+\infty[$.

(ii) Let $s \in]1,+\infty[$. *Let* F_s *be the function from* $]0,+\infty[$ *to itself defined by*

$$F_s(\xi) \equiv \int_{\xi}^{+\infty} e^{-1/u} u^{-s}\, du \qquad \forall \xi \in]0,+\infty[\,.$$

If $\gamma \in]0, s-1]$, *then*

$$D_{s,\gamma} \equiv \sup_{\xi\in]0,+\infty[} \xi^{\gamma} F_s(\xi) < +\infty\,.$$

(iii) Let $s \in]1,+\infty[$. *Let* $\tilde{F}_s$ *be the function from* $]0,+\infty[$ *to itself defined by*

$$\tilde{F}_s(\xi) \equiv \int_{0}^{\xi} e^{-1/u} u^{-s}\, du \qquad \forall \xi \in]0,+\infty[\,.$$

If $\gamma \in [0,+\infty[$, *then*

$$\tilde{D}_{s,\gamma} \equiv \sup_{\xi\in]0,+\infty[} \xi^{-\gamma} \tilde{F}_s(\xi) < +\infty\,.$$

(iv) Let $s \in]1,+\infty[$. *Then* $M_s \equiv \int_0^{+\infty} e^{-1/u} u^{-s}\, du < +\infty$.

(v) Let $b_1 \in]0,+\infty[$, $b_2 \in]b_1,+\infty[$, $m \in \mathbb{N}\setminus\{0\}$. *Then*

$$C(b_1,b_2,m) \equiv \sup_{\eta\in]0,+\infty[} e^{\frac{\eta}{b_2}-\frac{\eta}{b_1}} \sum_{j=0}^{m} \eta^j < +\infty\,.$$

Proof. For statement (i), we refer for example to Kress [13, (9.17)]. Next we prove (ii). Since $s > 1$, the function $e^{-1/u}u^{-s}$ is integrable in $]0,+\infty[$. Then assumption $\gamma > 0$ implies that $\lim_{\xi\to 0} \xi^{\gamma} F_s(\xi) = 0$, and assumption $\gamma \leqslant s-1$ and de l'Hôpital rule imply that $\lim_{\xi\to+\infty} \xi^{\gamma} F_s(\xi) \in \mathbb{R}$. Hence, statement (ii) holds true. By de l'Hôpital rule, we have $\lim_{\xi\to 0} \xi^{-\gamma}\tilde{F}_s(\xi) = 0$, $\lim_{\xi\to+\infty} \xi^{-\gamma}\tilde{F}_s(\xi) \in \mathbb{R}$ and thus statement (iii) follows. Statement (iv) is well known.

In order to prove statement (v) it suffices to note that the argument of the supremum is continuous in η and has limiting values 1 and 0 at 0 and $+\infty$, respectively. □

Also, we denote by Γ the Euler Γ-function (cf. *e.g.*, Folland [4, p. 58].) Finally, we point out the following immediate consequence of the triangular inequality

$$\frac{1}{2}|t'-\tau| \leqslant |t''-\tau| \leqslant 2|t'-\tau| \qquad \forall \tau \in \mathbb{R}\setminus]t'-2|t'-t''|, t'+2|t'-t''|[\,, \tag{3.1}$$

for all t', $t'' \in \mathbb{R}$, $t' \neq t''$.

4 Preliminary inequalities for the fundamental solution of the heat operator

The function Φ_n from $\mathbb{R}^{n+1} \setminus \{(0,0)\}$ to $\mathbb{R}$ defined by

$$\Phi_n(t,x) \equiv \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times\mathbb{R}^n\,, \\ 0 & \text{if } (t,x) \in (]-\infty,0]\times\mathbb{R}^n)\setminus\{(0,0)\}\,, \end{cases}$$

is well known to be the fundamental solution for the heat operator $\partial_t - \Delta$ in $\mathbb{R}^{n+1}$. Then we have the following elementary inequalities for Φ_n.

Lemma 4.1. *Let $T \in]-\infty,+\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$. Then the following statements hold.*

(i)

$$C_{0,G} \equiv \sup_{(t,x)\in]0,+\infty[\times\mathbb{R}^n,\,|x|\leqslant \mathrm{diam}\,(G)} |\Phi_n(t,x)| t^{n/2} e^{\frac{|x|^2}{4t}} < +\infty\,.$$

(ii)

$$\tilde{C}_{0,G} \equiv \sup\Big\{ |\Phi_n(t-\tau,x'-y) - \Phi_n(t-\tau,x''-y)| \frac{|t-\tau|^{(n/2)+1}}{|x'-y|\,|x'-x''|} e^{\frac{|x'-y|^2}{16(t-\tau)}} : x',x''\in G, x'\neq x'', y\in G\setminus\mathbb{B}_n(x',2|x'-x''|)\,, t,\tau\in\overline{]-\infty,T[},\ \tau<t \Big\} < +\infty\,.$$

(iii) Let $a \in]8,+\infty[$. Then

$$\tilde{C}'_{0,a,G} \equiv \sup\Big\{ |\Phi_n(t'-\tau,x-y) - \Phi_n(t''-\tau,x-y)| \frac{|t'-\tau|^{(n/2)+1}}{|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : x,y\in G, x\neq y, t',t''\in\overline{]-\infty,T[}, t'<t'', \tau < t'-2|t'-t''| \Big\} < +\infty\,.$$

Proof. Statement (i) is an immediate consequence of the definition of Φ_n. We now consider statement (ii). Let $t,\tau\in\overline{]-\infty,T[}$, $\tau<t$, x', $x''\in G$, $x'\neq x''$, $y\in G\setminus\mathbb{B}_n(x',2|x'-x''|)$. Then Lemma 3.1 implies that

$$\begin{aligned} &|\Phi_n(t-\tau,x'-y) - \Phi_n(t-\tau,x''-y)| \\ &\qquad = \frac{1}{(4\pi)^{n/2}(t-\tau)^{n/2}} \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \\ &\qquad \leqslant \frac{1}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} \rho_{2,y}(x',x'') e^{-\frac{\rho^2_{1,y}(x',x'')}{4(t-\tau)}} |x'-x''| \\ &\qquad \leqslant \frac{1}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} 2|x'-y| e^{-\frac{|x'-y|^2}{16(t-\tau)}} |x'-x''|\,, \end{aligned}$$

and accordingly, (ii) follows. Next we consider (iii). Let $t', t'' \in \overline{]-\infty, T[}$, $t' < t''$, $\tau < t' - 2|t' - t''|$, x, $y \in G$, $x \neq y$. By the Mean Value Theorem there exists $\xi \in]t', t''[$ such that

$$
\begin{aligned}
&|\Phi_n(t' - \tau, x - y) - \Phi_n(t'' - \tau, x - y)| \\
&\quad = \frac{1}{(4\pi)^{n/2}} \left| \frac{1}{(t' - \tau)^{(n/2)}} e^{-\frac{|x-y|^2}{4(t'-\tau)}} - \frac{1}{(t'' - \tau)^{(n/2)}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} \right| \\
&\quad = \frac{|t' - t''|}{(4\pi)^{n/2}} \left| \frac{-n/2}{(\xi - \tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi - \tau)^{n/2}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x - y|^2}{4(\xi - \tau)^2} \right| .
\end{aligned}
$$

Then by inequality (3.1), and by the inequalities $|t' - \tau| \leqslant |\xi - \tau| \leqslant |t'' - \tau|$, and by Lemma 3.4 (v), we have

$$
\begin{aligned}
&\frac{|t' - t''|}{(4\pi)^{n/2}} \left| \frac{-n/2}{(\xi - \tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi - \tau)^{n/2}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x - y|^2}{4(\xi - \tau)^2} \right| \\
&\quad \leqslant \frac{|t' - t''|}{(4\pi)^{n/2}} \left[\frac{n/2}{(t' - \tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} + \frac{1}{(t' - \tau)^{n/2}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} \frac{|x - y|^2}{4(t' - \tau)^2} \right] \\
&\quad \leqslant \frac{|t' - t''|}{(4\pi)^{n/2}} \left[\frac{n/2}{(t' - \tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{8(t'-\tau)}} + \frac{1}{(t' - \tau)^{n/2}} e^{-\frac{|x-y|^2}{8(t'-\tau)}} \frac{|x - y|^2}{4(t' - \tau)^2} \right] \\
&\quad \leqslant \frac{|t' - t''|(n/2)}{(4\pi)^{n/2}(t' - \tau)^{(n/2)+1}} C(8, a, 1) e^{-\frac{|x-y|^2}{a(t'-\tau)}} ,
\end{aligned}
$$

and thus statement (iii) holds true. □

Next we consider the spatial gradient of the fundamental solution. Clearly,

$$
D_x \Phi_n(t, x) \equiv \begin{cases} \frac{-x}{2(4\pi)^{n/2} t^{(n/2)+1}} e^{-\frac{|x|^2}{4t}} & \text{if } (t, x) \in]0, +\infty[\times \mathbb{R}^n , \\ 0 & \text{if } (t, x) \in (]-\infty, 0] \times \mathbb{R}^n) \setminus \{(0, 0)\} . \end{cases}
$$

Then we have the following.

Lemma 4.2. *Let $T \in]-\infty, +\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$. Then the following statements hold.*

(i)

$$
C_{0,1,G} \equiv \sup_{(t,x) \in]0,+\infty[\times \mathbb{R}^n, \, |x| \leqslant \operatorname{diam}(G)} |D_x \Phi_n(t, x)| \frac{t^{(n/2)+1}}{|x|} e^{\frac{|x|^2}{4t}} < +\infty .
$$

(ii) Let $a \in]16, +\infty[$. Then

$$
\begin{aligned}
&\tilde{C}_{0,1,a,G} \\
&\quad \equiv \sup \Bigg\{ |D_x \Phi_n(t - \tau, x' - y) - D_x \Phi_n(t - \tau, x'' - y)| \frac{|t - \tau|^{(n/2)+1} e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x' - x''|} : \\
&\qquad\qquad x', x'' \in G, x' \neq x'', y \in G \setminus \mathbb{B}_n(x', 2|x' - x''|) , \\
&\qquad\qquad t, \tau \in \overline{]-\infty, T[}, \ \tau < t \Bigg\} < +\infty .
\end{aligned}
$$

(iii) *Let* $a \in]8,+\infty[$. *Then*

$$\tilde{C}'_{0,1,a,G} \equiv \sup\Big\{ |D_x\Phi_n(t'-\tau,x-y) - D_x\Phi_n(t''-\tau,x-y)| \frac{|t'-\tau|^{(n/2)+2}}{|x-y|\,|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : x,y \in G, x \neq y, t',t'' \in \overline{]-\infty,T[}, t' < t'', \tau < t' - 2|t'-t''| \Big\} < +\infty\,.$$

Proof. Statement (i) is an immediate consequence of the formula for $D_x\Phi_n$. We now consider statement (ii). Let t, $\tau \in \overline{]-\infty,T[}$, $\tau < t$, x', $x'' \in G$, $x' \neq x''$, $y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)$. By the triangular inequality, we have

$$\begin{aligned} &|D_x\Phi_n(t-\tau,x'-y) - D_x\Phi_n(t-\tau,x''-y)| \\ &\qquad = \frac{1}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} \left| (x'-y)e^{-\frac{|x'-y|^2}{4(t-\tau)}} - (x''-y)e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \\ &\qquad \leqslant \frac{1}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} \Big\{ e^{-\frac{|x'-y|^2}{4(t-\tau)}} |x'-x''| \\ &\qquad\quad + |x''-y| \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \Big\}\,. \end{aligned} \tag{4.1}$$

Now Lemma 3.1 implies that

$$\begin{aligned} &\left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \\ &\qquad \leqslant \frac{2\rho_{2,y}(x',x'')}{4(t-\tau)} e^{-\frac{\rho^2_{1,y}(x',x'')}{4(t-\tau)}} |x'-x''| \leqslant \frac{|x'-y|\,|x'-x''|}{(t-\tau)} e^{-\frac{|x'-y|^2}{16(t-\tau)}}\,. \end{aligned} \tag{4.2}$$

Hence, Lemmas 3.1 (i), 3.4 (v) imply that the right hand side of (4.1) is less or equal to

$$\begin{aligned} &\frac{e^{-\frac{|x'-y|^2}{16(t-\tau)}}}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} \left\{ |x'-x''| + 2\frac{|x'-y|^2}{t-\tau}|x'-x''| \right\} \\ &\qquad \leqslant C(16,a,1) \frac{e^{-\frac{|x'-y|^2}{a(t-\tau)}}}{(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} |x'-x''|\,, \end{aligned}$$

and thus statement (ii) holds true. Next we consider statement (iii). Let x, $y \in G$, $x \neq y$, t', $t'' \in \overline{]-\infty,T[}$, $t' < t''$, $\tau < t' - 2|t'-t''|$. By the Mean Value Theorem, there exists $\xi \in]t',t''[$ such that

$$\begin{aligned} &|D_x\Phi_n(t'-\tau,x-y) - D_x\Phi_n(t''-\tau,x-y)| \\ &\quad = \frac{|x-y|}{2(4\pi)^{n/2}} \left| \frac{1}{(t'-\tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(t'-\tau)}} - \frac{1}{(t''-\tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} \right| \\ &\quad = \frac{|t'-t''|\,|x-y|}{2(4\pi)^{n/2}} \left| \frac{-(n/2)-1}{(\xi-\tau)^{(n/2)+2}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x-y|^2}{4(\xi-\tau)^2} \right|\,. \end{aligned}$$

Then by inequality (3.1), and by the inequalities $t' - \tau \leqslant \xi - \tau \leqslant t'' - \tau$, and by Lemma 3.4 (v), we have

$$\left| \frac{-(n/2)-1}{(\xi-\tau)^{(n/2)+2}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{(n/2)+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x-y|^2}{4(\xi-\tau)^2} \right| \tag{4.3}$$
$$\leqslant \frac{e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{(t'-\tau)^{(n/2)+2}} \left(\frac{n}{2} + 1 + \frac{1}{4} \frac{|x-y|^2}{t'-\tau} \right)$$
$$\leqslant \frac{e^{-\frac{|x-y|^2}{8(t'-\tau)}}}{(t'-\tau)^{(n/2)+2}} \left(\frac{n}{2} + 1 \right) \left(1 + \frac{|x-y|^2}{t'-\tau} \right)$$
$$\leqslant C(8,a,1) \frac{e^{-\frac{|x-y|^2}{a(t'-\tau)}}}{(t'-\tau)^{(n/2)+2}} \left(\frac{n}{2} + 1 \right),$$

and thus statement (iii) holds true. □

Next we consider time derivative of the fundamental solution. Clearly,

$$\partial_t \Phi_n(t,x) \equiv \begin{cases} \frac{e^{-\frac{|x|^2}{4t}} \left(-\frac{n}{2} + \frac{1}{4} \frac{|x|^2}{t} \right)}{(4\pi)^{n/2} t^{(n/2)+1}} & \text{if } (t,x) \in]0,+\infty[\times \mathbb{R}^n, \\ 0 & \text{if } (t,x) \in (]-\infty,0] \times \mathbb{R}^n) \setminus \{(0,0)\}. \end{cases}$$

Then we have the following.

Lemma 4.3. *Let $T \in]-\infty,+\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$. Then the following statements hold.*

(i) Let $a \in]4,+\infty[$. Then

$$C_{1,0,a,G} \equiv \sup_{(t,x)\in]0,+\infty[\times\mathbb{R}^n,\, |x|\leqslant \mathrm{diam}\,(G)} |\partial_t \Phi_n(t,x)| t^{(n/2)+1} e^{\frac{|x|^2}{at}} < +\infty.$$

(ii) Let $a \in]16,+\infty[$. Then

$$\tilde{C}_{1,0,a,G} \equiv \sup\Bigg\{ |\partial_t \Phi_n(t-\tau, x'-y) - \partial_t \Phi_n(t-\tau, x''-y)| \frac{|t-\tau|^{(n/2)+2} e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-y|\,|x'-x''|} : x', x'' \in G, x' \neq x'', y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|), t, \tau \in \overline{]-\infty,T[},\ \tau < t \Bigg\} < +\infty.$$

(iii) Let $a \in]8,+\infty[$. Then

$$\tilde{C}'_{1,0,a,G} \equiv \sup\Bigg\{ |\partial_t \Phi_n(t'-\tau, x-y) - \partial_t \Phi_n(t''-\tau, x-y)| \frac{|t'-\tau|^{(n/2)+2}}{|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : x, y \in G, x \neq y, t', t'' \in \overline{]-\infty,T[}, t' < t'', \tau < t' - 2|t'-t''| \Bigg\} < +\infty.$$

Proof. Statement (i) is an immediate consequence of the formula for $\partial_t\Phi_n$ and of Lemma 3.4 (v). We now consider statement (ii). Let $t,\ \tau \in \overline{]-\infty,T[}$, $\tau < t$, $x',\ x'' \in G$, $x' \neq x''$, $y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)$. By the triangular inequality, we have

$$
\begin{aligned}
|\partial_t\Phi_n(t-\tau,x'-y) - \partial_t\Phi_n(t-\tau,x''-y)| &= \frac{1}{(4\pi)^{n/2}(t-\tau)^{(n/2)+1}} \\
&\times \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x'-y|^2}{t-\tau}\right) - e^{-\frac{|x''-y|^2}{4(t-\tau)}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x''-y|^2}{t-\tau}\right)\right| \\
&\leqslant \frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\frac{1}{4(t-\tau)}\left||x'-y|^2-|x''-y|^2\right| \\
&+\frac{\frac{n}{2}\left(1+\frac{|x''-y|^2}{t-\tau}\right)}{(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\left|e^{-\frac{|x'-y|^2}{4(t-\tau)}}-e^{-\frac{|x''-y|^2}{4(t-\tau)}}\right| .
\end{aligned}
$$

By the Mean Value Theorem and by Lemma 3.1 (iii), we have

$$
\left||x'-y|^2-|x''-y|^2\right| \leqslant 2\rho_{2,y}(x',x'')|x'-x''| \leqslant 4|x'-y|\,|x'-x''|\,,
$$

and by Lemma 3.1 (ii), (iii), we have

$$
\begin{aligned}
&\left|e^{-\frac{|x'-y|^2}{4(t-\tau)}}-e^{-\frac{|x''-y|^2}{4(t-\tau)}}\right| \\
&\qquad\leqslant \frac{\rho_{2,y}(x',x'')}{2(t-\tau)}e^{-\frac{\rho^2_{1,y}(x',x'')}{4(t-\tau)}}|x'-x''| \leqslant \frac{|x'-y|}{(t-\tau)}e^{-\frac{|x'-y|^2}{16(t-\tau)}}|x'-x''|\,.
\end{aligned}
$$

Hence, Lemmas 3.1 and 3.4 (v) imply that

$$
\begin{aligned}
&|\partial_t\Phi_n(t-\tau,x'-y) - \partial_t\Phi_n(t-\tau,x''-y)| \\
&\qquad\leqslant \frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\frac{1}{4(t-\tau)}4|x'-y|\,|x'-x''| \\
&\qquad+\frac{\frac{n}{2}\left(1+\frac{|x'-y|^2}{t-\tau}\right)}{(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\frac{|x'-y|}{(t-\tau)}e^{-\frac{|x'-y|^2}{16(t-\tau)}}|x'-x''| \\
&\qquad\leqslant \frac{|x'-y|\,|x'-x''|}{(4\pi)^{n/2}(t-\tau)^{(n/2)+2}}\left\{e^{-\frac{|x'-y|^2}{4(t-\tau)}}+(n/2)C(16,a,1)e^{-\frac{|x'-y|^2}{a(t-\tau)}}\right\},
\end{aligned}
$$

and thus statement (ii) holds true. Next we consider statement (iii). Let $x,\ y \in G$, $x \neq y$, t', $t'' \in \overline{]-\infty,T[}$, $t' < t''$, $\tau < t' - 2|t'-t''|$. By the Mean Value Theorem, there exists $\xi \in]t',t''[$

such that

$$
\begin{aligned}
&|\partial_t\Phi_n(t'-\tau,x-y)-\partial_t\Phi_n(t''-\tau,x-y)| \\
&\quad = \frac{1}{(4\pi)^{n/2}}\left|\frac{e^{-\frac{|x-y|^2}{4(t'-\tau)}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(t'-\tau)}\right)}{(t'-\tau)^{(n/2)+1}}-\frac{e^{-\frac{|x-y|^2}{4(t''-\tau)}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(t''-\tau)}\right)}{(t''-\tau)^{(n/2)+1}}\right| \\
&\quad \leqslant \frac{1}{(4\pi)^{n/2}}\left|\frac{\left(\frac{n}{2}+1\right)e^{-\frac{|x-y|^2}{4(\xi-\tau)}}}{(\xi-\tau)^{(n/2)+2}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)}\right)\right||t'-t''| \\
&\qquad +\frac{1}{(4\pi)^{n/2}}\left|\frac{e^{-\frac{|x-y|^2}{4(\xi-\tau)}}}{(\xi-\tau)^{(n/2)+1}}\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)^2}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)}\right)\right||t'-t''| \\
&\qquad +\frac{1}{(4\pi)^{n/2}}\left|\frac{e^{-\frac{|x-y|^2}{4(\xi-\tau)}}}{(\xi-\tau)^{(n/2)+1}}\left(-\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)^2}\right)\right||t'-t''|\,.
\end{aligned}
$$

Then by inequality (3.1), and by the inequalities $t'-\tau\leqslant\xi-\tau\leqslant t''-\tau$, and by Lemma 3.4 (v), we have

$$
\begin{aligned}
&|\partial_t\Phi_n(t'-\tau,x-y)-\partial_t\Phi_n(t''-\tau,x-y)| \\
&\quad \leqslant \frac{|t'-t''|}{(4\pi)^{n/2}}\Bigg\{\frac{\left(\frac{n}{2}+1\right)\frac{n}{2}e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{(t'-\tau)^{(n/2)+2}}\left(1+\frac{|x-y|^2}{t'-\tau}\right) \\
&\qquad +\frac{\frac{n}{2}|x-y|^2e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{4(t'-\tau)^{(n/2)+3}}\left(1+\frac{|x-y|^2}{t'-\tau}\right)+\frac{|x-y|^2e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{4(t'-\tau)^{(n/2)+3}}\Bigg\} \\
&\quad \leqslant \frac{|t'-t''|}{(4\pi)^{n/2}}\frac{3\frac{n}{2}\left(\frac{n}{2}+1\right)}{(t'-\tau)^{(n/2)+2}}\left[1+\frac{|x-y|^2}{t'-\tau}+\left(\frac{|x-y|^2}{t'-\tau}\right)^2\right]e^{-\frac{|x-y|^2}{8(t'-\tau)}} \\
&\quad \leqslant \frac{|t'-t''|}{(4\pi)^{n/2}}\frac{3\frac{n}{2}\left(\frac{n}{2}+1\right)}{(t'-\tau)^{(n/2)+2}}C(8,a,2)e^{-\frac{|x-y|^2}{a(t'-\tau)}}\,,
\end{aligned}
$$

and thus statement (iii) holds true. □

5 Preliminary inequalities on the kernel of the double layer heat potential

We now turn to introduce some inequalities for the kernel of the double layer heat potential. We do so by means of the following.

Lemma 5.1. *Let $T\in]-\infty,+\infty]$, $\alpha\in]0,1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i)

$$
b_{\Omega,\alpha}\equiv\sup\Bigg\{\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right|\frac{|t-\tau|^{(n/2)+1}}{|x-y|^{1+\alpha}}e^{\frac{|x-y|^2}{4(t-\tau)}}: \\
x,y\in\partial\Omega,x\neq y,t,\tau\in\overline{]-\infty,T[},\tau<t\Bigg\}<+\infty\,.
$$

Here

$$\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\equiv -D_x\Phi_n(t-\tau,x-y)\nu(y)\,,$$

where $D_x\Phi_n$ denotes the Jacobian matrix of Φ_n with respect to the (spatial) second variable.

(ii) Let $a\in]16,+\infty[$. Then

$$\tilde{b}_{a,\Omega,\alpha}\equiv\sup\Biggl\{\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x'-y)-\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x''-y)\right| \times\frac{|t-\tau|^{(n/2)+1}e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-y|^{\alpha}|x'-x''|}:\,x',x''\in\partial\Omega,x'\neq x'', y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)\,,t,\tau\in\overline{]-\infty,T[},\ \tau<t\Biggr\}<+\infty\,.$$

(iii) Let $a\in]8,+\infty[$. Then

$$\tilde{b}'_{a,\Omega,\alpha}\equiv\sup\Biggl\{\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t'-\tau,x-y)-\frac{\partial}{\partial\nu(y)}\Phi_n(t''-\tau,x-y)\right| \times\frac{|t'-\tau|^{(n/2)+2}}{|x-y|^{1+\alpha}\,|t'-t''|}e^{\frac{|x-y|^2}{a(t'-\tau)}}: x,y\in\partial\Omega,x\neq y,t',t''\in\overline{]-\infty,T[},t'<t'', \tau<t'-2|t'-t''|\Biggr\}<+\infty\,.$$

Proof. Let $x,y\in\partial\Omega$, $x\neq y$, t, $\tau\in\overline{]-\infty,T[}$, $\tau<t$. By Lemma 3.2, we have

$$\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right|=\left|\frac{(x-y)^t\nu(y)e^{-\frac{|x-y|^2}{4(t-\tau)}}}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\right| \leqslant\frac{c_{\Omega,\alpha}|x-y|^{1+\alpha}e^{-\frac{|x-y|^2}{4(t-\tau)}}}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\,,$$

and thus statement (i) holds true. We now consider statement (ii). Let t, $\tau\in\overline{]-\infty,T[}$, $\tau<t$, x', $x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Then the triangular inequality implies that

$$\begin{aligned}&\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x'-y)-\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x''-y)\right| \qquad (5.1)\\ &=\frac{1}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\left|(x'-y)^t\nu(y)e^{-\frac{|x'-y|^2}{4(t-\tau)}}-(x''-y)^t\nu(y)e^{-\frac{|x''-y|^2}{4(t-\tau)}}\right|\\ &\leqslant\frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\left|(x'-y)^t\nu(y)-(x''-y)^t\nu(y)\right|\\ &\quad+\frac{|(x''-y)^t\nu(y)|}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\left|e^{-\frac{|x'-y|^2}{4(t-\tau)}}-e^{-\frac{|x''-y|^2}{4(t-\tau)}}\right|\,.\end{aligned}$$

Since $|x'-x''|\leqslant|x'-y|$, Lemma 3.2, implies that

$$
\begin{aligned}
|(x'-x'')^t\nu(y)| &\leqslant |(x'-x'')^t(\nu(y)-\nu(x'))| + |(x'-x'')^t\nu(x')| \\
&\leqslant |x'-x''|\,|\nu|_\alpha|x'-y|^\alpha + c_{\Omega,\alpha}|x'-x''|^{1+\alpha} \\
&\leqslant |x'-x''|\,|x'-y|^\alpha(|\nu|_\alpha + c_{\Omega,\alpha})\,,
\end{aligned}
$$

and

$$
|(x''-y)^t\nu(y)| \leqslant c_{\Omega,\alpha}|x''-y|^{1+\alpha} \leqslant c_{\Omega,\alpha}2^{1+\alpha}|x'-y|^{1+\alpha}\,.
$$

Then Lemmas 3.1 (i) and 3.4 (v) and inequality (4.2) imply that the right hand side of (5.1) is less or equal to

$$
\begin{aligned}
&\frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}|x'-x''|\,|x'-y|^\alpha(|\nu|_\alpha + c_{\Omega,\alpha}) \\
&\qquad + \frac{c_{\Omega,\alpha}2^{1+\alpha}|x'-y|^{1+\alpha}e^{-\frac{|x'-y|^2}{16(t-\tau)}}}{2(4\pi)^{n/2}(t-\tau)^{(n/2)+1}}\,\frac{|x'-y|\,|x'-x''|}{(t-\tau)} \\
&\qquad \leqslant \max\left\{\frac{(|\nu|_\alpha + c_{\Omega,\alpha})}{2(4\pi)^{n/2}}, \frac{c_{\Omega,\alpha}2^{1+\alpha}}{2(4\pi)^{n/2}}\right\}\frac{|x'-x''|\,|x'-y|^\alpha e^{-\frac{|x'-y|^2}{16(t-\tau)}}}{(t-\tau)^{(n/2)+1}} \\
&\qquad \times\left\{1 + \frac{|x'-y|^2}{t-\tau}\right\} \\
&\qquad \leqslant \max\left\{\frac{(|\nu|_\alpha + c_{\Omega,\alpha})}{2(4\pi)^{n/2}}, \frac{c_{\Omega,\alpha}2^{1+\alpha}}{2(4\pi)^{n/2}}\right\}C(16,a,1)\frac{|x'-x''|\,|x'-y|^\alpha e^{-\frac{|x'-y|^2}{a(t-\tau)}}}{(t-\tau)^{(n/2)+1}}\,,
\end{aligned}
$$

and thus statement (ii) holds true. Next we consider statement (iii). Let x, $y\in\partial\Omega$, $x\neq y$, t', $t''\in\overline{]-\infty,T[}$, $t'<t''$, $\tau<t'-2|t'-t''|$. By the Mean Value Theorem, there exists $\xi\in]t',t''[$ such that

$$
\begin{aligned}
&\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t'-\tau,x-y) - \frac{\partial}{\partial\nu(y)}\Phi_n(t''-\tau,x-y)\right| \\
&\qquad = \frac{|(x-y)^t\nu(y)|}{2(4\pi)^{n/2}}\left|\frac{e^{-\frac{|x-y|^2}{4(t'-\tau)}}}{(t'-\tau)^{(n/2)+1}} - \frac{e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{(t''-\tau)^{(n/2)+1}}\right| \\
&\qquad = \frac{|(x-y)^t\nu(y)|\,|t'-t''|}{2(4\pi)^{n/2}} \\
&\qquad\quad \times\left|\frac{-(n/2)-1}{(\xi-\tau)^{(n/2)+2}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{(n/2)+1}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}}\frac{|x-y|^2}{4(\xi-\tau)^2}\right|\,.
\end{aligned}
$$

Then by inequality (3.1), and by Lemma 3.2 and inequality (4.3), we have

$$
\begin{aligned}
&\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t'-\tau,x-y) - \frac{\partial}{\partial\nu(y)}\Phi_n(t''-\tau,x-y)\right| \\
&\qquad \leqslant \frac{c_{\Omega,\alpha}|x-y|^{1+\alpha}\,|t'-t''|}{2(4\pi)^{n/2}} \\
&\qquad\quad \times\left|\frac{-(n/2)-1}{(\xi-\tau)^{(n/2)+2}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{(n/2)+1}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}}\frac{|x-y|^2}{4(\xi-\tau)^2}\right| \\
&\qquad \leqslant \frac{c_{\Omega,\alpha}[(n/2)+1]}{2(4\pi)^{n/2}}C(8,a,1)\frac{|x-y|^{1+\alpha}|t'-t''|}{(t'-\tau)^{(n/2)+2}}e^{-\frac{|x-y|^2}{a(t'-\tau)}}\,,
\end{aligned}
$$

and thus statement (iii) holds true. □

6 Time dependent boundary norms for kernels

For each subset A of $\mathbb{R}\times\mathbb{R}^n$, we find convenient to set

$$\Delta_A \equiv \{(t,x,\tau,y)\in A\times A:\ t=\tau, x=y\}\,.$$

For each $T\in]-\infty,+\infty]$ and $G\subseteq\mathbb{R}^n$, we now introduce a class of functions on $(G_T)^2\setminus\Delta_{G_T}$ which may carry a singularity as the variable tends to a point of the diagonal, just as in the case of the kernels of integral operators corresponding to layer heat potentials.

Definition 3. Let $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. Let G be a nonempty bounded subset of $\mathbb{R}^n$. Let

$$\gamma\equiv(\gamma_1,\gamma_2,\gamma_1',\gamma_2',\gamma_l',\gamma_1'',\gamma_2'',\gamma_l'')\in\mathbb{R}^8\,. \tag{6.1}$$

We denote by $\mathcal{K}_{\gamma,a}(G_T)$ the set of continuous functions K from $(G_T)^2\setminus\Delta_{G_T}$ to $\mathbb{C}$ such that

$$K(t,x,\tau,y)=0 \qquad \text{if } (t,x,\tau,y)\in(G_T)^2\setminus\Delta_{G_T}, \tau\geqslant t\,,$$

and such that

$$\begin{aligned}
&\|K\|_{\mathcal{K}_{\gamma,a}(G_T)}\\
&\quad\equiv\sup\Bigg\{|K(t,x,\tau,y)|\frac{|t-\tau|^{\gamma_1}}{|x-y|^{\gamma_2}}e^{\frac{|x-y|^2}{a(t-\tau)}}:\\
&\qquad\qquad x,y\in G, x\neq y, t,\tau\in\overline{]-\infty,T[}, \tau<t\Bigg\}\\
&\quad+\sup\Bigg\{\ |K(t,x',\tau,y)-K(t,x'',\tau,y)|\ \frac{|t-\tau|^{\gamma_1'}e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-y|^{\gamma_2'}|x'-x''|^{\gamma_l'}}:\\
&\quad x',x''\in G, x'\neq x'', y\in G\setminus\mathbb{B}_n(x',2|x'-x''|)\,, t,\tau\in\overline{]-\infty,T[},\ \tau<t\Bigg\}\\
&\quad+\sup\Bigg\{\ |K(t',x,\tau,y)-K(t'',x,\tau,y)|\ \frac{|t'-\tau|^{\gamma_1''}}{|x-y|^{\gamma_2''}\,|t'-t''|^{\gamma_l''}}e^{\frac{|x-y|^2}{a(t'-\tau)}}:\\
&\quad x,y\in G, x\neq y, t',t''\in\overline{]-\infty,T[}, t'<t'', \tau<t'-2|t'-t''|\Bigg\}<+\infty\,.
\end{aligned}$$

One can easily verify that $(\mathcal{K}_{\gamma,a}(G_T),\|\cdot\|_{\mathcal{K}_{\gamma,a}(G_T)})$ is a Banach space.

Remark 3. Let $a\in]16,+\infty[$, $T\in]-\infty,+\infty]$.

(i) Let G be a nonempty subset of $\mathbb{R}^n$. Then Lemma 4.1 implies that the kernel $\Phi_n(t-\tau,x-y)$ belongs to $\mathcal{K}_{\gamma,a}(G_T)$, with $\gamma\equiv(\gamma_1,\gamma_2,\gamma_1',\gamma_2',\gamma_l',\gamma_1'',\gamma_2'',\gamma_l'')$ and

$$\begin{aligned}
&\gamma_1=\frac{n}{2}\,,\quad \gamma_2=0\,,\quad \gamma_1'=\frac{n}{2}+1\,,\quad \gamma_2'=1\,,\quad \gamma_l'=1\,,\\
&\gamma_1''=\frac{n}{2}+1\,,\quad \gamma_2''=0\,,\quad \gamma_l''=1\,.
\end{aligned}$$

(ii) Let G be a nonempty subset of $\mathbb{R}^n$. Then Lemma 4.3 implies that the kernel $\partial_t \Phi_n(t-\tau, x-y)$ belongs to $\mathcal{K}_{\gamma,a}(G_T)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2}+1\,, \quad \gamma_2 = 0\,, \quad \gamma_1' = \frac{n}{2}+2\,, \quad \gamma_2' = 1\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2}+2\,, \quad \gamma_2'' = 0\,, \quad \gamma_l'' = 1\,.$$

(iii) Let G be a nonempty subset of $\mathbb{R}^n$. Let $r \in \{1,\dots,n\}$. Then Lemma 4.2 implies that the kernel $\partial_{x_r}\Phi_n(t-\tau, x-y)$ belongs to $\mathcal{K}_{\gamma,a}(G_T)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2}+1\,, \quad \gamma_2 = 1\,, \quad \gamma_1' = \frac{n}{2}+1\,, \quad \gamma_2' = 0\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2}+2\,, \quad \gamma_2'' = 1\,, \quad \gamma_l'' = 1\,.$$

(iv) Let $\alpha \in]0,1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then Lemma 5.1 implies that the kernel $\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau, x-y)$ belongs to $\mathcal{K}_{\gamma,a}(\partial_T\Omega)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2}+1\,, \quad \gamma_2 = 1+\alpha\,, \quad \gamma_1' = \frac{n}{2}+1\,, \quad \gamma_2' = \alpha\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2}+2\,, \quad \gamma_2'' = 1+\alpha\,, \quad \gamma_l'' = 1\,.$$

7 Integral operators on the space of essentially bounded functions

For each $\theta \in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $]0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} r^\theta |\ln r| & r \in]0, r_\theta]\,, \\ r_\theta^\theta |\ln r_\theta| & r \in]r_\theta, +\infty[\,, \end{cases}$$

where

$$r_\theta \equiv e^{-1/\theta} \qquad \forall \theta \in]0,1]\,.$$

Obviously, $\omega_\theta(\cdot)$ satisfies (2.1). We also note that if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then the following continuous imbedding holds

$$C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

for all $\theta' \in]0,\theta[$. We now consider the properties of an integral operator with a kernel in the class $\mathcal{K}_{\gamma,a}(\partial_T\Omega)$ and acting on the space of essentially bounded functions on $\partial_T\Omega$.

Proposition 7.1. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'') \in \mathbb{R}^8$. Then the following statements hold.*

(i) Let $\gamma_1 > 1$, $2\gamma_1 - \gamma_2 - 2 \in]-\infty, n-1[$. If $(K,\mu) \in \mathcal{K}_{\gamma,a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$, and if $(t,x) \in \partial_T\Omega$, then the function $K(t,x,\cdot,\cdot)\mu(\cdot,\cdot)$ is integrable in $\partial_T\Omega$ and the function $u[\partial_T\Omega, K, \mu]$ from $\partial_T\Omega$ to $\mathbb{C}$ defined by

$$u[\partial_T\Omega, K, \mu](t,x) \equiv \int_{-\infty}^t \int_{\partial\Omega} K(t,x,\tau,y)\mu(\tau,y)\, d\sigma_y d\tau \qquad \forall (t,x) \in \partial_T\Omega\,, \tag{7.1}$$

is bounded. Moreover, the bilinear map from $\mathcal{K}_{\gamma,a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ to $B(\partial_T\Omega)$, which takes (K,μ) to $u[\partial_T\Omega, K, \mu]$ is bilinear and continuous.

(ii) *Let* $\gamma_1 > 1$, $2\gamma_1 - \gamma_2 - 2 \in [n-2, n-1[$, $\gamma_1' > 1$, $\gamma_l' + (n-1) - (2\gamma_1' - \gamma_2' - 2) > 0$, $\gamma_l' \in]0,1]$. *Let*

$$\omega(r) \equiv \begin{cases} r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),\gamma_l'+(n-1)-(2\gamma_1'-\gamma_2'-2)\}} & \text{if } 2\gamma_1' - \gamma_2' - 2 > n-1\,, \\ \max\{r^{(n-1)-(2\gamma_1-\gamma_2-2)}, \omega_{\gamma_l'}(r)\} & \text{if } 2\gamma_1' - \gamma_2' - 2 = n-1\,, \\ r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),\gamma_l'\}} & \text{if } 2\gamma_1' - \gamma_2' - 2 < n-1\,, \end{cases}$$

for all $r \in]0,+\infty[$. *Then the bilinear map from* $\mathcal{K}_{\gamma,a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ *to* $B\left(\overline{]-\infty,T[}, C^{0,\omega(\cdot)}(\partial\Omega)\right)$ *which takes* (K,μ) *to* $u[\partial_T\Omega, K, \mu]$ *is continuous (cf. Remark 2.)*

Proof. Let $(t,x) \in \partial_T\Omega$. Then we have

$$\begin{aligned} &\left| \int_{-\infty}^t \int_{\partial\Omega} K(t,x,\tau,y)\mu(\tau,y)\, d\sigma_y d\tau \right| \\ &\qquad \leqslant \int_{-\infty}^t \int_{\partial\Omega} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{L^\infty(\partial_T\Omega)} \frac{|x-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t-\tau)}}\, d\sigma_y d\tau \\ &\qquad = \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{L^\infty(\partial_T\Omega)} \int_{\partial\Omega} \int_0^{+\infty} \frac{|x-y|^{\gamma_2+2} a^{-1+\gamma_1}}{u^{\gamma_1}|x-y|^{2\gamma_1}} e^{-1/u}\, du d\sigma_y \\ &\qquad = \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{L^\infty(\partial_T\Omega)} a^{-1+\gamma_1} \int_0^{+\infty} u^{-\gamma_1} e^{-1/u}\, du \int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}\,, \end{aligned}$$

and the integrals in the right hand side converges for $2\gamma_1 - \gamma_2 - 2 < n-1$ and $\gamma_1 > 1$. Then Lemma 3.3 (i) implies the validity of statement (i).

Next we consider statement (ii). Let $t \in \overline{]-\infty,T[}$, $x', x'' \in \partial\Omega$. By statement (i) and Remark 1, there is no loss of generality in assuming that $0 < |x'-x''| \leqslant r_{\gamma_l'}$. Then the inclusion $\mathbb{B}_n(x', 2|x'-x''|) \subseteq \mathbb{B}_n(x'', 3|x'-x''|)$ and the triangular inequality imply that

$$\begin{aligned} &|u[\partial_T\Omega, K, \mu](t,x') - u[\partial_T\Omega, K, \mu](t,x'')| \qquad (7.2) \\ &\qquad \leqslant \|\mu\|_{L^\infty(\partial_T\Omega)} \Big\{ \int_{-\infty}^t \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(t,x',\tau,y)|\, d\sigma_y d\tau \\ &\qquad + \int_{-\infty}^t \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(t,x'',\tau,y)|\, d\sigma_y d\tau \\ &\qquad + \int_{-\infty}^t \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} |K(t,x',\tau,y) - K(t,x'',\tau,y)|\, d\sigma_y d\tau \Big\} \\ &\quad \leqslant \|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \Big\{ \int_{-\infty}^t \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2}}{(t-\tau)^{\gamma_1}} e^{-\frac{|x'-y|^2}{a(t-\tau)}}\, d\sigma_y d\tau \\ &\qquad + \int_{-\infty}^t \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2}}{(t-\tau)^{\gamma_1}} e^{-\frac{|x''-y|^2}{a(t-\tau)}}\, d\sigma_y d\tau \\ &\qquad + |x'-x''|^{\gamma_l'} \int_{-\infty}^t \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x'-y|^{\gamma_2'}}{(t-\tau)^{\gamma_1'}} e^{-\frac{|x'-y|^2}{a(t-\tau)}}\, d\sigma_y d\tau \Big\}\,. \end{aligned}$$

Then by setting $a(t-\tau) = u|x'-y|^2$, $a(t-\tau) = u|x''-y|^2$, $a(t-\tau) = u|x'-y|^2$ in the first, and second and third integrals in the right hand side of (7.2), respectively, we deduce that the right hand side of (7.2) equals

$$\|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\bigg\{\int_0^{+\infty}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega}\frac{|x'-y|^{\gamma_2+2}a^{-1+\gamma_1}}{u^{\gamma_1}|x'-y|^{2\gamma_1}}e^{-\frac{1}{u}}\,d\sigma_y du$$

$$+\int_0^{+\infty}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega}\frac{|x''-y|^{\gamma_2+2}a^{-1+\gamma_1}}{u^{\gamma_1}|x''-y|^{2\gamma_1}}e^{-\frac{1}{u}}\,d\sigma_y du$$

$$+|x'-x''|^{\gamma_l'}\int_0^{+\infty}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{|x'-y|^{\gamma_2'+2}a^{-1+\gamma_1'}}{u^{\gamma_1'}|x'-y|^{2\gamma_1'}}e^{-\frac{1}{u}}\,d\sigma_y du\bigg\}$$

$$\leqslant\|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\bigg\{\frac{\Gamma(\gamma_1-1)}{a^{1-\gamma_1}}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega}\frac{d\sigma_y}{|x'-y|^{2\gamma_1-\gamma_2-2}}$$

$$+\frac{\Gamma(\gamma_1-1)}{a^{1-\gamma_1}}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega}\frac{d\sigma_y}{|x''-y|^{2\gamma_1-\gamma_2-2}}$$

$$+\frac{\Gamma(\gamma_1'-1)}{a^{1-\gamma_1'}}|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\bigg\}$$

$$\leqslant\|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\bigg\{2\frac{\Gamma(\gamma_1-1)}{a^{1-\gamma_1}}c''_{\Omega,\alpha}|x'-x''|^{(n-1)-(2\gamma_1-\gamma_2-2)}$$

$$+\frac{\Gamma(\gamma_1'-1)}{a^{1-\gamma_1'}}|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\bigg\}.$$

We now distinguish three cases. In case $2\gamma_1'-\gamma_2'-2>n-1$, Lemma 3.3 (iii) implies that

$$|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\leqslant c'''_{\Omega,2\gamma_1'-\gamma_2'-2}|x'-x''|^{\gamma_l'+(n-1)-(2\gamma_1'-\gamma_2'-2)}.$$

In case $2\gamma_1'-\gamma_2'-2=n-1$, Lemma 3.3 (iv) implies that

$$|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\leqslant|x'-x''|^{\gamma_l'}c^{iv}_\Omega\,|\ln|x'-x''||\,.$$

In case $2\gamma_1'-\gamma_2'-2<n-1$, Lemma 3.3 (i) implies that

$$|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\leqslant c'_{\Omega,2\gamma_1'-\gamma_2'-2}|x'-x''|^{\gamma_l'}.$$

Then the above inequalities imply the validity of statement (ii). □

Proposition 7.2. *Let* $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$. *Let* $\gamma\equiv(\gamma_1,\gamma_2,\gamma_1',\gamma_2',\gamma_l',\gamma_1'',\gamma_2'',\gamma_l'')\in\mathbb{R}^8$. *Let* $\gamma_1>1$, $2\gamma_1-\gamma_2-2\in]-\infty,n-1[$. *Let*

$$h\in\left]0,\frac{(n-1)-(2\gamma_1-\gamma_2-2)}{2}\right[\cap]0,1].$$

Let $\gamma_1''>1$, $\gamma_l''\in]0,1]$, $\max\left\{0,\frac{(2\gamma_1''-\gamma_2''-2)-(n-1)}{2}\right\}<\min\{\gamma_1''-1,\gamma_l''\}$,

$$h'\in\left]\max\left\{0,\frac{(2\gamma_1''-\gamma_2''-2)-(n-1)}{2}\right\},\min\{\gamma_1''-1,\gamma_l''\}\right].$$

Then the bilinear map from $\mathcal{K}_{\gamma,a}(\partial_T\Omega)\times L^\infty(\partial_T\Omega)$ *to* $C_b^{0,\min\{h,\gamma_l''-h'\}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$, *which takes* (K,μ) *to* $u[\partial_T\Omega,K,\mu]$ *is continuous (cf. Remark 2.)*

Proof. By Proposition 7.1 (i), it suffices to estimate the Hölder quotient of $u[\partial_T\Omega, K, \mu]$ in the time variable. Let $x \in \partial\Omega$, t', $t'' \in \overline{]-\infty, T[}$, $t' < t''$. By Remark 1 and Proposition 7.1 (i), there is no loss of generality in assuming that $0 < |t' - t''| \leqslant 1/e$. Then the inclusion $]t' - 2|t' - t''|, t' + 2|t' - t''|[\subseteq]t'' - 3|t' - t''|, t'' + 3|t' - t''|[$ and the triangular inequality imply that

$$\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t', x) - u[\partial_T\Omega, K, \mu](t'', x)| \qquad (7.3)\\
&\leqslant \int_{t'-2|t'-t''|}^{t'+2|t'-t''|} \int_{\partial\Omega} |K(t', x, \tau, y) - K(t'', x, \tau, y)|\, |\mu(\tau, y)|\, d\sigma_y d\tau \\
&+ \int_{-\infty}^{t'-2|t'-t''|} \int_{\partial\Omega} |K(t', x, \tau, y) - K(t'', x, \tau, y)|\, |\mu(\tau, y)|\, d\sigma_y d\tau \\
&\leqslant \|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \int_{t'-2|t'-t''|}^{t'} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t'-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t'-\tau)}}\, d\sigma_y d\tau \right. \\
&+ \int_{t''-3|t'-t''|}^{t''} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t''-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t''-\tau)}}\, d\sigma_y d\tau \\
&\left. + \int_{\partial\Omega} \int_{-\infty}^{t'-2|t'-t''|} \frac{|x-y|^{\gamma_2''}}{|t'-\tau|^{\gamma_1''}} |t' - t''|^{\gamma_l''} e^{-\frac{|x-y|^2}{a(t'-\tau)}}\, d\sigma_y d\tau \right\}.
\end{aligned}$$

Then by setting $a(t' - \tau) = u|x - y|^2$, $a(t'' - \tau) = u|x - y|^2$, $a(t' - \tau) = u|x - y|^2$ in the first, and second and third integrals in the right hand side of (7.3), respectively, we deduce that the right hand side of (7.3) equals

$$\begin{aligned}
&\|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \int_{\partial\Omega} \int_0^{2a\frac{|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2} a^{\gamma_1-1}}{u^{\gamma_1}|x-y|^{2\gamma_1}} e^{-1/u}\, du d\sigma_y \right. \\
&+ \int_{\partial\Omega} \int_0^{3a\frac{|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2} a^{\gamma_1-1}}{u^{\gamma_1}|x-y|^{2\gamma_1}} e^{-1/u}\, du d\sigma_y \\
&\left. + \int_{\partial\Omega} \int_{2a\frac{|t'-t''|}{|x-y|^2}}^{+\infty} \frac{|x-y|^{\gamma_2''+2} a^{\gamma_1''-1}}{u^{\gamma_1''}|x-y|^{2\gamma_1''}} |t' - t''|^{\gamma_l''} e^{-1/u}\, du d\sigma_y \right\}.
\end{aligned}$$

Next we note that our assumptions of h and h' imply that $2\gamma_1 - \gamma_2 - 2 + 2h < n - 1$ and that $2\gamma_1'' - \gamma_2'' - 2 - 2h' < n - 1$. Then by Lemmas 3.3 and 3.4 (ii), (iii), the right hand side of (7.3) is less or equal to

$$\begin{aligned}
&\|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \int_{\partial\Omega} \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} \left(2a\frac{|t'-t''|}{|x-y|^2}\right)^h \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}} \right. \\
&+ \int_{\partial\Omega} \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} \left(3a\frac{|t'-t''|}{|x-y|^2}\right)^h \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}} \\
&\left. + \int_{\partial\Omega} D_{\gamma_1'',h'} \left(2a\frac{|t'-t''|}{|x-y|^2}\right)^{-h'} a^{\gamma_1''-1} |t'-t''|^{\gamma_l''} \frac{d\sigma_y}{|x-y|^{2\gamma_1''-\gamma_2''-2}} \right\} \\
&\leqslant \|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} (2a)^h c'_{\Omega,2\gamma_1-\gamma_2-2+2h} \right. \\
&\left. + \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} (3a)^h c'_{\Omega,2\gamma_1-\gamma_2-2+2h} + D_{\gamma_1'',h'} a^{\gamma_1''-1} (2a)^{-h'} c'_{\Omega,2\gamma_1''-\gamma_2''-2-2h'} \right\} \\
&\times |t' - t''|^{\min\{h, \gamma_l''-h'\}},
\end{aligned}$$

and thus the statement holds true. □

Next we prove the following.

Lemma 7.1. *Let* $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$. *Let* G *be a subset of* $\mathbb{R}^n$. *Let* $\gamma \equiv (\gamma_1,\gamma_2,\gamma_1',\gamma_2',\gamma_l',\gamma_1'',\gamma_2'',\gamma_l'') \in \mathbb{R}^8$. *Let* $\gamma_1 > 1$, $2\gamma_1-\gamma_2-2 \in]-\infty,n-1[$. *Let* $K \in C^0((G_T \times \partial_T\Omega)\setminus \Delta_{\partial_T\Omega})$ *be such that*

$$K(t,x,\tau,y) = 0 \text{ if } (t,x,\tau,y) \in (G_T \times \partial_T\Omega)\setminus \Delta_{\partial_T\Omega}, \tau \geqslant t\,,$$

and such that

$$\varkappa_{\gamma_1,\gamma_2} \equiv \sup\Bigg\{ |K(t,x,\tau,y)| \frac{|t-\tau|^{\gamma_1}}{|x-y|^{\gamma_2}} e^{\frac{|x-y|^2}{a(t-\tau)}} : \\ (t,x,\tau,y) \in (G_T \times \partial_T\Omega)\setminus \Delta_{\partial_T\Omega} \Bigg\} < +\infty\,.$$

If $\mu \in L^\infty(\partial_T\Omega)$, *and if* $(t,x) \in G_T$, *then the function* $K(t,x,\cdot,\cdot)\mu(\cdot,\cdot)$ *is integrable in* $\partial_T\Omega$ *and the function* $u^\sharp[G_T,\partial_T\Omega,K,\mu]$ *from* G_T *to* $\mathbb{C}$ *defined by*

$$u^\sharp[G_T,\partial_T\Omega,K,\mu](t,x) \equiv \int_{-\infty}^t \int_{\partial\Omega} K(t,x,\tau,y)\mu(\tau,y)\, d\sigma_y d\tau\,,$$

for all $(t,x) \in G_T$, *is continuous. If* $\sup_{x\in G}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}} < +\infty$, *then the following inequality holds*

$$|u^\sharp[G_T,\partial_T\Omega,K,\mu](t,x)| \\ \leqslant \Gamma(\gamma_1-1)a^{\gamma_1-1}\sup_{x\in G}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}\varkappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}\,, \tag{7.4}$$

for all $(t,x) \in G_T$.

Proof. Let $(t,x) \in G_T$. Then we have

$$\begin{aligned}
&\int_{-\infty}^t \int_{\partial\Omega} |K(t,x,\tau,y)\mu(\tau,y)|\, d\sigma_y d\tau \\
&\qquad \leqslant \varkappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)} \int_{-\infty}^t\int_{\partial\Omega}\frac{|x-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}}e^{-\frac{|x-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \\
&\qquad = \varkappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}\int_{\partial\Omega}\int_0^{+\infty}\frac{|x-y|^{\gamma_2+2}a^{\gamma_1-1}}{u^{\gamma_1}|x-y|^{2\gamma_1}}e^{-1/u}\,du d\sigma_y \\
&\qquad = \varkappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}a^{\gamma_1-1}\int_0^{+\infty}u^{-\gamma_1}e^{-1/u}\,du\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}\,.
\end{aligned}$$

Then our assumptions imply the convergence of the integrals in the right hand side and the validity of inequality (7.4). The continuity of $u^\sharp[G_T,\partial_T\Omega,K,\mu]$ follows by the Vitali Convergence Theorem. □

8 Applications to layer heat potentials with essentially bounded densities

Theorem 8.1. *Let $\alpha \in]0,1]$, $\beta \in]0,\alpha[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

(i) If $\mu \in L^\infty(\partial_T\Omega)$, then the double layer heat potential

$$w[\partial_T\Omega,\mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \ \forall (t,x)\in\partial_T\Omega\,, \qquad (8.1)$$

belongs to $B(\overline{]-\infty,T[},C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega))$. Moreover, the operator from $L^\infty(\partial_T\Omega)$ to $B(\overline{]-\infty,T[},C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega))$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous (cf. Remark 2.)

(ii) The operator from $L^\infty(\partial_T\Omega)$ to $C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous (cf. Remark 2.)

(iii) The operator from $L^\infty(\partial_T\Omega)$ to $C^{\beta/2;\beta}(\partial_T\Omega)$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous.

Proof. (i) Let $a \in]16,+\infty[$. Then we already know that $\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)$ belongs to $\mathcal{K}_{\gamma,a}(\partial_T\Omega)$ with γ as in Remark 3 (iv). Clearly,

$$\gamma_1 = (n/2)+1 > 1\,, \qquad 2\gamma_1-\gamma_2-2 = (n-1)-\alpha \in [n-2,n-1[\,,$$

and

$$\gamma_1' = (n/2)+1>1\,, \qquad 2\gamma_1'-\gamma_2'-2 = n-\alpha \begin{cases} >(n-1) & \text{if } \alpha<1\,,\\ =(n-1) & \text{if } \alpha=1\,.\end{cases}$$

$$\gamma_l' + (n-1) - (2\gamma_1'-\gamma_2'-2) = 1+(n-1)-(n-\alpha) = \alpha > 0\,, \qquad \gamma_l' = 1\,.$$

If $\alpha<1$, then Proposition 7.1 (ii) implies that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to

$$B\left(\overline{]-\infty,T[},C^{0,\min\{(n-1)-[(n-1)-\alpha],\alpha\}}(\partial\Omega)\right) = B\left(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega)\right) = B(\overline{]-\infty,T[},C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega))\,.$$

If $\alpha=1$, then Proposition 7.1 (ii) implies that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[},C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega)\right)$. Hence, statement (i) follows. Next we prove statement (ii) and we plan to apply Proposition 7.2. We note that $\gamma_1>1$ and that

$$\left]0,\frac{(n-1)-(2\gamma_1-\gamma_2-2)}{2}\right[\cap]0,1] =]0,\alpha/2[\,.$$

Then we can choose $h\equiv\beta/2$. Next we note that $\gamma_1''>1$, $\gamma_l''=1$ and that

$$\begin{aligned}&(2\gamma_1''-\gamma_2''-2) = n+1-\alpha\,,\\ &\frac{(2\gamma_1''-\gamma_2''-2)-(n-1)}{2} = \frac{2-\alpha}{2} = 1-\frac{\alpha}{2} < 1\,,\\ &\min\{\gamma_1''-1,\gamma_l''\} = \min\{(n/2)+1,1\} = 1\,.\end{aligned}$$

Next we choose $h' \equiv 1-h = 1-(\beta/2)$. Clearly $h' \in]1-(\alpha/2),1[$. Then Proposition 7.2 implies that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to

$$C_b^{0,\min\{h,\gamma_l''-h'\}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right) = C_b^{0,\beta/2}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right) ,$$

and thus statement (ii) holds true. Then Proposition 2.1, statements (i), (ii) and the continuity of the imbedding of $C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega)$ into $C^{0,\beta}(\partial\Omega)$, imply the validity of statement (iii). □

Next we turn to analyze a class of integral operators which we need to study the properties of an integral operator related to the kernel $D_x\Phi_n(t-\tau,x-y)$, and we introduce the following.

Lemma 8.1. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let $Z \in C^0([(\mathrm{cl}\Omega)_T \times \partial_T\Omega]\setminus\Delta_{\partial_T\Omega})$ be such that*

$$Z(t,x,\tau,y) = 0 \qquad \text{if } (t,x,\tau,y) \in [(\mathrm{cl}\Omega)_T \times \partial_T\Omega]\setminus\Delta_{\partial_T\Omega}, \tau \geqslant t\,,$$

and such that

$$\zeta \equiv \sup\Biggl\{ |Z(t,x,\tau,y)|\frac{|t-\tau|^{\frac{n}{2}+1}}{|x-y|}e^{\frac{|x-y|^2}{a(t-\tau)}} : \\ (t,x,\tau,y) \in [(\mathrm{cl}\Omega)_T \times \partial_T\Omega]\setminus\Delta_{\partial_T\Omega}\Biggr\} < +\infty\,.$$

Let $f \in C^{0,\theta}(\mathrm{cl}\Omega)$. Let $H^\sharp[Z,f]$ be the function from $[(\mathrm{cl}\Omega)_T \times \partial_T\Omega]\setminus\Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by

$$H^\sharp[Z,f](t,x,\tau,y) \equiv (f(x)-f(y))Z(t,x,\tau,y) \quad \forall (t,x,\tau,y) \in [(\mathrm{cl}\Omega)_T \times \partial_T\Omega]\setminus\Delta_{\partial_T\Omega}\,.$$

If $\mu \in L^\infty(\partial_T\Omega)$ and if $(t,x) \in (\mathrm{cl}\Omega)_T$, then the function $H^\sharp[Z,f](t,x,\cdot,\cdot)\mu(\cdot,\cdot)$ is Lebesgue integrable in $\partial_T\Omega$ and the function $Q^\sharp[Z,f,\mu]$ from $(\mathrm{cl}\Omega)_T$ to $\mathbb{C}$ defined by

$$Q^\sharp[Z,f,\mu](t,x) \equiv \int_{-\infty}^t\int_{\partial\Omega} H^\sharp[Z,f](t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall (t,x) \in (\mathrm{cl}\Omega)_T\,,$$

is continuous and bounded.

Proof. We plan to apply Lemma 7.1. By definition of ζ, and by the Hölder continuity of f, we have

$$\left|H^\sharp[Z,f](t,x,\tau,y)\right| \leqslant \frac{|f|_\theta|x-y|^{1+\theta}}{|t-\tau|^{(n/2)+1}}\zeta e^{-\frac{|x-y|^2}{a(t-\tau)}}$$

for all $(t,x,\tau,y) \in (\mathrm{cl}\Omega)_T \times \partial_T\Omega\setminus\Delta_{\partial_T\Omega}$. Next we note that

$$(n/2)+1 > 1\,, \qquad 2\left((n/2)+1\right)-(1+\theta)-2 = n-1-\theta < n-1\,,$$

and that the Vitali Convergence Theorem implies the continuity of the function $\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{(n-1)-\theta}}$ in the variable $x \in \mathrm{cl}\Omega$ and accordingly that $\sup_{x\in\mathrm{cl}\Omega}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{(n-1)-\theta}} < +\infty$. Then Lemma 7.1 implies the validity of the statement. □

Lemma 8.2. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let*

$$\begin{aligned}\gamma_n &\equiv ((n/2)+1,1,(n/2)+1,0,1,(n/2)+2,1,1)\,, \\ \gamma_{n,\theta} &\equiv ((n/2)+1,1+\theta,(n/2)+1,1,\theta,(n/2)+2,1+\theta,1)\,,\end{aligned} \tag{8.2}$$

(cf. Remark 3 (iii).) Then the following statements hold.

(i) *The map H from $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\theta}(\partial\Omega)$ to $\mathcal{K}_{\gamma_n,\theta,4a}(\partial_T\Omega)$, which takes (Z,g) to the function $H[Z,g]$ from $(\partial_T\Omega)^2\setminus\Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by*

$$H[Z,g](t,x,\tau,y)\equiv(g(x)-g(y))Z(t,x,\tau,y)\quad\forall(t,x,\tau,y)\in(\partial_T\Omega)^2\setminus\Delta_{\partial_T\Omega}\,,\tag{8.3}$$

is bilinear and continuous.

(ii) *Let $\theta_1\in]0,\theta[$. The map Q from $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\theta}(\partial\Omega)\times L^\infty(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[},C^{0,\omega_\theta(\cdot)}(\partial\Omega)\right)\cap C_b^{0,\theta_1/2}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$, which takes (Z,g,μ) to the function $Q[Z,g,\mu]$ from $\partial_T\Omega$ to $\mathbb{C}$ defined by*

$$Q[Z,g,\mu](t,x)\equiv\int_{-\infty}^t\int_{\partial\Omega}H[Z,g](t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau\quad\forall(t,x)\in\partial_T\Omega\,,\tag{8.4}$$

is trilinear and continuous (cf. Remark 2.)

Proof. Let $x,y\in\partial\Omega$, $x\neq y$, $t,\tau\in\overline{]-\infty,T[}$, $\tau<t$. Then we have

$$|(g(x)-g(y))Z(t,x,\tau,y)|\leqslant|g|_\theta|x-y|^\theta\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\frac{|x-y|}{|t-\tau|^{(n/2)+1}}e^{-\frac{|x-y|^2}{a(t-\tau)}}\,.\tag{8.5}$$

Let t, $\tau\in\overline{]-\infty,T[}$, $\tau<t$, x', $x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Then Lemma 3.1 (i) and the definition of $\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}$ imply that

$$\begin{aligned}
&|(g(x')-g(y))Z(t,x',\tau,y)-(g(x'')-g(y))Z(t,x'',\tau,y)|\\
&\quad\leqslant|g(x')-g(y)|\,|Z(t,x',\tau,y)-Z(t,x'',\tau,y)|+|g(x')-g(x'')|\,|Z(t,x'',\tau,y)|\\
&\quad\leqslant|g|_\theta\frac{|x'-y|^\theta|x'-x''|}{|t-\tau|^{(n/2)+1}}e^{-\frac{|x'-y|^2}{a(t-\tau)}}\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\\
&\qquad+\frac{|x'-x''|^\theta|x''-y|}{|t-\tau|^{(n/2)+1}}e^{-\frac{|x''-y|^2}{a(t-\tau)}}|g|_\theta\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\\
&\quad\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\left\{\frac{|x'-y|^\theta}{|t-\tau|^{(n/2)+1}}|x'-x''|+|x'-x''|^\theta\frac{2|x'-y|}{|t-\tau|^{(n/2)+1}}\right\}e^{-\frac{|x'-y|^2}{4a(t-\tau)}}\,.
\end{aligned}$$

Since $|x'-x''|\leqslant|x'-y|$, we have $|x'-x''|^{1-\theta}\leqslant|x'-y|^{1-\theta}$. Hence,

$$\begin{aligned}
&|x'-y|^\theta|x'-x''|+|x'-x''|^\theta 2|x'-y|\\
&\qquad\leqslant|x'-y|\,|x'-x''|^\theta+|x'-x''|^\theta 2|x'-y|=3|x'-y|\,|x'-x''|^\theta\,,
\end{aligned}$$

and accordingly

$$\begin{aligned}
&|(g(x')-g(y))Z(t,x',\tau,y)-(g(x'')-g(y))Z(t,x'',\tau,y)|\\
&\qquad\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}3\frac{|x'-y|}{|t-\tau|^{(n/2)+1}}|x'-x''|^\theta e^{-\frac{|x'-y|^2}{4a(t-\tau)}}\,.
\end{aligned}\tag{8.6}$$

Now let x, $y\in\partial\Omega$, $x\neq y$, t', $t''\in\overline{]-\infty,T[}$, $t'<t''$, $\tau<t'-2|t'-t''|$. Then the Hölder continuity of g and the definiton of $\|\cdot\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}$ imply that

$$\begin{aligned}
&|(g(x)-g(y))Z(t',x,\tau,y)-(g(x)-g(y))Z(t'',x,\tau,y)|\\
&\qquad\leqslant|x-y|^\theta|g|_\theta|Z(t',x,\tau,y)-Z(t'',x,\tau,y)|\\
&\qquad\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}|x-y|^\theta\frac{|x-y|}{|t'-\tau|^{(n/2)+2}}|t'-t''|e^{-\frac{|x-y|^2}{a(t'-\tau)}}\,.
\end{aligned}\tag{8.7}$$

Then inequalities (8.5)–(8.7) imply the validity of statement (i). Next we consider statement (ii). By Proposition 7.1 (ii) with $\gamma = \gamma_{n,\theta}$, the map $u[\partial_T\Omega, \cdot, \cdot]$ is bilinear and continuous from $\mathcal{K}_{\gamma_{n,\theta},4a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ to $B\left(\overline{]-\infty, T[}, C^{0,\max\{r^\theta,\omega_\theta(\cdot)\}}(\partial\Omega)\right)$. Indeed,

$$
\begin{aligned}
&\gamma_1 = (n/2) + 1 > 1\,,\\
&2\gamma_1 - \gamma_2 - 2 = 2((n/2)+1) - (1+\theta) - 2 = (n-1) - \theta \in [n-2, n-1[\,,\\
&\gamma_1' = (n/2) + 1 > 1\,,\\
&\gamma_l' = \theta \in]0,1]\,,\\
&2\gamma_1' - \gamma_2' - 2 = 2((n/2)+1) - 1 - 2 = n-1\,,\\
&(n-1) - (2\gamma_1 - \gamma_2 - 2) = (n-1) - [2((n/2)+1) - (1+\theta) - 2] = \theta\,,\\
&\gamma_l' + (n-1) - (2\gamma_1' - \gamma_2' - 2) = \theta + (n-1) - (n-1) = \theta > 0\,,
\end{aligned}
$$

and $C^{0,\max\{r^\theta,\omega_\theta(\cdot)\}}(\partial\Omega) = C^{0,\omega_\theta(\cdot)}(\partial\Omega)$.

Next we wish to apply Proposition 7.2 with $\gamma = \gamma_{n,\theta}$. Clearly, $\gamma_1 = (n/2)+1 > 1$. Moreover, we have seen above that $2\gamma_1 - \gamma_2 - 2 = (n-1) - \theta \in [n-2, n-1[$. Then we can choose

$$
h \equiv \theta_1/2 \in]0, \theta/2[= \left]0, \frac{(n-1) - [(n-1) - \theta]}{2}\right[\cap]0,1]\,.
$$

Next we observe that $\gamma_1'' = (n/2) + 2 > 1$, $\gamma_l'' = 1$ and that

$$
\begin{aligned}
&\frac{(2\gamma_1'' - \gamma_2'' - 2) - (n-1)}{2} = \frac{2((n/2)+2) - (1+\theta) - 2 - (n-1)}{2} = 1 - (\theta/2) > 0,\\
&\gamma_1'' - 1 = ((n/2)+2) - 1 = (n/2) + 1 > 1\,,\\
&\min\{\gamma_1'' - 1, \gamma_l''\} = \min\{(n/2)+1, 1\} = 1 > 1 - (\theta/2)\,.
\end{aligned}
$$

Since $1 - (\theta/2) < 1 - (\theta_1/2) = 1 - h < 1$, we can choose

$$
h' \in]1 - (\theta/2), 1[\,,
$$

close enough to $1 - (\theta/2)$ so that $h' < 1 - (\theta_1/2) = 1 - h$, *i.e.*, such that $h < 1 - h'$. Then $\min\{h, \gamma_l'' - h'\} = \min\{h, 1 - h'\} = h$ and Proposition 7.2 implies that the map $u[\partial_T\Omega, \cdot, \cdot]$ is bilinear and continuous from $\mathcal{K}_{\gamma_{n,\theta},4a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ to $C_b^{0,\theta_1/2}\left(\overline{]-\infty, T[}, C^0(\partial\Omega)\right)$. Hence, statement (i) implies the validity of statement (ii). □

Remark 4. Under the assumptions of the previous proposition, Proposition 2.1 implies that $B\left(\overline{]-\infty, T[}, C^{0,\omega_\theta(\cdot)}(\partial\Omega)\right) \cap C_b^{0,\theta_1/2}\left(\overline{]-\infty, T[}, C^0(\partial\Omega)\right)$ is continuously imbedded into $C^{\theta_1/2,\theta_1}(\partial_T\Omega)$.

Then Remark 3 (iii), Lemma 8.2 and Remark 4 immediately imply the validity of the following.

Theorem 8.2. *Let $T \in]-\infty, +\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$, $\theta_1 \in]0, \theta[$. Let $r \in \{1, \ldots, n\}$. Then the map $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), \cdot, \cdot]$ from $C^{0,\theta}(\partial\Omega) \times L^\infty(\partial_T\Omega)$ to $C^{\theta_1/2,\theta_1}(\partial_T\Omega)$ which takes (g, μ) to the function*

$$
\begin{aligned}
&Q[\partial_{x_r}\Phi_n(t-\tau, x-y), g, \mu](t,x) \qquad (8.8)\\
&\quad \equiv \int_{-\infty}^t \int_{\partial\Omega} (g(x) - g(y))\partial_{x_r}\Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \quad \forall (t,x) \in \partial_T\Omega\,,
\end{aligned}
$$

is bilinear and continuous (cf. Remark 2.)

Next we turn to analyze a class of integral operators which we need to study the properties of an integral operator related to the kernel $\partial_t \Phi_n(t-\tau, x-y)$, and we introduce the following two statements.

Lemma 8.3. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let $\tilde{Z} \in C^0([(\mathrm{cl}\Omega)_T \times \partial_T\Omega] \setminus \Delta_{\partial_T\Omega})$ be such that*

$$\tilde{Z}(t,x,\tau,y) = 0 \qquad \text{if } (t,x,\tau,y) \in [(\mathrm{cl}\Omega)_T \times \partial_T\Omega] \setminus \Delta_{\partial_T\Omega}, \tau \geqslant t\,,$$

and such that

$$\tilde{\zeta} \equiv \sup\Big\{ |\tilde{Z}(t,x,\tau,y)||t-\tau|^{\frac{n}{2}+1} e^{\frac{|x-y|^2}{a(t-\tau)}} : (t,x,\tau,y) \in [(\mathrm{cl}\Omega)_T \times \partial_T\Omega] \setminus \Delta_{\partial_T\Omega} \Big\} < +\infty\,.$$

Let $f \in C^{0,\theta}(\mathrm{cl}\Omega)$ and $\varphi \in C_b^{0,1/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))$. Let $\tilde{H}^\sharp[\tilde{Z}, f, \varphi]$ be the function from $[(\mathrm{cl}\Omega)_T \times \partial_T\Omega] \setminus \Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by

$$\tilde{H}^\sharp[\tilde{Z}, f, \varphi](t,x,\tau,y) \equiv (f(x)-f(y))\tilde{Z}(t,x,\tau,y)(\varphi(\tau,y)-\varphi(t,y)) \qquad \forall (t,x,\tau,y) \in [(\mathrm{cl}\Omega)_T \times \partial_T\Omega] \setminus \Delta_{\partial_T\Omega}\,.$$

If $(t,x) \in (\mathrm{cl}\Omega)_T$, then the function $\tilde{H}^\sharp[\tilde{Z}, f, \varphi](t,x,\cdot,\cdot)$ is Lebesgue integrable in $\partial_T\Omega$ and the function $\tilde{Q}^\sharp[Z, f, \varphi]$ from $(\mathrm{cl}\Omega)_T$ to $\mathbb{C}$ defined by

$$\tilde{Q}^\sharp[\tilde{Z}, f, \varphi](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega} \tilde{H}^\sharp[\tilde{Z}, f, \varphi](t,x,\tau,y)\, d\sigma_y d\tau \qquad \forall (t,x) \in (\mathrm{cl}\Omega)_T\,,$$

is continuous and bounded.

Proof. We plan to apply Lemma 7.1. By definition of $\tilde{\zeta}$ and of $\tilde{H}[\tilde{Z}, f, \varphi]$, we have

$$\left|\tilde{H}^\sharp[\tilde{Z}, \tilde{f}](t,x,\tau,y)\right| \leqslant \frac{|f|_\theta \|\varphi\|_{C_b^{0,1/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))}|x-y|^\theta}{|t-\tau|^{(n+1)/2}} \tilde{\zeta} e^{-\frac{|x-y|^2}{a(t-\tau)}}\,,$$

for all $(t,x,\tau,y) \in (\mathrm{cl}\Omega)_T \times \partial_T\Omega \setminus \Delta_{\partial_T\Omega}$. Next we note that

$$(n+1)/2 > 1\,, \qquad 2((n+1)/2) - \theta - 2 = n-1-\theta < n-1\,,$$

and that Vitali Convergence Theorem implies the continuity of the function $\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{(n-1)-\theta}}$ in the variable $x \in \mathrm{cl}\Omega$ and accordingly that $\sup_{x\in\mathrm{cl}\Omega} \int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{(n-1)-\theta}} < +\infty$. Then Lemma 7.1 with $\mu \equiv 1$ implies the validity of the statement. □

Proposition 8.1. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let $b \in [1/2,1]$, $b_1 \in]0,b[$. If $n=2$, then we further assume that $b<1$. Let*

$$\begin{aligned} \gamma_n^\sharp &\equiv ((n/2)+1, 0, (n/2)+2, 1, 1, (n/2)+2, 0, 1)\,, \\ \gamma_{n,\theta,b}^\sharp &\equiv ((n/2)+1-b, \theta, (n/2)+1-b, 0, \theta, (n/2)+1-(b-b_1), \theta, b_1)\,, \end{aligned} \tag{8.9}$$

(cf. Remark 3 (ii).) Then the following statements hold.

(i) *The map* $\tilde{H}$ *from* $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)\times C^{0,\theta}(\partial\Omega)\times C^{0,b;0,1}(\partial_T\Omega)$ *to* $\mathcal{K}_{\gamma_{n,\theta,b}^\sharp,5a}(\partial_T\Omega)$*, which takes* (Z,g,φ) *to the function from* $(\partial_T\Omega)^2\setminus\Delta_{\partial_T\Omega}$ *to* $\mathbb{C}$ *defined by*

$$\tilde{H}[Z,g,\varphi](t,x,\tau,y)\equiv(g(x)-g(y))Z(t,x,\tau,y)(\varphi(\tau,y)-\varphi(t,y))\,,$$

for all $(t,x,\tau,y)\in(\partial_T\Omega)^2\setminus\Delta_{\partial_T\Omega}$ *is trilinear and continuous.*

(ii) *Let* $2b+\theta\leqslant 2$*. Let*

$$\tilde{\omega}_{b,\theta}(r)\equiv\begin{cases} r^\theta & \text{if } b\in]1/2,1]\,,\\ \omega_\theta(r) & \text{if } b=1/2\,,\end{cases}$$

for all $r\in]0,+\infty[$*. The map* $\tilde{Q}$ *from* $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)\times C^{0,\theta}(\partial\Omega)\times C^{0,b;0,1}(\partial_T\Omega)$ *to* $B(\overline{]-\infty,T[},C^{0,\tilde{\omega}_{b,\theta}(\cdot)}(\partial\Omega))$ *which takes* (Z,g,φ) *to the function from* $\partial_T\Omega$ *to* $\mathbb{C}$ *defined by*

$$\tilde{Q}[Z,g,\varphi](t,x)\equiv\int_{-\infty}^{t}\int_{\partial\Omega}\tilde{H}[Z,g,\varphi](\tau,y)\,d\sigma_y d\tau\,,$$

for all $(t,x)\in\partial_T\Omega$ *is trilinear and continuous.*

(iii) *The interval* $]\max\{0,[1-\theta-2(b-b_1)]/2\},\min\{(n/2)-(b-b_1),b_1\}]$ *is not empty and the map* $\tilde{Q}$ *from* $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)\times C^{0,\theta}(\partial\Omega)\times C^{0,b;0,1}(\partial_T\Omega)$ *to* $C_b^{0,\min\{h,b_1-b_2\}}(\overline{]-\infty,T[},C^0(\partial\Omega))$ *is trilinear and continuous for all*

$$h\in]0,(2b+\theta-1)/2[\,,b_2\in]\max\{0,[1-\theta-2(b-b_1)]/2\},\min\{(n/2)-(b-b_1),b_1\}]\,.$$

Proof. We first consider statement (i). Let $x,y\in\partial\Omega$, $x\neq y$, t, $\tau\in\overline{]-\infty,T[}$, $\tau<t$. Then we have

$$\begin{aligned}
&|(g(x)-g(y))Z(t,x,\tau,y)|\,|\varphi(\tau,y)-\varphi(t,y)| \qquad (8.10)\\
&\quad\leqslant |g|_\theta\frac{|x-y|^\theta}{|t-\tau|^{(n/2)+1}}e^{-\frac{|x-y|^2}{a(t-\tau)}}\|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}|\tau-t|^b\\
&\quad=|g|_\theta\|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}\frac{|x-y|^\theta}{|t-\tau|^{(n/2)+1-b}}e^{-\frac{|x-y|^2}{a(t-\tau)}}\,.
\end{aligned}$$

Let t, $\tau\in\overline{]-\infty,T[}$, $\tau<t$, x', $x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Then Lemma 3.1 (i), and the definition of $\|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)}$, and the Hölder continuity of g, φ, and the triangular inequality imply that

$$\begin{aligned}
&|(g(x')-g(y))Z(t,x',\tau,y)(\varphi(\tau,y)-\varphi(t,y)) \qquad (8.11)\\
&\qquad\qquad -(g(x'')-g(y))Z(t,x'',\tau,y)(\varphi(\tau,y)-\varphi(t,y))|\\
&\quad\leqslant\Big\{|g(x')-g(y)|\,|Z(t,x',\tau,y)-Z(t,x'',\tau,y)|\\
&\qquad\qquad +|g(x')-g(x'')|\,|Z(t,x'',\tau,y)|\Big\}|\varphi(\tau,y)-\varphi(t,y)|\\
&\quad\leqslant |g|_\theta\|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}\Big\{\frac{|x'-y|^{\theta+1}}{|t-\tau|^{(n/2)+2}}|x'-x''|e^{-\frac{|x'-y|^2}{a(t-\tau)}}\\
&\qquad\qquad +\frac{|x'-x''|^\theta}{|t-\tau|^{(n/2)+1}}e^{-\frac{|x''-y|^2}{a(t-\tau)}}\Big\}|t-\tau|^b\,.
\end{aligned}$$

Since $|x'-x''|\leqslant|x'-y|$, we have $|x'-x''|^{1-\theta}\leqslant|x'-y|^{1-\theta}$. Moreover, Lemma 3.1 (i) implies that $|x''-y|\geqslant\frac{1}{2}|x'-y|$. Then Lemma 3.4 (v) implies that the right hand side of (8.11) is less or equal to

$$
\begin{aligned}
&|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}\biggl\{\frac{|x'-y|^2|x'-x''|^\theta}{|t-\tau|^{(n/2)+2}}e^{-\frac{|x'-y|^2}{a(t-\tau)}} \qquad (8.12)\\
&\qquad+\frac{|x'-x''|^\theta}{|t-\tau|^{(n/2)+1}}e^{-\frac{|x'-y|^2}{4a(t-\tau)}}\biggr\}|t-\tau|^b\\
&\quad\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}\frac{|x'-x''|^\theta}{|t-\tau|^{(n/2)+1-b}}\biggl\{\frac{|x'-y|^2}{t-\tau}+1\biggr\}e^{-\frac{|x'-y|^2}{4a(t-\tau)}}\\
&\quad\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}\frac{|x'-x''|^\theta}{|t-\tau|^{(n/2)+1-b}}C(4a,5a,1)e^{-\frac{|x'-y|^2}{5a(t-\tau)}}\,.
\end{aligned}
$$

Let $x,\,y\in\partial\Omega$, $x\neq y$, $t',\,t''\in\overline{]-\infty,T[}$, $t'<t''$, $\tau<t'-2|t'-t''|$. Then inequality (3.1) and the triangular inequality imply that

$$
\begin{aligned}
&|(g(x)-g(y))Z(t',x,\tau,y)(\varphi(\tau,y)-\varphi(t',y)) \qquad (8.13)\\
&\qquad\qquad-(g(x)-g(y))Z(t'',x,\tau,y)(\varphi(\tau,y)-\varphi(t'',y))|\\
&\leqslant|(g(x)-g(y))Z(t',x,\tau,y)|\,|(\varphi(\tau,y)-\varphi(t',y))-(\varphi(\tau,y)-\varphi(t'',y))|\\
&\quad+|\varphi(\tau,y)-\varphi(t'',y)|\,|(g(x)-g(y))Z(t',x,\tau,y)-(g(x)-g(y))Z(t'',x,\tau,y)|\\
&\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}\biggl\{\frac{|x-y|^\theta|t'-t''|^b}{(t'-\tau)^{(n/2)+1}}e^{-\frac{|x-y|^2}{a(t'-\tau)}}\\
&\quad+\frac{|\tau-t''|^b|x-y|^\theta|t'-t''|}{(t'-\tau)^{(n/2)+2}}e^{-\frac{|x-y|^2}{a(t'-\tau)}}\biggr\}\\
&\leqslant|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}|t'-t''|^{b_1}\biggl\{\frac{|x-y|^\theta|t'-t''|^{b-b_1}}{(t'-\tau)^{(n/2)+1}}\\
&\quad+2^b\frac{|x-y|^\theta|t'-t''|^{1-b_1}}{(t'-\tau)^{(n/2)+2-b}}\biggr\}e^{-\frac{|x-y|^2}{a(t'-\tau)}}\\
&\leqslant2|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}|t'-t''|^{b_1}\biggl\{\frac{|x-y|^\theta}{(t'-\tau)^{(n/2)+1-(b-b_1)}}\\
&\quad+\frac{|x-y|^\theta}{(t'-\tau)^{(n/2)+2-b-(1-b_1)}}\biggr\}e^{-\frac{|x-y|^2}{a(t'-\tau)}}\\
&\leqslant4|g|_\theta\|Z\|_{\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)}\|\varphi\|_{C^{0,b;0,1}(\partial_T\Omega)}|t'-t''|^{b_1}\frac{|x-y|^\theta}{(t'-\tau)^{(n/2)+1-(b-b_1)}}e^{-\frac{|x-y|^2}{a(t'-\tau)}}\,.
\end{aligned}
$$

Then inequalities (8.10)–(8.13) imply the validity of statement (i).

We now prove statement (ii). We distinguish case $b\in]1/2,1]$ and case $b=1/2$ and we first consider case $b\in]1/2,1]$. By Proposition 7.1 (ii) with $\gamma=\gamma^\sharp_{n,\theta,b}$, the map $u[\partial_T\Omega,\cdot,1]$ is linear and continuous from $\mathcal{K}_{\gamma^\sharp_{n,\theta,b},5a}(\partial_T\Omega)$ to

$$
\begin{aligned}
&B(\overline{]-\infty,T[},C^{0,\min\{(n-1)-(n-2b-\theta),\theta\}}(\partial\Omega))\\
&\qquad=B(\overline{]-\infty,T[},C^{0,\min\{2b+\theta-1,\theta\}}(\partial\Omega))=B(\overline{]-\infty,T[},C^{0,\theta}(\partial\Omega))\,.
\end{aligned}
$$

Indeed, by our assumptions we have $1 < 2b + \theta \leqslant 2$ and

$$\begin{aligned} &\gamma_1 = (n/2) + 1 - b > 1\,, \\ &2\gamma_1 - \gamma_2 - 2 = 2[(n/2) + 1 - b] - \theta - 2 \\ &\qquad\qquad = n - 2b - \theta = (n-1) - (2b + \theta - 1) \in [n-2, n-1[\,, \\ &\gamma_1' = (n/2) + 1 - b > 1\,, \\ &\gamma_l' = \theta \in]0,1]\,, \\ &\gamma_l' + (n-1) - (2\gamma_1' - \gamma_2' - 2) = \theta + (n-1) - (n-2b) = \theta - 1 + 2b > 0\,. \end{aligned} \tag{8.14}$$

Moreover, assumption $b > 1/2$ implies that

$$2\gamma_1' - \gamma_2' - 2 = 2((n/2) + 1 - b) - 2 = n - 2b < n - 1\,,$$

and that

$$(n-1) - (2\gamma_1 - \gamma_2 - 2) = 2b + \theta - 1 > \theta = \gamma_l'\,.$$

Hence, statement (i) implies the validity of statement (ii) for $b \in]1/2, 1]$. Next we consider the case $b = 1/2$. Since

$$2\gamma_1' - \gamma_2' - 2 = n - 1\,, \qquad (n-1) - (2\gamma_1 - \gamma_2 - 2) = \theta\,,$$

statement (i) and Proposition 7.1 (ii) imply the validity of statement (ii).

We now turn to prove statement (iii). We first note that the assumptions $b \in [1/2, 1]$ and $\theta > 0$ imply that $[1 - \theta - 2(b - b_1)]/2 < b_1$ and that accordingly the interval $]\max\{0, [1-\theta-2(b-b_1)]/2\}, \min\{(n/2)-(b-b_1), b_1\}]$ is not empty. Indeed, $[1-\theta-2(b-b_1)]/2 < (n/2) - (b - b_1)$. Next we plan to exploit Proposition 7.2 with $\gamma = \gamma^\sharp_{n,\theta,b}$. By assumption and by the equalities in (8.14), we have

$$h \in]0, (2b + \theta - 1)/2[=]0, [(n-1) - (2\gamma_1 - \gamma_2 - 2)]/2[\cap]0, 1]\,.$$

Next we observe that $\gamma_1'' = (n/2) + 1 - (b - b_1) > 1$, $\gamma_l'' = b_1 \in]0, 1]$ and that

$$\begin{aligned} \frac{(2\gamma_1'' - \gamma_2'' - 2) - (n-1)}{2} &= \frac{n + 2 - 2(b - b_1) - \theta - 2 - (n-1)}{2} \\ &= \frac{(1-\theta) - 2(b - b_1)}{2} < \gamma_1'' - 1 = (n/2) + 1 - (b - b_1) - 1 \\ &= (n/2) - (b - b_1) > 0\,, \\ \frac{(2\gamma_1'' - \gamma_2'' - 2) - (n-1)}{2} &= \frac{(1-\theta) - 2(b - b_1)}{2} < \gamma_l'' = b_1 > 0\,. \end{aligned}$$

Indeed, $(1 - \theta) - 2(b - b_1) < n - 2(b - b_1)$ and $(1 - \theta) - 2(b - b_1) < 2b_1$. By assumption,

$$b_2 \in]\max\{0, [(1-\theta) - 2(b - b_1)]/2\}, \min\{(n/2) - (b - b_1), b_1\}]\,.$$

Then the map $u[\partial_T\Omega, \cdot, 1]$ from $\mathcal{K}_{\gamma^\sharp_{n,\theta,b}, 5a}(\partial_T\Omega)$ to $C_b^{0,\min\{h, b_1 - b_2\}}(\overline{]-\infty, T[}, C^0(\partial\Omega))$ is linear and continuous. Hence, statement (i) implies the validity of statement (iii). $\square$

9 Integral operators on the space of Hölder continuous functions

Next we consider the action of $u[\partial_T\Omega, K, \cdot]$ in case the functional variable μ is Hölder continuous.

Proposition 9.1. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma \in \mathbb{R}^8$ be as in (6.1). Let γ_l', $\gamma_l'' \in]0,1]$. Let $\gamma_1 > 1$, $2\gamma_1 - \gamma_2 - 2 \in [n-2, n-1[$. Then the following statements hold.*

(i) Let $\eta_2 \in]0,1[$. Let $\gamma_1' - (\eta_2/2) > 1$, $(n-1) - (2\gamma_1' - \gamma_2' - 2 - \eta_2) + \gamma_l' > 0$. Let

$$\omega(r) \equiv \begin{cases} r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),\gamma_l'\}} \\ \qquad\qquad \text{if } 2\gamma_1' - \gamma_2' - 2 - \eta_2 < n-1\,, \\ \max\{r^{(n-1)-(2\gamma_1-\gamma_2-2)}, \omega_{\gamma_l'}(r)\} \\ \qquad\qquad \text{if } 2\gamma_1' - \gamma_2' - 2 - \eta_2 = n-1\,, \\ r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),(n-1)-(2\gamma_1'-\gamma_2'-2-\eta_2)+\gamma_l'\}} \\ \qquad\qquad \text{if } 2\gamma_1' - \gamma_2' - 2 - \eta_2 > n-1\,, \end{cases}$$

for all $r \in]0,+\infty[$. Then there exists $c_1 > 0$ such that the function $u[\partial_T\Omega, K, \mu]$ defined by (7.1) satisfies the following inequality

$$\begin{aligned} &|u[\partial_T\Omega, K, \mu](t,x') - u[\partial_T\Omega, K, \mu](t,x'')| \\ &\qquad \leqslant c_1 \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C^{\eta_2/2;\eta_2}(\partial_T\Omega)} \omega(|x'-x''|) \\ &\qquad\quad + \|\mu\|_{L^\infty(\partial_T\Omega)} |u[\partial_T\Omega, K, 1](t,x') - u[\partial_T\Omega, K, 1](t,x'')|\,, \end{aligned} \tag{9.1}$$

for all x', $x'' \in \partial\Omega$, $t \in \overline{]-\infty,T[}$, and for all $(K,\mu) \in \mathcal{K}_{\gamma,a}(\partial_T\Omega) \times C^{\eta_2/2;\eta_2}(\partial_T\Omega)$.

(ii) Let $\eta_1 \in]0,2[$, $\eta_2 \in]0,\eta_1]$. Let

$$\begin{aligned} &\gamma_1 - (\eta_2/2) > 1\,, \\ &2\gamma_1 - \gamma_2 - 2 + (\eta_1 - \eta_2) < n-1\,, \\ &\gamma_1'' - (\eta_2/2) > 1\,, \\ &\gamma_l'' < \gamma_1'' - 1 + 2^{-1}(\eta_1 - \eta_2)\,, \\ &\eta_1 < 2\gamma_l''\,, \\ &2\gamma_1'' - \gamma_2'' - 2 - 2\gamma_l'' + (\eta_1 - \eta_2) < (n-1)\,. \end{aligned}$$

Then there exists $c_2 > 0$ such that the function $u[\partial_T\Omega, K, \mu]$ defined by (7.1) satisfies the following inequality

$$\begin{aligned} &|u[\partial_T\Omega, K, \mu](t',x) - u[\partial_T\Omega, K, \mu](t'',x)| \\ &\qquad \leqslant c_2 \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{\eta_1/2} \\ &\qquad\quad + |u[\partial_T\Omega, K, \mu(t',\cdot)](t',x) - u[\partial_T\Omega, K, \mu(t',\cdot)](t'',x)|\,, \end{aligned} \tag{9.2}$$

for all $x \in \partial\Omega$, t', $t'' \in \overline{]-\infty,T[}$, $t' < t''$, and for all $(K,\mu) \in \mathcal{K}_{\gamma,a}(\partial_T\Omega) \times C_b^{0,\eta_2/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))$.

Proof. We first consider inequality (9.1). Let x', $x'' \in \partial\Omega$, $t \in \overline{]-\infty,T[}$. By Remark 1 and by Proposition 7.1 (i), it suffices to consider case $0 < |x'-x''| < r_{\gamma_l'}$. By the triangular inequality and by the inclusion $\mathbb{B}_n(x', 2|x'-x''|) \subseteq \mathbb{B}_n(x'', 3|x'-x''|)$, we have

$$\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t,x') - u[\partial_T\Omega, K, \mu](t,x'')| \\
&\quad \leqslant |[u[\partial_T\Omega, K, \mu](t,x') - \mu(t,x')u[\partial_T\Omega, K, 1](t,x')] \\
&\qquad -[u[\partial_T\Omega, K, \mu](t,x'') - \mu(t,x')u[\partial_T\Omega, K, 1](t,x'')]| \\
&\qquad +|\mu(t,x')|\,|u[\partial_T\Omega, K, 1](t,x') - u[\partial_T\Omega, K, 1](t,x'')| \\
&\quad \leqslant \int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(t,x',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} |K(t,x',\tau,y)-K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad +\|\mu\|_{L^\infty(\partial_T\Omega)}|u[\partial_T\Omega, K, 1](t,x') - u[\partial_T\Omega, K, 1](t,x'')|\,.
\end{aligned} \tag{9.3}$$

We now estimate the sum of the first two terms in the right hand side of (9.3). By Lemma 3.3 (ii), we have

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(t,x',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\quad \leqslant 2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)}\Big\{\int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}} e^{-\frac{|x'-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}} e^{-\frac{|x''-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau\Big\} \\
&\quad \leqslant 2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)} \\
&\qquad \times\Big\{\int_{0}^{+\infty}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2+2}a^{-1+\gamma_1}}{|u|^{\gamma_1}|x'-y|^{2\gamma_1}} e^{-1/u}\,d\sigma_y du \\
&\qquad + \int_{0}^{+\infty}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2+2}a^{-1+\gamma_1}}{|u|^{\gamma_1}|x''-y|^{2\gamma_1}} e^{-1/u}\,d\sigma_y du\Big\} \\
&\quad \leqslant 4\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)}\frac{\Gamma(\gamma_1-1)}{a^{\gamma_1-1}}c''_{\Omega,2\gamma_1-\gamma_2-2}|x'-x''|^{(n-1)-(2\gamma_1-\gamma_2-2)}\,.
\end{aligned} \tag{9.4}$$

We now consider the third term in the right hand side of (9.3).

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} |K(t,x',\tau,y)-K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad \leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x'-y|^{\gamma'_2}}{|t-\tau|^{\gamma'_1}}|x'-x''|^{\gamma'_l}e^{-\frac{|x'-y|^2}{a(t-\tau)}} \\
&\qquad \times[|\mu(\tau,y)-\mu(\tau,x')| + |\mu(\tau,x')-\mu(t,x')|]\,d\sigma_y d\tau \\
&\qquad \leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C^{\eta_2/2;\eta_2}(\partial_T\Omega)}
\end{aligned}$$

$$\times\Big\{\int_0^{+\infty}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{|x'-y|^{\gamma_2'+2+\eta_2}a^{-1+\gamma_1'}}{u^{\gamma_1'}|x'-y|^{2\gamma_1'}}|x'-x''|^{\gamma_l'}e^{-1/u}\,d\sigma_y du$$
$$+\int_0^{+\infty}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{|x'-y|^{\gamma_2'+2}a^{-1+\gamma_1'-(\eta_2/2)}}{u^{\gamma_1'-(\eta_2/2)}|x'-y|^{2\gamma_1'-\eta_2}}|x'-x''|^{\gamma_l'}e^{-1/u}\,d\sigma_y du\Big\}$$
$$\leqslant 2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C^{\eta_2/2;\eta_2}(\partial_T\Omega)}|x'-x''|^{\gamma_l'}$$
$$\times\max\Big\{\frac{\Gamma(\gamma_1'-1)}{a^{\gamma_1'-1}},\frac{\Gamma(\gamma_1'-\frac{\eta_2}{2}-1)}{a^{\gamma_1'-\frac{\eta_2}{2}-1}}\Big\}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\,. \tag{9.5}$$

At this point we distinguish three cases. If $2\gamma_1'-\gamma_2'-2-\eta_2<n-1$, then Lemma 3.3 (i) implies that

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leqslant\int_{\partial\Omega}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leqslant c'_{\Omega,2\gamma_1'-\gamma_2'-2-\eta_2}\,,$$

and thus inequalities (9.3)–(9.5) imply that there exists $c_1>0$ such that inequality (9.1) holds with $\omega(r)$ as in statement (i). If $2\gamma_1'-\gamma_2'-2-\eta_2=n-1$, then Lemma 3.3 (iv) implies that

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leqslant c_\Omega^{iv}|\ln|x'-x''||\,,$$

and thus inequalities (9.3)–(9.5) imply that there exists $c_1>0$ such that inequality (9.1) holds with $\omega(r)$ as in statement (i). If $2\gamma_1'-\gamma_2'-2-\eta_2>n-1$, then Lemma 3.3 (iii) implies that

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leqslant c'''_{\Omega,2\gamma_1'-\gamma_2'-2-\eta_2}|x'-x''|^{(n-1)-(2\gamma_1'-\gamma_2'-2-\eta_2)}\,,$$

and thus inequalities (9.3)–(9.5) imply that there exists $c_1>0$ such that inequality (9.1) holds with $\omega(r)$ as in statement (i).

Next we consider statement (ii). Let $x\in\partial\Omega$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$. By Remark 1 and by Proposition 7.1 (i), it suffices to consider case $0<|t'-t''|<1$. By inequalities (3.1), and by the inclusion $]t'-2|t'-t''|,t'+2|t'-t''|[\subseteq]t''-3|t'-t''|,t''+3|t'-t''|[$, we have

$$\begin{aligned}&|u[\partial_T\Omega,K,\mu](t',x)-u[\partial_T\Omega,K,\mu](t'',x)|\\&\quad\leqslant|[u[\partial_T\Omega,K,\mu](t',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t',x)]\\&\qquad-[u[\partial_T\Omega,K,\mu](t'',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t'',x)]|\\&\qquad+|u[\partial_T\Omega,K,\mu(t',\cdot)](t',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t'',x)|\\&\quad\leqslant\int_{t'-2|t'-t''|}^{t'+2|t'-t''|}\int_{\partial\Omega}|K(t',x,\tau,y)|\,|\mu(\tau,y)-\mu(t',y)|\,d\sigma_yd\tau\\&\qquad+\int_{t''-3|t'-t''|}^{t''+3|t'-t''|}\int_{\partial\Omega}|K(t'',x,\tau,y)|\,|\mu(\tau,y)-\mu(t',y)|\,d\sigma_yd\tau\\&\qquad+\int_{-\infty}^{t'-2|t'-t''|}\int_{\partial\Omega}|(K(t',x,\tau,y)-K(t'',x,\tau,y))(\mu(\tau,y)-\mu(t',y))|\,d\sigma_yd\tau\\&\qquad+|u[\partial_T\Omega,K,\mu(t',\cdot)](t',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t'',x)|\,.\end{aligned}\tag{9.6}$$

We now estimate the first two summands in the right hand side of (9.6). By Lemmas 3.3 (i) and 3.4 (iii), and by the elementary inequality $|t'-\tau|^{\eta_2/2} \leqslant |t'-t''|^{\eta_2/2} + |t''-\tau|^{\eta_2/2}$, we have

$$
\begin{aligned}
&\int_{t'-2|t'-t''|}^{t'+2|t'-t''|} \int_{\partial\Omega} |K(t',x,\tau,y)|\, |\mu(\tau,y)-\mu(t',y)|\, d\sigma_y d\tau \qquad (9.7)\\
&\quad + \int_{t''-3|t'-t''|}^{t''+3|t'-t''|} \int_{\partial\Omega} |K(t'',x,\tau,y)|\, |\mu(\tau,y)-\mu(t',y)|\, d\sigma_y d\tau \\
&\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} \\
&\quad \times \Big\{ \int_{t'-2|t'-t''|}^{t'} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t'-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t'-\tau)}} |t'-\tau|^{(\eta_2/2)}\, d\sigma_y d\tau \\
&\quad + |t'-t''|^{\eta_2/2} \int_{t''-3|t'-t''|}^{t''} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t''-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t''-\tau)}}\, d\sigma_y d\tau \\
&\quad + \int_{t''-3|t'-t''|}^{t''} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t''-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t''-\tau)}} |t''-\tau|^{\eta_2/2}\, d\sigma_y d\tau \Big\} \\
&\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} \\
&\quad \times \Big\{ \int_{\partial\Omega} \int_0^{\frac{2a|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2} a^{-1+\gamma_1-(\eta_2/2)}}{u^{\gamma_1-(\eta_2/2)} |x-y|^{2\gamma_1-\eta_2}} e^{-1/u}\, du d\sigma_y \\
&\quad + |t'-t''|^{\eta_2/2} \int_{\partial\Omega} \int_0^{\frac{3a|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2}}{u^{\gamma_1}|x-y|^{2\gamma_1}} a^{-1+\gamma_1} e^{-1/u}\, du d\sigma_y \\
&\quad + \int_{\partial\Omega} \int_0^{\frac{3a|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2} a^{-1+\gamma_1-(\eta_2/2)}}{u^{\gamma_1-(\eta_2/2)} |x-y|^{2\gamma_1-\eta_2}} e^{-1/u}\, du d\sigma_y \Big\} \\
&\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} \max\{a^{-1+\gamma_1-(\eta_2/2)}, a^{-1+\gamma_1}\} \\
&\quad \times \Big\{ 2 \int_{\partial\Omega} \tilde{D}_{\gamma_1-(\eta_2/2),\eta_1/2} \left(\frac{3a|t'-t''|}{|x-y|^2}\right)^{\eta_1/2} \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2-\eta_2}} \\
&\quad + |t'-t''|^{\eta_2/2} \int_{\partial\Omega} \tilde{D}_{\gamma_1,(\eta_1-\eta_2)/2} \left(\frac{3a|t'-t''|}{|x-y|^2}\right)^{(\eta_1-\eta_2)/2} \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}} \Big\} \\
&\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} \max\{a^{-1+\gamma_1-(\eta_2/2)}, a^{-1+\gamma_1}\} \\
&\quad \times \Big\{ 2\tilde{D}_{\gamma_1-(\eta_2/2),\eta_1/2} (3a)^{\eta_1/2} c'_{\Omega,2\gamma_1-\gamma_2-2+(\eta_1-\eta_2)} \\
&\quad + (3a)^{(\eta_1-\eta_2)/2} \tilde{D}_{\gamma_1,(\eta_1-\eta_2)/2} c'_{\Omega,2\gamma_1-\gamma_2-2+(\eta_1-\eta_2)} \Big\} |t'-t''|^{\eta_1/2}.
\end{aligned}
$$

We now estimate the third term in the right hand side of (9.6). By Lemmas 3.3 (i) and 3.4 (ii), we have

$$
\begin{aligned}
&\int_{-\infty}^{t'-2|t'-t''|} \int_{\partial\Omega} |(K(t',x,\tau,y)-K(t'',x,\tau,y))(\mu(\tau,y)-\mu(t',y))|\, d\sigma_y d\tau \\
&\qquad \leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} \\
&\qquad \int_{-\infty}^{t'-2|t'-t''|} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2''}}{|t'-\tau|^{\gamma_1''-(\eta_2/2)}} |t'-t''|^{\gamma_1'''} e^{-\frac{|x-y|^2}{a(t'-\tau)}}\, d\sigma_y d\tau
\end{aligned}
$$

$$\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\gamma_l''}$$
$$\int_{\partial\Omega}\int_{\frac{2a|t'-t''|}{|x-y|^2}}^{+\infty}\frac{|x-y|^{\gamma_2''+2}a^{-1+\gamma_1''-(\eta_2/2)}}{u^{\gamma_1''-(\eta_2/2)}|x-y|^{2\gamma_1''-\eta_2}}e^{-1/u}\,d\sigma_y du$$
$$\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}D_{\gamma_1''-(\eta_2/2),r}|t'-t''|^{\gamma_l''}a^{-1+\gamma_1''-(\eta_2/2)}$$
$$\times\int_{\partial\Omega}\left(\frac{2a|t'-t''|}{|x-y|^2}\right)^{-r}\frac{d\sigma_y}{|x-y|^{2\gamma_1''-\gamma_2''-2-\eta_2}}$$
$$\leqslant \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\eta_2/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}D_{\gamma_1''-(\eta_2/2),r}|t'-t''|^{\gamma_l''-r}$$
$$\times(2a)^{-r}a^{-1+\gamma_1''-(\eta_2/2)}c'_{\Omega,2\gamma_1''-\gamma_2''-2-\eta_2-2r}\,, \tag{9.8}$$

for all $r\in]0,\gamma_1''-(\eta_2/2)-1[$, provided that $2\gamma_1''-\gamma_2''-2-\eta_2-2r<(n-1)$. We now wish to select r so that $\gamma_l''-r=\eta_1/2$, *i.e.*, $r=\gamma_l''-\eta_1/2$. To do so, we must verify that

$$0<\gamma_l''-(\eta_1/2)\,,\qquad \gamma_l''-(\eta_1/2)<\gamma_1''-(\eta_2/2)-1\,, \tag{9.9}$$

and that $2\gamma_1''-\gamma_2''-2-\eta_2-2r<(n-1)$. Now we can rewrite inequalities (9.9) as

$$\eta_1<2\gamma_l''\,,\qquad \gamma_l''<\gamma_1''-1+2^{-1}(\eta_1-\eta_2)\,,$$

and we observe that such inequalities hold by assumption. Moreover, if we set $r=\gamma_l''-\eta_1/2$, then our assumptions imply that

$$2\gamma_1''-\gamma_2''-2-\eta_2-2r=2\gamma_1''-\gamma_2''-2-\eta_2-2\gamma_l''+\eta_1<(n-1)\,,$$

Hence, we conclude that we can choose r as above, and that accordingly inequalities (9.6)–(9.8) imply the validity of statement (ii). □

Lemma 9.1. *Let $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\alpha\in]0,1[$, $\beta\in]0,\alpha[$. Let γ_n be defined as in (8.2). Then the following statements hold.*

(i) There exists $q_1\in]0,+\infty[$ such that

$$\begin{aligned}|Q[Z,g,\mu](t,x')-Q[Z,g,\mu](t,x'')|&\\ \leqslant q_1\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C^{\beta/2;\beta}(\partial_T\Omega)}|x'-x''|^{\alpha}&\\ +\|\mu\|_{L^\infty(\partial_T\Omega)}|Q[Z,g,1](t,x')-Q[Z,g,1](t,x'')|\,,&\end{aligned} \tag{9.10}$$

for all x', $x''\in\partial\Omega$, $t\in\overline{]-\infty,T[}$, and for all $(Z,g,\mu)\in\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\alpha}(\partial\Omega)\times C^{\beta/2;\beta}(\partial_T\Omega)$ (cf. (8.4).)

(ii) There exists $q_2\in]0,+\infty[$ such that

$$\begin{aligned}|Q[Z,g,\mu](t',x)-Q[Z,g,\mu](t'',x)|&\\ \leqslant q_2\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\alpha/2}&\\ +|Q[Z,g,\mu(t',\cdot)](t',x)-Q[Z,g,\mu(t',\cdot)](t'',x)|\,,&\end{aligned} \tag{9.11}$$

for all $x\in\partial\Omega$, t', $t''\in\overline{]-\infty,T[}$, $t'<t''$, and for all $(Z,g,\mu)\in\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\alpha}(\partial\Omega)\times C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$.

(iii) There exists $q_3 \in]0,+\infty[$ *such that*

$$|Q[Z,g,\mu](t',x) - Q[Z,g,\mu](t'',x)| \qquad (9.12)$$
$$\leqslant q_3 \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{(1+\alpha)/2}$$
$$+ |Q[Z,g,\mu(t',\cdot)](t',x) - Q[Z,g,\mu(t',\cdot)](t'',x)| \,,$$

for all $x \in \partial\Omega$, $t', t'' \in \overline{]-\infty,T[}$, $t' < t''$, *and for all* $(Z,g,\mu) \in \mathcal{K}_{\gamma_n,a}(\partial_T\Omega) \times C^{0,\alpha}(\partial\Omega) \times C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$.

(iv) There exists $q_4 \in]0,+\infty[$ *such that*

$$|Q[Z,g,\mu](t',x) - Q[Z,g,\mu](t'',x)| \qquad (9.13)$$
$$\leqslant q_4 ||Z||_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} ||g||_{C^{0,\alpha}(\partial\Omega)} ||\mu||_{C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{(1+\beta)/2}$$
$$+ |Q[Z,g,\mu(t',\cdot)](t',x) - Q[Z,g,\mu(t',\cdot)](t'',x)|,$$

for all $x \in \partial\Omega, , t', t'' \in \overline{]-\infty,T[}, t' < t''$, *and for all* $(Z,g,\mu) \in \mathcal{K}_{\gamma_n,a}(\partial_T\Omega) \times C^{0,\alpha}(\partial\Omega) \times C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$.

Proof. We first consider statement (i). Let $\gamma_{n,\theta}$, be defined as in (8.2) with $\theta = \alpha$. By Proposition 9.1 (i) with $\gamma = \gamma_{n,\theta}$, $\theta = \alpha$, $\eta_2 = \beta$, there exists $c_1 > 0$ such that inequality (9.1) holds with $\omega(r) \equiv r^\alpha$ for all $(K,\mu) \in \mathcal{K}_{\gamma_n,\alpha,4a}(\partial_T\Omega) \times C^{\beta/2;\beta}(\partial_T\Omega)$. Indeed, $\gamma_l' = \alpha$, $\gamma_l'' = 1$,

$$\begin{aligned}
&\gamma_1 = (n/2)+1 > 1\,,\\
&2\gamma_1 - \gamma_2 - 2 = 2((n/2)+1) - (1+\alpha) - 2 = (n-1) - \alpha \in [n-2, n-1[\,,\\
&\gamma_1' - (\eta_2/2) = (n/2) + 1 - (\beta/2) > 1\,,\\
&(n-1) - (2\gamma_1' - \gamma_2' - 2 - \eta_2) + \gamma_l'\\
&\qquad\qquad = (n-1) - [2((n/2)+1) - 1 - 2] + \alpha + \beta = \alpha + \beta > 0\,,\\
&2\gamma_1' - \gamma_2' - 2 - \eta_2 = 2((n/2)+1) - 1 - 2 - \beta = (n-1) - \beta < (n-1)\,.
\end{aligned}$$

Then inequality (9.10) follows by Lemma 8.2 (i) with $\theta = \alpha$ and by the equality $u[H[Z,g],\mu] = Q[Z,g,\mu]$. Next we consider statement (ii). By Proposition 9.1 (ii) with $\gamma = \gamma_{n,\theta}$, $\theta = \alpha$, $\eta_1 = \alpha$, $\eta_2 = \beta$, there exists $c_2 > 0$ such that inequality (9.2) holds for all $(K,\mu) \in \mathcal{K}_{\gamma_n,\alpha,4a}(\partial_T\Omega) \times C^{\beta/2;\beta}(\partial_T\Omega)$. Indeed, $\gamma_l' = \alpha$, $\gamma_l'' = 1$,

$$\begin{aligned}
&\gamma_1 - (\eta_2/2) = (n/2) + 1 - (\beta/2) > 1\,,\\
&2\gamma_1 - \gamma_2 - 2 + (\eta_1 - \eta_2) = (n-1) - \alpha + (\alpha - \beta) = (n-1) - \beta < (n-1)\,,\\
&\gamma_1'' - (\eta_2/2) = (n/2) + 2 - (\beta/2) > 1\,,\\
&\gamma_l'' - \gamma_1'' + 1 - 2^{-1}(\eta_1 - \eta_2)\\
&\qquad = 1 - ((n/2)+2) + 1 - 2^{-1}(\alpha-\beta) = -(n/2) - 2^{-1}(\alpha - \beta) < 0\,,\\
&\eta_1 - 2\gamma_l'' = \alpha - 2 < 0\\
&2\gamma_1'' - \gamma_2'' - 2 - 2\gamma_l'' + (\eta_1 - \eta_2)\\
&\qquad = 2((n/2)+2) - (1+\alpha) - 2 - 2 + (\alpha - \beta) = (n-1) - \beta < (n-1)\,.
\end{aligned}$$

Then inequality (9.11) follows by Lemma 8.2 (i) with $\theta = \alpha$ and by the equality $u[H[Z,g],\mu] = Q[Z,g,\mu]$. Next we consider statement (iii). Let $\gamma_{n,\alpha}$ be as in (8.2) with $\theta = \alpha$. By Proposition

9.1 (ii) with $\gamma = \gamma_{n,\alpha}$, $\eta_1 = (1+\alpha)$, $\eta_2 = (1+\beta)$, there exists $c_3 > 0$ such that

$$
\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t', x) - u[\partial_T\Omega, K, \mu](t'', x)| \\
&\qquad \leqslant c_3 \|K\|_{\mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))} |t' - t''|^{(1+\alpha)/2} \\
&\qquad\quad + |u[\partial_T\Omega, K, \mu(t', \cdot)](t', x) - u[\partial_T\Omega, K, \mu(t', \cdot)](t'', x)| ,
\end{aligned}
\tag{9.14}
$$

for all $x \in \partial\Omega$, t', $t'' \in \overline{]-\infty, T[}$, $t' < t''$, and for all $(K, \mu) \in \mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega) \times C_b^{0,(1+\beta)/2}(\overline{]-\infty, T[}, C^0(\partial\Omega))$. Indeed,

$$
\begin{aligned}
&\gamma_1 - (\eta_2/2) = (n/2) + 1 - 2^{-1}(1+\beta) > 1 , \\
&2\gamma_1 - \gamma_2 - 2 + (\eta_1 - \eta_2) \\
&\qquad = 2((n/2)+1) - 1 - \alpha - 2 + (1+\alpha) - (1+\beta) = n - 1 - \alpha + (\alpha - \beta) \\
&\qquad < (n-1) , \\
&\gamma_1'' - (\eta_2/2) = (n/2) + 2 - 2^{-1}(1+\beta) > 1 , \\
&\gamma_l'' - \gamma_1'' + 1 - 2^{-1}(\eta_1 - \eta_2) \\
&\qquad = 1 - ((n/2)+2) + 1 - 2^{-1}(\alpha - \beta) = -(n/2) - 2^{-1}(\alpha - \beta) < 0 , \\
&\eta_1 - 2\gamma_l'' = (1+\alpha) - 2 < 0 , \\
&2\gamma_1'' - \gamma_2'' - 2 - 2\gamma_l'' + ((1+\alpha) - (1+\beta)) \\
&\qquad = 2((n/2)+2) - 1 - \alpha - 2 - 2 + ((1+\alpha) - (1+\beta)) \\
&\qquad = (n-1) - \alpha + (\alpha - \beta) < (n-1) .
\end{aligned}
$$

Then inequality (9.12) follows by Lemma 8.2 (i) with $\theta = \alpha$ and by inequality (9.14) and by the equality $u[H[Z, g], \mu] = Q[Z, g, \mu]$.

Finally we consider statement (iv). Next we plan to apply Proposition 9.1 (ii) with $\eta_1 = 1 + \beta, \eta_2 = 1$, $\gamma = \gamma_{n,\alpha}$. As above we can verify that all the assumption of Proposition 9.1 (ii) are satisfied and that accordingly there exists $q_4 > 0$ such that

$$
\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t', x) - u[\partial_T\Omega, K, \mu](t'', x)| \\
&\qquad \leqslant q_4 ||K||_{\mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega)} ||\mu||_{C_b^{0,1/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))} |t' - t''|^{(1+\beta)/2} \\
&\qquad\quad + |u[\partial_T\Omega, K, \mu(t', \cdot)](t', x) - u[\partial_T\Omega, K, \mu(t', \cdot)](t'', x)|,
\end{aligned}
\tag{9.15}
$$

for all $x \in \partial\Omega, t', t'' \in \overline{]-\infty, T[}, t' < t''$, and for all $(K, \mu) \in \mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega) \times C_b^{0,1/2}(\overline{]-\infty, T[}, C^0(\partial\Omega))$. Then the inequality (9.13) follows by Lemma 8.2 (i) and by inequality (9.15) and by the equality $u[H[Z, g], \mu] = Q[Z, g, \mu]$. □

10 Applications to layer heat potentials with Hölder continuous densities

Let

$$
S_n(x) \equiv \begin{cases} \frac{1}{s_n} \ln |x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2 , \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2 , \end{cases}
$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. S_n is well known to be the fundamental solution of the Laplace operator. Then we have the following known elementary lemma for densities which do not depend on time.

Lemma 10.1. *Let $\alpha \in]0,1]$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $\varphi \in L^\infty(\partial\Omega)$, then*

$$w[\partial_T\Omega,\varphi](t,x) = -\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} S_n(x-y)\varphi(y)\,d\sigma_y\,, \tag{10.1}$$

for all $(x,t) \in \partial_T\Omega$ (cf. (8.1).)

Proof. By setting $(t-\tau) = u|x-y|^2$ in the integral (8.1) which defines the double layer heat potential, we obtain

$$w[\partial_T\Omega,\varphi](t,x) = -\frac{s_n}{2(4\pi)^{n/2}} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} S_n(x-y)\psi(y)\,d\sigma_y\,,$$

for all $(x,t) \in \partial_T\Omega$, where

$$\psi(y) \equiv \int_0^{+\infty} \frac{e^{-\frac{1}{4u}}}{u^{(n/2)+1}}\varphi(y)\,du \qquad \forall y \in \partial\Omega\,,$$

(see also Kress [13, p. 157].) Since $\int_0^{+\infty} \frac{e^{-\frac{1}{4u}}}{u^{(n/2)+1}}\,du = 4^{(n/2)}\Gamma(n/2)$ and $s_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, the formula of the statement holds true. □

Then we have the following statement.

Theorem 10.1. *Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

(i) If $\mu \in C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$, then $w[\partial_T\Omega,\mu] \in C_b^{0,\alpha/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. Moreover, the operator from $C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $C_b^{0,\alpha/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous (cf. Remark 2.)

(ii) If $\mu \in C^{\beta/2;\beta}(\partial_T\Omega)$, then $w[\partial_T\Omega,\mu] \in C^{\alpha/2;\alpha}(\partial_T\Omega)$. Moreover, the operator from $C^{\beta/2;\beta}(\partial_T\Omega)$ to $C^{\alpha/2;\alpha}(\partial_T\Omega)$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous.

(iii) If $\mu \in C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$, then $w[\partial_T\Omega,\mu] \in C_b^{0,(1+\alpha)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. Moreover, the operator from $C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $C_b^{0,(1+\alpha)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous (cf. Remark 2.)

(iv) If $\mu \in C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$, then $w[\partial_T\Omega,\mu] \in C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. Moreover the operator from $C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which takes μ to $w[\partial_T\Omega,\mu]$ is linear and continuous (cf. Remark 2.)

Proof. We first consider statement (i). By Theorem 8.1 (i), we already know that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to $B(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega))$, and accordingly from $C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $B(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega))$.

Next we plan to apply Proposition 9.1 (ii) with $\eta_1 = \alpha$, $\eta_2 = \beta$, $a \in]16,+\infty[$. By Remark 3 (iv), we already know that the kernel $\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y) \in \mathcal{K}_{\gamma,a}(\partial_T\Omega)$, with γ as in Remark

3 (iv). Then we observe that

$$
\begin{aligned}
&\gamma_1-(\eta_2/2)=(n/2)+1-(\beta/2)>1\,,\\
&2\gamma_1-\gamma_2-2=2((n/2)+1)-(1+\alpha)-2=(n-1)-\alpha\in[n-2,(n-1)[\,,\\
&2\gamma_1-\gamma_2-2+(\eta_1-\eta_2)=(n-1)-\alpha+(\alpha-\beta)=(n-1)-\beta<(n-1)\,,\\
&\gamma_1''-(\eta_2/2)=((n/2)+2)-(\beta/2)>1\,,\\
&\gamma_l''-\gamma_1''+1-2^{-1}(\eta_1-\eta_2)=1-((n/2)+2)+1-2^{-1}(\alpha-\beta)<0\,,\\
&\eta_1-2\gamma_l''=\alpha-2<0\,,\\
&2\gamma_1''-\gamma_2''-2-2\gamma_l''+(\eta_1-\eta_2)\\
&\qquad\qquad=2((n/2)+2)-(1+\alpha)-2-2+(\alpha-\beta)=(n-1)-\beta<(n-1)\,.
\end{aligned}
$$

Then Proposition 9.1 (ii) implies the existence of $c_2>0$ such that

$$
\begin{aligned}
&|w[\partial_T\Omega,\mu](t',x)-w[\partial_T\Omega,\mu](t'',x)| \\
&\quad\leqslant c_2\left\|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\alpha/2}\\
&\qquad+|w[\partial_T\Omega,\mu(t',\cdot)](t',x)-w[\partial_T\Omega,\mu(t',\cdot)](t'',x)|\,,
\end{aligned}
\tag{10.2}
$$

for all $x\in\partial\Omega$, t', $t''\in\overline{]-\infty,T[}$, $t'<t''$, and for all $\mu\in C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. By Lemma 10.1, we have

$$
w[\partial_T\Omega,\mu(t',\cdot)](t,x)=-\int_{\partial\Omega}\frac{\partial}{\partial\nu(y)}S_n(x-y)\mu(t',y)\,d\sigma_y\,,
$$

for all $x\in\partial\Omega$, t, $t'\in\overline{]-\infty,T[}$ and for all $\mu\in C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. Then the second summand in the right hand side of (10.2) equals 0 and inequality (10.2) implies that statement (i) holds true. Statement (ii) is an immediate consequence of statement (i) and of Theorem 8.1 (i) and of Proposition 2.1.

We now consider statement (iii). Since $C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ is continuously imbedded into $L^\infty(\overline{]-\infty,T[},C^0(\partial\Omega))$, Theorem 8.1 (i) implies that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $B(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega))$. Next we plan to apply Proposition 9.1 (ii) with $\eta_1=1+\alpha$, $\eta_2=1+\beta$, $a\in]16,+\infty[$, γ as in Remark 3 (iv). As above, we can verify that all the assumptions of Proposition 9.1 (ii) are satisfied and that accordingly there exists $\tilde{c}_2>0$ such that

$$
\begin{aligned}
&|w[\partial_T\Omega,\mu](t',x)-w[\partial_T\Omega,\mu](t'',x)| \\
&\quad\leqslant \tilde{c}_2\left\|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{(1+\alpha)/2}\\
&\qquad+|w[\partial_T\Omega,\mu(t',\cdot)](t',x)-w[\partial_T\Omega,\mu(t',\cdot)](t'',x)|\,,
\end{aligned}
\tag{10.3}
$$

for all $x\in\partial\Omega$, t', $t''\in\overline{]-\infty,T[}$, $t'<t''$, and for all $\mu\in C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. Then again by (10.1), we conclude that the second summand in the right hand side of (10.3) vanishes and that accordingly statement (iii) holds true.

We now consider statement (iv). Since $C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ is continuously imbedded into $L^\infty(\overline{]-\infty,T[},C^0(\partial\Omega))$, Theorem 8.1 (i) implies that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $B(\overline{]-\infty,,T[},C^{0,\alpha}(\partial\Omega))$. Next we plan to apply Proposition 9.1 (ii) with $\eta_1=1+\beta,\eta_2=1,a\in]16,+\infty[,\gamma$ as in Remark 3 (iv). As above, we can verify that all the assumptions of Proposition 9.1 (ii) are satisfied and that accordingly there exists $\hat{c}_2 > 0$ such that

$$
\begin{aligned}
&|w[\partial_T\Omega,\mu](t',x) - w[\partial_T\Omega,\mu](t'',x)| \\
&\quad \leqslant \hat{c}_2 \left|\left| \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right|\right|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} ||\mu||_{C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{(1+\beta)/2} \\
&\quad +|w[\partial_T\Omega,\mu(t',\cdot)](t',x) - w[\partial_T\Omega,\mu(t',\cdot)](t'',x)|,
\end{aligned}
\tag{10.4}
$$

for all $x \in \partial\Omega, t', t'' \in \overline{]-\infty,T[}, t' < t''$, and for all $\mu \in C_b^{0,1/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))$. Then again by (10.1), we conclude that the second summand in the right hand side of (10.4) vanishes and that accordingly statement (iv) holds true. □

Next we analyze an integral operator with kernel $\partial_{x_r}\Phi_n(t-\tau,x-y)$ for $r \in \{1,\dots,n\}$ and we prove the following.

Theorem 10.2. *Let $T \in]-\infty,+\infty]$. Let $\alpha \in]0,1[, \beta \in]0,\alpha[$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold*

(i) The map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\alpha}(\partial\Omega)\times C^{\beta/2;\beta}(\partial_T\Omega)$ to $C^{\alpha/2;\alpha}(\partial_T\Omega)$ which takes (g,μ) to $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu]$ is bilinear and continuous (cf. (8.8).)

(ii) The map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\alpha}(\partial\Omega)\times C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $C_b^{0,(1+\alpha)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which takes (g,μ) to $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu]$ is bilinear and continuous.

(iii) The map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\alpha}(\partial\Omega)\times C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $C_b^{0,(1+\beta)/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which takes (g,μ) to $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu]$ is bilinear and continuous.

Proof. Let $Z(t,x,\tau,y) \equiv \partial_{x_r}\Phi_n(t-\tau,x-y)$. Let $a \in]16,+\infty[$. We now prove statement (i). By Theorem 8.2 the map $Q[Z,\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C_b^{0,1/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$ to $C^0(\partial_T\Omega)$. By Remark 3 (iii), we have $Z \in \mathcal{K}_{\gamma_n,a}(\partial_T\Omega)$ with γ_n as in (8.2). Then Lemma 9.1 (i) implies that there exists $q_1 > 0$ such that

$$
\begin{aligned}
&|Q[Z,g,\mu](t,x') - Q[Z,g,\mu](t,x'')| \\
&\quad \leqslant q_1||Z||_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}||g||_{C^{0,\alpha}(\partial\Omega)}||\mu||_{C^{\beta/2;\beta}(\partial_T\Omega))}|x'-x''|^{\alpha} \\
&\quad +||\mu||_{L^\infty(\partial_T\Omega)}|Q[Z,g,1](t,x') - Q[Z,g,1](t,x'')|,
\end{aligned}
\tag{10.5}
$$

for all $x', x'' \in \partial\Omega, t \in \overline{]-\infty,T[}$, and for all $(g,\mu) \in C^{0,\alpha}(\partial\Omega)\times C^{\beta/2;\beta}(\partial_T\Omega)$. By performing the change of variables $(t-\tau) = u|x-y|^2$, we have

$$
\begin{aligned}
Q[Z,g,1](t,x) &= \int_{-\infty}^{t}\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\,d\sigma_y d\sigma_\tau \\
&= -\int_0^{+\infty}\frac{e^{-\frac{1}{4u}}}{2(4\pi)^{n/2}u^{(n/2)+1}}\,dus_n\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial x_r}S_n(x-y)\,d\sigma_y \\
&= -\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial x_r}S_n(x-y)\,d\sigma_y \qquad \forall (x,t)\in\partial_T\Omega\,,
\end{aligned}
\tag{10.6}
$$

(see also the proof of Lemma 10.1.) Then known properties of harmonic layer potentials imply that there exists $q_1' > 0$ such that

$$|Q[Z,g,1](t,x') - Q[Z,g,1](t,x'')| \leqslant q_1'|x'-x''|^\alpha, \tag{10.7}$$

for all $x', x'' \in \partial\Omega, t \in \overline{]-\infty,T[}$ (cf. *e.g.* Schauder [18, Hilfsatz VII, p. 112], [2, §8].) Next we apply Lemma 9.1 (ii). Then there exists $q_2 > 0$ such that

$$\begin{aligned} &|Q[Z,g,\mu](t',x) - Q[Z,g,\mu](t'',x)| \\ &\quad \leqslant q_2 \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C_b^{0,\beta/2}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{\alpha/2} \\ &\quad + |Q[Z,g,\mu(t',\cdot)](t',x) - Q[Z,g,\mu(t',\cdot)](t'',x)|, \end{aligned} \tag{10.8}$$

for all $x \in \partial\Omega, t', t'' \in \overline{]-\infty,T[}, t' < t''$, and for all $(g,\mu) \in C^{0,\alpha}(\partial\Omega) \times C_b^{0,\beta/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))$. By exploiting the same change of variables of (10.6), we can show that

$$|Q[Z,g,\mu(t',\cdot)](t',x) - Q[Z,g,\mu(t',\cdot)](t'',x)| = 0\,, \tag{10.9}$$

for all $x \in \partial\Omega, t', t'' \in \overline{]-\infty,T[}, t' < t''$. Then by Proposition 2.1 and by inequalities (10.5), (10.7), (10.8) and equality (10.9), we conclude that statement (i) holds true. Statement (ii) is a consequence of Lemma 9.1 (iii) and of equality (10.9).

Finally statement (iii) is a consequence of Lemma 9.1 (iv) again together with equality (10.9). □

Next we analyze an integral operator with kernel $\partial_t\Phi_n(t-\tau,x-y)$.

Theorem 10.3. *Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$. Then the following statements hold.*

(i) The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\alpha}(\partial\Omega)\times C^{0,(1+\beta)/2;0,1}(\partial_T\Omega)$ to $C^{\alpha/2;\alpha}(\partial_T\Omega)$ which takes (g,μ) to the function

$$\begin{aligned} &\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),g,\mu](t,x) \\ &\quad \equiv \int_0^t \int_{\partial\Omega} (g(x)-g(y))\partial_t\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))d\sigma_y d\tau\,, \end{aligned}$$

for all $(t,x) \in \partial_T\Omega$ is bilinear and continuous.

(ii) The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,1/2;0,1}(\partial_T\Omega)$ to $C^{\beta/2;\beta}(\partial_T\Omega)$.

(iii) The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,1}(\partial\Omega)\times C^{0,1;0,1}(\partial_T\Omega)$ to $C_b^{0,(1+\alpha)/2}(\overline{]-\infty,T[}, C^0(\partial\Omega))$.

Proof. Let $a \in]16,+\infty[$. Then Remark 3 (ii) implies that $\partial_t\Phi_n(t-\tau,x-y)$ belongs to $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)$ with $\gamma_n^\sharp$ as in (8.9). We now prove statement (i). If $\beta_1 \in]0,\beta]$, then $C^{0,(1+\beta)/2;0,1}(\partial_T\Omega)$ is continuously imbedded into $C^{0,(1+\beta_1)/2;0,1}(\partial_T\Omega)$. Thus there is no loss of generality in assuming that $\alpha+\beta<1$. Then we can apply Proposition 8.1 with $b=(1+\beta)/2$, $\theta=\alpha$ and $b_1 \in](\alpha/2),(\alpha+\beta)/2[\subseteq](\alpha/2),(1+\beta)/2[$. Indeed, $2b+\theta = 1+\beta+\alpha \in]1,2]$. Proposition 8.1 (ii) implies that $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,(1+\beta)/2;0,1}(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega)\right)$. Then we note that

$$h\equiv\alpha/2\in]0,(2b+\theta-1)/2[=]0,(\alpha+\beta)/2[\,,$$

and that $1\geqslant b-b_1\geqslant(1-\alpha)/2$, $(n/2)-[2^{-1}(1+\beta)-b_1]\geqslant(n/2)-2^{-1}(1+\beta)+(\alpha/2)\geqslant(\alpha/2)-(\beta/2)$, and that accordingly

$$\begin{aligned}&]\max\{0,[1-\theta-2(b-b_1)]/2\},\min\{(n/2)-[b-b_1],b_1\}[\\&\qquad=]0,\min\{(n/2)-[2^{-1}(1+\beta)-b_1],b_1\}[\supseteq]0,2^{-1}(\alpha-\beta)[\,.\end{aligned}$$

Since $b_1>\alpha/2$, then we can choose $b_2\in]0,2^{-1}(\alpha-\beta)[$ such that $b_1-b_2>\alpha/2$, and thus $\min\{h,b_1-b_2\}=\alpha/2$, and Proposition 8.1 (iii) implies that $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,(1+\beta)/2;0,1}(\partial_T\Omega)$ to $C_b^{0,\alpha/2}(\overline{]-\infty,T[},C^0(\partial\Omega))$. Then Proposition 2.1 implies that statement (i) holds true.

Next we consider statement (ii). We apply Proposition 8.1 (ii) with $b=1/2,\theta=\alpha$. Then the map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,1/2;0,1}(\partial_T\Omega)$ to $B(\overline{]-\infty,T[},C^{0,\omega_\alpha(\cdot)}(\partial\Omega))$. By the continuity of the embedding of $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ into $C^{0,\beta}(\partial\Omega)$, the same map is continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,1/2;0,1}(\partial_T\Omega)$ to $B(\overline{]-\infty,T[},C^{0,\beta}(\partial\Omega))$. Then we plan to apply Proposition 8.1 (iii) with $b=1/2,\theta=\alpha,b_1\in]\beta/2,\alpha/2[$. We take

$$h\equiv\beta/2\in\,]0,(2b+\theta-1)/2[\,=\,]0,\alpha/2[\ ,$$

and we choose

$$\begin{aligned}b_2\in\,&]\max\{0,[1-\theta-2(b-b_1)]/2\},\min\{(n/2)-(b-b_1),b_1\}]\\&=\,]\max\{0,b_1-\alpha/2\},\min\{(n-1)/2+b_1,b_1\}]\\&=]0,b_1[\,,\end{aligned}$$

where we have exploited the membership of b_1 in $]\beta/2,\alpha/2[$. We note that since $b_1>\beta/2$ we can choose $b_2\in]0,b_1[$ such that $b_1-b_2>\beta/2$. Hence, $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,1/2;0,1}(\partial_T\Omega)$ to $C_b^{0,\beta/2}(\overline{]-\infty,T]},C^0(\partial\Omega))$. Combining Proposition 2.1 and the two result above we can conclude that $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,1/2;0,1}(\partial_T\Omega)$ to $C^{\beta/2,\beta}(\partial_T\Omega)$. Hence, statement (ii) holds true.

Next we turn to prove statement (iii). We plan to apply Proposition 8.1 (iii) with $b\in]2^{-1}(1+\alpha),1[$, $\theta=1$, $b_1\in]2^{-1}(1+\alpha),b[$. We note that

$$h\equiv2^{-1}(1+\alpha)\in]0,(2b+\theta-1)/2[=]0,b[\,,$$

and that $b-b_1\leqslant1-2^{-1}(1+\alpha)=2^{-1}(1-\alpha)$, and that accordingly

$$\begin{aligned}&]\max\{0,[1-\theta-2(b-b_1)]/2\},\min\{(n/2)-[b-b_1],b_1\}[\\&\quad=]0,\min\{(n/2)-[b-b_1],b_1\}[\supseteq]0,\min\{1-2^{-1}(1-\alpha),b_1\}[\\&\quad=]0,2^{-1}(1+\alpha)[\,.\end{aligned}$$

Since $b_1>2^{-1}(1+\alpha)$, we can choose $b_2\in]0,2^{-1}(1+\alpha)[$ such that $b_1-b_2>2^{-1}(1+\alpha)$ and thus $\min\{h,b_1-b_2\}=2^{-1}(1+\alpha)$. Hence, Proposition 8.1 (iii) implies that statement (iii) holds true. □

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