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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate "For the merits in the development of science in the Republic of Kazakhstan", in 2007 and 2014 the state grant "The best university teacher", in 2016 the Order "Kurmet" (= "Honour").

R. Oinarov was born in the village Kul'Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis "Continuity and Lipschitzness of nonlinear integral operators of Uryson's type" at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis "Weighted estimates of integral and differential operators" at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov's kernels* or *Oinarov condition.* On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user=NzXYMS4AAAAJhl=ruoi=ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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ON MULTIPERIODIC INTEGRALS OF A LINEAR SYSTEM WITH THE DIFFERENTIATION OPERATOR IN THE DIRECTION OF THE MAIN DIAGONAL IN THE SPACE OF INDEPENDENT VARIABLES

A.A. Kulzhumiyeva, Zh. Sartabanov

Communicated by Ya.T. Sultanayev

Dedicated to the 70^{th} *birthday of Professor Ryskul Oinarov*

Key words: multiperiodic solution, linear system, differentiation operator.

AMS Mathematics Subject Classification: 35B10.

Abstract. In the general theory of first order partial differential equations one of the effective ways of integration is a common method of finding solutions by the complete integral [4-5]. In this note we propose a method of research of problems of multiperiodic solutions of linear systems of equations with the same differentiation operator in the direction of a vector field, which is based on the results of the study multiperiodicity of their complete integral. Such an approach is not found in earlier studies based on the methods of fundamental works [8-9] on multiperiodic solutions of such systems. The elements of the proposed method are used in [1-3, 6-7]. In this note in order to establish the multiperiodicity of a complete integral Green's function is introduced in the absence of non-trivial integral multiperiodic solutions of homogeneous system.

1 Introduction

The operator D_e representing the sum of differential operators $\frac{\partial}{\partial t_j}$, $j = \overline{0, m}$ on the independent variable is called the differentiation operator in the direction $(1, e)$ of the main diagonal $t = e\tau$ of the space $(\tau, t) \in R \times R^m$, where $e = (1, \ldots, 1)$ – m-vector, $t = (t_1, \ldots, t_m)$. Consequently, we have

$$D_e = \frac{\partial}{\partial \tau} + \left\langle e, \frac{\partial}{\partial t} \right\rangle, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ – denotes the scalar product and $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}\right)$ is a vector operator.

We call the equation

$$\frac{dt}{d\tau} = e \tag{1.2}$$

characteristic for the operator D_e and its general solution

$$t = \sigma + e\tau \tag{1.3}$$

with an arbitrary constant vector $\sigma = (\sigma_1, ..., \sigma_m)$ is called the characteristics of the operator D_e.

Then equation (1.2) by (1.3) has the integral

$$t - e\tau = \sigma, \tag{1.4}$$

which we call the base for operator D_e, because for an arbitrary differentiable function h the function $h(\sigma)$ is a general integral $h(t - e\tau)$ of equation (1.3) and it is the general solution of equation

$$D_e h = 0. \tag{1.5}$$

We consider the problem for a linear system of equations

$$D_e x = P(\tau, t)x + f(\tau, t) \tag{1.6}$$

with differentiation operator (1.1), where $x = (x_1, ..., x_n)$ – the complete sought-for integral.

The solution $x = x(\tau, t, \sigma, c)$ of system (1.6) with the arbitrary constant vectors $\sigma = (\sigma_1, ..., \sigma_m)$ and $c = (c_1, ..., c_n)$ is called its complete integral.

We set the problem of studying on the existence of a complete integral, satisfying the below condition

$$x(\tau + \theta, t + q\omega, \sigma, c) = x(\tau, t, \sigma, c), \quad q \in Z^m, \tag{1.7}$$

where $(\tau, t) \in R \times R^m$, $\theta = \omega_0$, $\omega_1, ..., \omega_m$ are positive rationally incommensurable constants, $\omega = (\omega_1, ..., \omega_m)$ a vector-period, $q = (q_1, ..., q_m) \in Z \times ... \times Z = Z^m$, Z the set of integers, $q\omega = (q_1\omega_1, ..., q_m\omega_m)$ a multiple vector-period, $P(\tau, t)$ an $n \times n$-matrix and $f(\tau, t)$ an n-vector-function of class $C^{(0,1)}_{\tau,t}(R \times R^m)$ for continuous in $\tau \in R$, continuously differentiable in $t \in R^m$ and multiperiodic in (τ, t) with the period (θ, ω). Consequently,

$$P(\tau + \theta, t + q\omega) = P(\tau, t) \in C^{(0,1)}_{\tau,t}(R \times R^m), \quad q \in Z^m, \tag{1.8}$$

$$f(\tau + \theta, t + q\omega) = f(\tau, t) \in C^{(0,1)}_{\tau,t}(R \times R^m), \quad q \in Z^m. \tag{1.9}$$

Therefore, our basic problem is to study the integral $x(\tau, t, \sigma, c)$ of system (1.6) satisfying condition (1.7).

This problem, in other words, may be called multiperiodic problem (1.6)-(1.7).

2 Multiperiodic integrals of the linear homogeneous system

We consider the homogeneous system

$$D_e x = P(\tau, t)x \tag{2.1}$$

corresponding to system (1.6).

Under condition (1.8) the complete integral of system (2.1) has the form

$$x(\tau, t, \sigma, c) = X(\tau, t, \sigma)c, \tag{2.2}$$

where $X(\tau, t, \sigma)$ is the matriciant of system (2.1):

$$D_e X(\tau, t, \sigma) = P(\tau, t)X(\tau, t, \sigma), \quad X(0, t, \sigma) = E \tag{2.3}$$

with the $n \times n$ identity matrix E, has the properties

$$X(\tau, t, \sigma + q\omega) = X(\tau, t + q\omega, \sigma) = X(\tau, t, \sigma), \quad q \in Z^m, \tag{2.4}$$

$$X(\tau+\theta,t,\sigma)=X(\tau,t,\sigma)X(\theta,\sigma,\sigma), \tag{2.5}$$

where σ is determined by (1.4).

The matriciant $X(\tau,t,\sigma)$ is constructed by considering of system (2.1) along solutions (1.3) of characteristic equation (1.2).

Here are some properties of complete integral (2.2) of system (2.1) satisfying condition (1.8).

2.1. There is a functional relationship between the constant vectors σ and c of the type

$$c=c(\sigma). \tag{2.6}$$

This property (2.6) is known from the general theory of partial differential equations [4-5].

2.2. In order that the complete integral of system (2.1) be ω-periodic in σ it is necessary and sufficient that the following condition is satisfied

$$c(\sigma+q\omega)=c(\sigma), \quad \sigma\in R^m, \quad q\in Z^m. \tag{2.7}$$

The proof follows from relations (2.2), (2.4) and (2.6).

2.3. Complete integral (2.2) of system (2.1) under condition (1.8) is ω-periodic in $t\in R^m$.

The proof of this property we have from relation about structure (2.2) of complete integral and property of the matriciant (2.4).

2.4. In order that the complete integral (2.2) of system (2.1) under condition (1.8) be θ-periodic in τ it is necessary and sufficient that vector-function (2.6) satisfies the system

$$[E-X(\theta,\sigma,\sigma)]c=0, \quad \sigma\in R^m. \tag{2.8}$$

The proof of property 2.4 follows from the equivalence of condition (2.8) and the determination of θ-periodicity of the integral in τ:

$$x(\tau,t,\sigma,c)-x(\tau+\theta,t,\sigma,c)=0, \quad (\tau,t,\sigma)\in R\times R^m\times R^m$$

taking into account property of matriciant (2.5).

2.5. In order that the complete integral $x(\tau,t,\sigma,c)$ of system (2.1) under condition (1.8) be (θ,ω)-periodic in (τ,t) it is necessary and sufficient that condition (2.8) is satisfied.

Proof. By property 2.3 the complete integral is ω-periodic in $t\in R^m$. Condition (2.8) of property 2.4 is equivalent to θ-periodicity of the integral $x(\tau,t,\sigma,c)$ of system (2.1). □

2.6. In order that the complete integral of system (2.1) under condition (1.8) be (θ,ω,ω)-periodic in (τ,t,σ) it is necessary and sufficient that vector-function $c=c(\sigma)$ is an ω-periodic solution of system (2.8).

Proof. Property 2.6 follows from the definition of equivalence of (θ,ω,ω)-periodicity of the integral

$$x(\tau,t,\sigma,c(\sigma))=x(\tau+\theta,t,\sigma,c(\sigma))=x(\tau,t+q\omega,\sigma,c(\sigma))=x(\tau,t,\sigma+q\omega,c(\sigma))$$

and conditions of all the properties 2.1-2.4. □

2.7. In order that under condition (1.8) system (2.1) does not have a non-trivial (θ, ω, ω)-periodic in (τ, t, σ) integral (2.2) it is necessary and sufficient that the following condition is satisfied

$$det[E - X(\theta, \sigma, \sigma)] \neq 0, \quad \sigma \in R^m. \tag{2.9}$$

The proof of property 2.7 follows since condition (2.9) is equivalent to the fact that system (2.8) has only the zero solution.

We will notice that condition (1.8) guarantees ω-periodicity of the integral in $t \in R^m$. Consequently, condition (2.9) is a condition of absence of θ-periodicity in $\tau \in R$ of the integral of system (2.1).

2.8. In order that under condition (1.8) integral (2.2) of system (2.1) be (θ, ω, ω)-periodic in (τ, t, σ) it is necessary and sufficient that condition (2.8) is satisfied for ω-periodic vector-functions.

The proof of property 2.8 follows from the fact that under condition (2.9) system (2.8) has only the zero ω-periodic in $\sigma \in R^m$ solution.

2.9. Under condition (1.8) complete integral (2.2) of system (2.1) defines the general solution

$$x(\tau, t, t - e\tau, u(t - e\tau)) = X(\tau, t, t - e\tau)u(t - e\tau) \tag{2.10}$$

with an arbitrary differentiable function u.

To prove 2.9 it suffices to apply the method of variation of Lagrange constants, assuming that the constants σ and c in (2.2) depend in (τ, t).

3 Green's function of a multiperiodic problem

Consider the interval $[\tau, \tau + \theta]$ of θ length with variable boundaries, depending in $\tau \in R$ and intervals

$$I_\alpha^- = [\tau, \alpha), \qquad I_\alpha^+ = (\alpha, \tau + \theta] \tag{3.1}$$

with some constant $\alpha \in R$.

Using the function $\varphi(\tau, t, \sigma)$, determined at $(\tau, t, \sigma) \in R \times R^m \times R^m$, we construct the function

$$\varphi_\alpha(s, \sigma + es, \sigma) = \begin{cases} \varphi(s, \sigma + es, \sigma), & s \in I_\alpha^-, \\ \varphi(s, \sigma + es - e\tau, \sigma), & s \in I_\alpha^+. \end{cases} \tag{3.2}$$

Note that as the point α we can take any fixed point, in particular, $\alpha = 0$. In the case of the fixed value τ instead of α it would be possible to take any intermediate point.

Further, we will assume that conditions (1.8) and (2.9) are satisfied.

Thus, in accordance with formulas (3.1) and (3.2) by using the inverse matrix $X^{-1}(\tau, t, \sigma)$ of the matriciant $X(\tau, t, \sigma)$ we construct the matrix function $X_\alpha^{-1}(s, \sigma + es, \sigma)$ and consider the matrix of the form

$$G_\alpha(s, \tau, t, \sigma) = X(\tau, t, \sigma)[E - X(\theta, \sigma, \sigma)]^{-1} X(\theta, \sigma, \sigma) X_\alpha^{-1}(s, \sigma + es, \sigma). \tag{3.3}$$

Matrix (3.3) has the following properties.

3.1. If $s \neq \alpha$ the matrix $G_\alpha(s, \tau, t, \sigma)$ satisfies system (2.1):

$$D_e G_\alpha(s, \tau, t, \sigma) = P(\tau, t) G_\alpha(s, \tau, t, \sigma), \quad s \neq \alpha. \tag{3.4}$$

The proof of identity (3.4) is carried out on the basis of (1.5), (2.3) and (3.3).

3.2. The difference of values matrix $G_\alpha(s,\tau,t,\sigma)$ at $s=\tau+\theta$ and $s=\tau$ is equal to the identity matrix E:

$$G_\alpha(\tau+\theta,\tau,t,\sigma)-G_\alpha(\tau,\tau,t,\sigma)=E. \tag{3.5}$$

The proof of identity (3.5) is associated with the difference of the form

$$\begin{aligned} &X_\alpha^{-1}(s,\sigma+es,\sigma)|_{s=\tau+\theta}-X_\alpha^{-1}(s,\sigma+es,\sigma)|_{s=\tau} \\ &=X^{-1}(s,\sigma+es-e\theta,\sigma)|_{s=\tau+\theta}-X^{-1}(s,\sigma+es,\sigma)|_{s=\tau} \\ &=X^{-1}(\tau+\theta,t,\sigma)-X^{-1}(\tau,t,\sigma)=X^{-1}(\theta,\sigma,\sigma)X^{-1}(\tau,t,\sigma)-X^{-1}(\tau,t,\sigma) \\ &=[X^{-1}(\theta,\sigma,\sigma)-E]X^{-1}(\tau,t,\sigma)=X^{-1}(\theta,\sigma,\sigma)[E-X(\theta,\sigma,\sigma)]X^{-1}(\tau,t,\sigma), \end{aligned} \tag{3.6}$$

which comes from formula (3.2) and properties (2.5) of the matriciant $X(\tau,t,\sigma)$. Further, putting difference (3.6) in relation (3.3) we will get property (3.5).

3.3. The matrix $G_\alpha(s,\tau,t,\sigma)$ has the property of ω-periodicity in t, σ and diagonal θ-periodicity in s, τ; in other words, the values of $G_\alpha(s,\tau,t,\sigma)$ at the points (s,τ,t,σ) and $(s+\theta,\tau+\theta,t+p\omega,\sigma+q\omega)$ for any $p,q\in Z^m$ are the same:

$$G_\alpha(s+\theta,\tau+\theta,t+p\omega,\sigma+q\omega)=G_\alpha(s,\tau,t,\sigma),\quad (s,\tau,t,\sigma)\in R\times R\times R^m\times R^m,\quad p,q\in Z^m. \tag{3.7}$$

Proof. In view of identity (2.4) of the matrices $X(\tau,t,\sigma)$, $X(\theta,\sigma,\sigma)$, $X^{-1}(s,\sigma+es,\sigma)$ and $X^{-1}(s,\sigma+es-e\theta,\sigma)$ entering (3.3) are ω-periodic in t and σ. Consequently, the matrix $G_\alpha(s,\tau,t,\sigma)$ is ω-periodic in t and σ.

Further, at the same time moving variables τ and s in θ, using property (2.5) of the matriciant $X(\tau,t,\sigma)$ and taking into account of the commutativity of the matrices $X(\theta,\sigma,\sigma)$ and $[E-X(\theta,\sigma,\sigma)]^{-1}$ we have

$$\begin{aligned} &G_\alpha(s+\theta,\tau+\theta,t,\sigma) \\ &=X(\tau,t,\sigma)X(\theta,\sigma,\sigma)[E-X(\theta,\sigma,\sigma)]^{-1}X(\theta,\sigma,\sigma)X^{-1}(\theta,\sigma,\sigma)X_\alpha(s,\sigma+es,\sigma) \\ &=G(s,\tau,t,\sigma). \end{aligned}$$

□

Corollary 3.1. *The matrix* $G_\alpha(\tau,\tau,t,\sigma)=G_\alpha^0(\tau,t,\sigma)$ *is multiperiodic in* (τ,t,σ) *with the period* (θ,ω,ω).

The proof of Corollary 3.1 follows from (3.7) with $s=\tau$.

3.5. The matrix $G_\alpha(s,\tau,t,\sigma)$ satisfies to the estimate

$$|G_\alpha(s,\tau,t,\sigma)|\leqslant\Delta e^{\delta|\tau-s|}, \tag{3.8}$$

where $|G_\alpha|$ is the Euclidean norm of the matrix G_α, $\delta=\|P\|$, $\Delta=\|G_\alpha^0\|$, $\|\cdot\|$ – maximization of quantity $|\cdot|$ in arguments.

Proof. As the matrix $G_\alpha(s,\tau,t,\sigma)$, in accordance with property (3.4) is a matrix solution of equation (2.1), it can be represented as

$$G_\alpha(s,\tau,t,\sigma)=Y(s,\tau,t,\sigma)G_\alpha(s,s,\sigma+es,\sigma), \tag{3.9}$$

where $Y(s,\tau,t,\sigma)=X(\tau,t,\sigma)X^{-1}(s,\sigma+es,\sigma)$ is a matrix solution of equation (2.1) with the initial condition $Y(s,s,t,\sigma)=E$, for which we have the estimate

$$|Y(s,\tau,t,\sigma)|\leqslant e^{\|P\||\tau-s|} \tag{3.10}$$

and the matrix $G(s, s, \sigma + es, \sigma)$ admits to the estimate

$$|G_\alpha(s, s, \sigma + es, \sigma)| \leqslant \|G_\alpha^0\|. \tag{3.11}$$

Inequality (3.8) follows from equality (3.9), estimates (3.10) and (3.11). □

The matrix $G_\alpha(s, s, t, \sigma)$ with constitutive properties 3.1-3.3 can be called Green's matrix of the multiperiodic problem for linear system (1.6) with condition (1.7).

The obtained results can be stated in the form of the following theorem.

Theorem 3.1. *Under conditions (1.8) and (2.9) for the multiperiodic problem for linear system (1.6) with condition (1.7) there is a matrix Green's function (3.3) satisfying estimate (3.8).*

4 Multiperiodic integral of linear inhomogeneous system

Considering system (1.6) along characteristics (1.3) of the operator D_e we have arrived at the complete integral of the form

$$x(\tau, t, \sigma, c) = X(\tau, t, \sigma)c + \tilde{x}(\tau, t, \sigma)$$

with constant vectors $\sigma = (\sigma_1, ..., \sigma_m)$ and $c = (c_1, ..., c_n)$, where $\tilde{x}(\tau, t, \sigma)$ is some integral of system (1.6).

Now, in order to solve the basic problem we will prove the below theorem.

Theorem 4.1. *Assume that conditions (1.8), (1.9) and (2.9) are satisfied. Then the multiperiodic problem for linear system (1.6) with condition (1.7) has a unique integral $x^*(\tau, t, \sigma)$, depending ω-periodically in σ, which can be represented in the form*

$$x^*(\tau, t, \sigma) = \int_\tau^{\tau+\theta} G_\alpha(s, \tau, t, \sigma) f_\alpha(s, \sigma + es) ds \tag{4.1}$$

and satisfies to the estimate

$$\|x^*\| \leqslant \frac{\Delta}{\delta}(e^{\delta\theta} - 1)\|f\|, \tag{4.2}$$

where $f_\alpha(s, \sigma + es)$ is the function obtained from the function $f(\tau, t)$ by formula (3.2).

Proof. Due to conditions (1.8), (2.9) and by Theorem 3.1, the problem has Green's function (3.3). To show that function (4.1) satisfies system (1.6), its we represent it as the sum of two integrals

$$x^*(\tau, t, \sigma) = \int_\tau^\alpha G_\alpha(s, \tau, t, \sigma) f(s, \sigma + es) ds + \int_\alpha^{\tau+\theta} G_\alpha(s, \tau, t, \sigma) f(s, \sigma + es + e\theta) ds$$

with the general constant boundary α.

Further, having applied the operator D_e, in view of property (3.4) of Green's function and smoothness (1.9) of the function $f(\tau, t)$, we have

$$D_e x^*(\tau, t, \sigma) = P(\tau, t)x^*(\tau, t, \sigma) + [G_\alpha(\tau + \theta, \tau, t, \sigma) - G_\alpha(\tau, \tau, t, \sigma)]f(\tau, t).$$

Next on the basis of property (3.4), we see that (4.1) is a solution to system (1.6).

We will show now that the solution of (4.1) satisfies (1.7). Indeed, from the condition of ω-periodicity (1.9) of the function $f(\tau,t)$ follows the ω-periodicity of $f_\alpha(s,\sigma+es)$ in $\sigma\in R^m$. According to relation (3.7), Green's function has the property ω-periodicity in t and σ, and therefore the function $x^*(\tau,t,\sigma)$ also has this property. To prove the θ-periodicity of the solution in τ, τ we shift in the θ and change the variable s to $s+\theta$. Then, by (4.1)

$$x^*(\tau+\theta,t,\sigma)=\int\limits_{\tau}^{\tau+\theta}G_\alpha(s+\theta,\tau+\theta,t,\sigma)f_\alpha(s+\theta,\sigma+es)ds,$$

which due to property (3.7) and the θ-periodicity of $f(\tau,t)$ in τ is equal to $x^*(\tau,t,\sigma)$.

Thus, the (θ,ω,ω)-periodicity $x^*(\tau,t,\sigma)$ in (τ,t,σ) is proved.

In view of estimate (3.8) and the inequality $|f(s,\sigma+es)|\leqslant\|f\|$ by (4.1) we have the estimate

$$|x^*(\tau,t,\sigma)|\leqslant\Delta\int\limits_{\tau}^{\tau+\theta}e^{\delta|\tau-s|}\|f\|ds,$$

which follows by estimate (4.2).

Now we prove uniqueness and the (θ,ω,ω)-periodicity of integral (4.1). Indeed, if we assume that there is another (θ,ω,ω)-periodic in (τ,t,σ) integral $x_*(\tau,t,\sigma)$, then their difference $x^*(\tau,t,\sigma)-x_*(\tau,t,\sigma)$ is a non-trivial integral of homogeneous system (2.1). The existence of such an integral, in of view 2.7 contradicts with condition (2.9). Consequently, the integral (4.1) is unique.

□

Concluding the study of the problem, we note that the complete integral $x(\tau,t,\sigma,c)$ of inhomogeneous system (1.6) under the hypotheses of Theorem 4.1 can be written as

$$x(\tau,t,\sigma,c)=X(\tau,t,\sigma)c+x^*(\tau,t,\sigma),\tag{4.3}$$

where $x^*(\tau,t,\sigma)$ is a solution of problem (1.6)-(1.7). The general solution in the form of Cauchy with the initial condition $x|_{\tau=0}=u(t)$ for arbitrary differentiable vector-function $u\in C_t^{(1)}(R^m)$, can be obtained by using the method of the variation arbitrary constants, from complete integral (4.3) as

$$x(\tau,t,t-e\tau,u(t-e\tau))=X(\tau,t,t-e\tau)u(t-e\tau)+x^*(\tau,t,t-e\tau).\tag{4.4}$$

Next formula (4.1), in accordance with relation (4.4), is a particular oscillatory solution

$$x^*(\tau,t,t-e\tau)=\int\limits_{\tau}^{\tau+\theta}G_\alpha(s,\tau,t,t-e\tau)f_\alpha(s,t-e\tau+es)ds.\tag{4.5}$$

It is obvious, that solution (4.5) is ω-periodic in t, but in view incommensurability of the periods $\theta=\omega_0$, $\omega_1,...,\omega_m$ it can not be θ-periodic in τ.

For $t=e\tau$ system of equations (1.6) reduces to the system of ordinary differential equations of the form

$$\frac{d\hat{x}}{d\tau}=P(\tau,e\tau)\hat{x}+f(\tau,e\tau),\tag{4.6}$$

which in accordance with formula (4.5) has the solution:

$$x^*(\tau, e\tau, 0) = \int_{\tau}^{\tau+\theta} G_\alpha(s, \tau, e\tau, 0) f_\alpha(s, es) ds. \tag{4.7}$$

According to Bohr's theorem, solution (4.7) is a quasi-periodic solution of system (4.6) with frequency basis $\nu_0 = \frac{2\pi}{\theta}$, $\nu_1 = \frac{2\pi}{\omega_1}$, ..., $\nu_m = \frac{2\pi}{\omega_m}$. This leads to the following statement.

Corollary 4.1. *Under the hypotheses of Theorem 4.1 quasi-periodic system (1.6) in τ has a unique quasi-periodic solution (4.7).*

In the conclusion we note that Theorem 4.1 in the terms of Green's function, originally proposed by a convenient method to the study of multiperiodic solutions of linear systems with operator D_e on their multiperiodic complete integrals.

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