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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate "For the merits in the development of science in the Republic of Kazakhstan", in 2007 and 2014 the state grant "The best university teacher", in 2016 the Order "Kurmet" (= "Honour").

R. Oinarov was born in the village Kul'Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis "Continuity and Lipschitzness of nonlinear integral operators of Uryson's type" at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis "Weighted estimates of integral and differential operators" at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov's kernels* or *Oinarov condition.* On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user=NzXYMS4AAAAJhl=ruoi=ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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SOME NEW INEQUALITIES FOR THE FOURIER TRANSFORM FOR FUNCTIONS IN GENERALIZED LORENTZ SPACES

A.N. Kopezhanova

Communicated by L.-E. Persson

Dedicated to the 70^{th} *birthday of Professor Ryskul Oinarov*

Key words: Fourier transform, Hausdorff-Young's inequality, generalized Lorentz spaces, weight function, generalized monotone function.

AMS Mathematics Subject Classification: 46E30, 42A38.

Abstract. The classical Hausdorff-Young and Hardy-Littlewood-Stein inequalities, relating functions on $\mathbb{R}$ and their Fourier transforms, are extended and complemented in various ways. In particular, a variant of the Hardy-Littlewood-Stein inequality covering the case $p \geqslant 2$ is proved and two-sided estimates are derived.

1 Introduction

Let

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx} dx,\ t \in \mathbb{R},$$

be the Fourier transform of a function $f \in L_1(\mathbb{R})$.

The following well-known inequalities relate some integral properties of functions and their Fourier transforms.

Let $1 < p < 2$, $p' = \frac{p}{p-1}$, and $0 < q \leqslant \infty$. Then we have the following inequalities

$$\|\widehat{f}\|_{L_{p'}(\mathbb{R})} \leqslant c_1 \|f\|_{L_p(\mathbb{R})}, \tag{1.1}$$

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \leqslant c_2 \|f\|_{L_{p,q}(\mathbb{R})}, \tag{1.2}$$

where $L_{p,q}(\mathbb{R})$ is the classical Lorentz space. These inequalities are called the Hausdorff-Young inequality and the Hardy-Littlewood-Stein inequality, respectively (see e.g. [5], [13] and [14]). There are similar inequalities for Fourier transform $\widehat{f} = \{\widehat{f}_n\}$ on the interval $[0, 1]$, where $\widehat{f}_n$ are the Fourier coefficients of functions with respect to a bounded orthonormal systems (see [5], [15] and some new developments related to this paper in [6] and [10]).

In [11] and [12] the authors introduced new function spaces, which they called net spaces. Using their properties, Hausdorff-Young type inequalities and its reverse inequalities in Lorentz spaces were obtained.

Let $1 < p < 2$ and $0 < q \leqslant \infty$. Then for the Fourier transform the following inequalities

$$c_1(p, q)\|H\|_{L_{p,q}(\mathbb{R},dx)} \leqslant \|\widehat{f}\|_{L_{p',q}(\mathbb{R},dx)} \leqslant c_2(p, q)\|f\|_{L_{p,q}(\mathbb{R},dx)} \tag{1.3}$$

hold, where $Hf(x)$ is the following Hardy-type operator

$$Hf(x)=\frac{1}{|x|}\int_{-|x|}^{|x|}f(t)dt.$$

Note that the left-hand side of inequality (1.3) holds for $1<p<\infty$. In particular, the following holds.

Let $1<p<2$ and $0<q\leqslant\infty$. If $|f(x)|\leqslant c|Hf(x)|$, then

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R},dx)}\sim\|f\|_{L_{p,q}(\mathbb{R},dx)}.$$

Let $2<p<\infty$ and $0<q\leqslant\infty$. If $|\widehat{f}(x)|\leqslant c|H\widehat{f}(x)|$, then

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R},dx)}\sim\|f\|_{L_{p,q}(\mathbb{R},dx)}.$$

The aim of this paper is to derive both upper and lower estimates of the norm of the Fourier transform in generalized Lorentz spaces. This means that also the reversed inequalities to (1.1) and (1.2) are obtained for the Fourier transform on $\mathbb{R}$.

The main results are formulated in Section 3. The proofs can be found in Section 4 and in Section 2 we have presented some necessary preliminaries.

Conventions The letter $c(c_1,c_2,\text{etc.})$ means a constant which does not dependent on the involved functions and it can be different in different occurences. Moreover, for $A,B>0$ the notation $A\sim B$ means that there exists positive constants a_1 and a_2 such that $a_1A\leqslant B\leqslant a_2A$.

2 Preliminaries

Let f be a measurable function on $\mathbb{R}$ and μ is the Lebesgue measure. The distribution function $m(\sigma,f)$ and the nonincreasing rearrangement f^* of a function f are defined as follows:

$$m(\sigma,f):=\mu\left\{x\in\mathbb{R}:|f(x)|>\sigma\right\},$$

$$f^*(t):=\inf\left\{\sigma:m(\sigma,f)\leqslant t\right\}.$$

Let ω be a nonnegative function on $[0,\infty)$. The generalized Lorentz space $\Lambda_q(\omega)$ consists of all functions f on $\mathbb{R}$ such that $\|f\|_{\Lambda_q(\omega)}<\infty$, where

$$\|f\|_{\Lambda_q(\omega,\mathbb{R})}:=\begin{cases}\left(\int_0^\infty\left(f^*(t)\omega(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} & \text{for}\quad 0<q<\infty,\\ \sup\limits_{t>0}f^*(t)\omega(t) & \text{for}\quad q=\infty,\end{cases}$$

where f^* is the nonincreasing rearrangement of the function f and ω denotes a positive and measurable function on $(0,\infty)$. These spaces $\Lambda_q(\omega)$ coincide with the classical Lorentz spaces L_{pq} in the case $\omega(t)=t^{\frac{1}{p}}$, $1<p<\infty$ (see [9] and also e.g. [2]).

Let $\mathfrak{M}$ be the class of all Lebesgue measurable functions on $(0,+\infty)$ and $\mathfrak{M}^+:=\{g\in\mathfrak{M}:\ g\geqslant 0\}$. $\mathfrak{M}^\downarrow$ denotes the cone of all nonincreasing functions from $\mathfrak{M}^+$. Suppose that $u,v,\omega\in\mathfrak{M}^+$. Let

$$G^1_{pq}=G^1_{pq}(\omega,u,v;\mathfrak{M}^\downarrow):=\sup_{g\in\mathfrak{M}^\downarrow}\frac{\left(\int\limits_0^\infty\left(\int\limits_0^t g(s)u(s)ds\right)^q\omega(t)dt\right)^{\frac{1}{q}}}{\left(\int\limits_0^\infty|g(t)|^p v(t)dt\right)^{\frac{1}{p}}},$$

and

$$G^2_{pq} = G^2_{pq}(\omega, u, \upsilon; \mathfrak{M}^{\downarrow}) := \sup_{g\in\mathfrak{M}^{\downarrow}} \frac{\left(\int\limits_0^\infty \left(\int\limits_t^\infty g(s)u(s)ds\right)^q \omega(t)dt\right)^{\frac{1}{q}}}{\left(\int\limits_0^\infty |g(t)|^p \upsilon(t)dt\right)^{\frac{1}{p}}}.$$

The constants G^1_{pq} and G^2_{pq} are obviously closely related (as operator norms) to the modern theory of Hardy type inequalities (see e.g. the books [7] and [8] and references therein).

In [1], [3], [8] and [7] the characterizations of these functionals in terms of weight functions were proved.

3 Main results

For our first main result we need to define the function $\overline{f}$ as follows

$$\overline{f}(t) := \sup_{y\geqslant t} \frac{1}{2y}\left|\int_{-y}^{y} f(s)ds\right|, \ t, y > 0.$$

Theorem 3.1. *Let $0 < p < \infty$, $0 < q < \infty$. Let ν, μ be weight functions such that*

$$G^1_{pq}\left(\frac{\nu^q(\frac{1}{t})}{t}, 1, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty,$$

$$G^2_{pq}\left(\frac{\left(t\nu(\frac{1}{t})\right)^q}{t}, \frac{1}{t}, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty.$$

Then, for all $f \in \Lambda_p(\mu, \mathbb{R})$, the following inequality

$$\|\overline{\widehat{f}}\|_{\Lambda_q(\nu,\mathbb{R})} \leqslant c_1\|f\|_{\Lambda_p(\mu,\mathbb{R})} \tag{3.1}$$

holds.

Inequality (3.1) (and similar ones futher on) is understood in the sense that if the right-hand side of the inequality is finite, then the left-hand side is also finite and the corresponding inequality holds.

Remark 1. For the case $\nu(t) = t^{\frac{1}{p'}}$, $\mu(t) = t^{\frac{1}{p}}$, $1 < p < \infty$, $0 < q < \infty$, the inequality (3.1) implies the following inequality

$$\|\overline{\widehat{f}}\|_{L_{p'q}} \leqslant c_2\|f\|_{L_{pq}}, \tag{3.2}$$

e.g. an estimate from below is obtained for the norm $\|f\|_{L_{pq}}$ by the Fourier transform of the function f. We especially emphasize that Hardy-Littlewood-Stein inequality (1.2) does not cover the case $2 \leqslant p < \infty$. Inequality (3.2) was obtained in [11].

Our next main result is a generalization of inequality (1.2).

Theorem 3.2. *Let $0 < p < \infty$, $0 < q < \infty$. Let ν, μ be the weight functions such that*

$$G^1_{pq}\left(\frac{\nu^q\left(\frac{1}{t}\right)}{t}, 1, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty,$$

$$G^2_{pq}\left(\frac{\left(t^{\frac{1}{2}}\nu\left(\frac{1}{t}\right)\right)^q}{t}, \frac{1}{t^{\frac{1}{2}}}, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty.$$

Then, for all $f \in \Lambda_p(\mu, \mathbb{R})$, *the following inequality*

$$\|\widehat{f}\|_{\Lambda_q(\nu,\mathbb{R})} \leqslant c_3\|f\|_{\Lambda_p(\mu,\mathbb{R})}$$

holds.

Corollary 3.1. *Let* $0 < p < \infty$. *Let* ν, μ *be the weight functions such that*

$$G^1_p\left(\frac{\nu^p\left(\frac{1}{t}\right)}{t}, 1, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty,$$

$$G^2_p\left(\frac{\left(t^{\frac{1}{2}}\nu\left(\frac{1}{t}\right)\right)^p}{t}, \frac{1}{t^{\frac{1}{2}}}, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty.$$

Then, for all $f \in \Lambda_p(\mu, \mathbb{R})$, *the following two-sided estimates*

$$c_4\|\overline{f}\|_{\Lambda_p(\mu,\mathbb{R})} \leqslant \|\widehat{f}\|_{\Lambda_p(\nu,\mathbb{R})} \leqslant c_5\|f\|_{\Lambda_p(\mu,\mathbb{R})}$$

holds.

Remark 2. In particular, if $\nu(t) = t^{\frac{1}{p'}}$, $\mu(t) = t^{\frac{1}{p}}$, $0 < q < \infty$, $1 < p < 2$, then we have

$$c_6\|\overline{f}\|_{L_{pq}} \leqslant \|\widehat{f}\|_{L_{p'q}} \leqslant c_7\|f\|_{L_{pq}}. \tag{3.3}$$

We note that the left-hand side inequality in (3.3) follows by the results in [11] and [12], where the net spaces are used.

Definition 1. We say that a function f on $\mathbb{R}$ is generalized monotone if there exists some constant $M > 0$ such that

$$|f(x)| \leqslant M\frac{1}{2x}\left|\int_{-x}^{x} f(t)dt\right|, \quad x > 0.$$

This condition is a more general condition than monotonicity, quasi-monotonicity and GM conditions of generalized monotonicity in [4].

Corollary 3.2. *Let* f *or* $\widehat{f}$ *be a generalized monotone function,* $0 < p < \infty$. *Let* ν, μ *be weight functions such that*

$$G^1_p\left(\frac{\nu^p\left(\frac{1}{t}\right)}{t}, 1, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty,$$

and

$$G^2_p\left(\frac{\left(t^{\frac{1}{2}}\nu\left(\frac{1}{t}\right)\right)^p}{t}, \frac{1}{t^{\frac{1}{2}}}, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow}\right) < \infty.$$

Then, for all $f \in \Lambda_p(\mu, \mathbb{R})$, *the following equivalence*

$$\|\widehat{f}\|_{\Lambda_p(\nu,\mathbb{R})} \sim \|f\|_{\Lambda_p(\mu,\mathbb{R})}$$

holds.

4 Proofs of the main results

Proof of Theorem 3.1: Let $f \in \Lambda_p(\mu, \mathbb{R})$. Let $y > 0$ and note that

$$I := \sup_{y \geqslant t} \frac{1}{2y} \left| \int_{-y}^{y} \widehat{f}(s) ds \right| = \sup_{y \geqslant t} \frac{1}{2y\sqrt{2\pi}} \left| \int_{-y}^{y} \int_{-\infty}^{+\infty} f(x) e^{-isx} dx ds \right|$$

$$\leqslant \sup_{y \geqslant t} \frac{1}{2y\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(x)| \left| \int_{-y}^{y} e^{-isx} ds \right| dx$$

$$= \sup_{y \geqslant t} \frac{1}{y\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \left| \frac{\sin yx}{x} \right| dx \leqslant \sup_{y \geqslant t} \frac{1}{y\sqrt{2\pi}} \int_{0}^{+\infty} f^*(x) \min\left(y, \frac{2}{x}\right) dx.$$

Consider $E_t \subset \left\{x \in \mathbb{R} : \ |f(x)| \geqslant f^*\left(\frac{1}{t}\right)\right\}$ such that $|E_t| = t$. Moreover, we define the functions f_0 and f_1 as follows:

$$f_0(x) = \begin{cases} f(x) - f^*(\frac{1}{t}), & \text{if} \quad x \in E(t), \\ 0, & \text{if} \quad x \notin E(t), \end{cases} \tag{4.1}$$

and

$$f_1(x) = \begin{cases} f^*(\frac{1}{t}), & \text{if} \quad x \in E(t), \\ f(x), & \text{if} \quad x \notin E(t). \end{cases} \tag{4.2}$$

Let $f = f_0 + f_1$ and by using the inequality (for $x > 0$) $f^*(x) \leqslant f_0^*\left(\frac{x}{2}\right) + f_1^*\left(\frac{x}{2}\right)$, we obtain the following estimate from above:

$$I \leqslant \sup_{y \geqslant t} \frac{1}{y\sqrt{2\pi}} \int_0^{\infty} \left(f_0^*\left(\frac{x}{2}\right) + f_1^*\left(\frac{x}{2}\right)\right) \min\left(y, \frac{2}{x}\right) dx$$

$$= \sup_{y \geqslant t} \frac{2}{y\sqrt{2\pi}} \int_0^{\infty} \left(f_0^*(x) + f_1^*(x)\right) \min\left(y, \frac{1}{x}\right) dx.$$

Now, by considering (4.1) and (4.2), we find that

$$\sup_{y \geqslant t} \frac{2}{y\sqrt{2\pi}} \left[\int_0^{\frac{1}{t}} \left(f^*(x) - f^*\left(\frac{1}{t}\right)\right) \min\left(y, \frac{1}{x}\right) dx \right.$$

$$\left. + \int_0^{\frac{1}{t}} f^*\left(\frac{1}{t}\right) \min\left(y, \frac{1}{x}\right) dx + \int_{\frac{1}{t}}^{\infty} f^*(x) \min\left(y, \frac{1}{x}\right) dx \right]$$

$$= \sup_{y \geqslant t} \frac{2}{y\sqrt{2\pi}} \int_0^{\frac{1}{t}} f^*(x) \min\left(y, \frac{1}{x}\right) dx + \sup_{y \geqslant t} \frac{2}{\sqrt{2\pi} y} \int_{\frac{1}{t}}^{\infty} f^*(x) \min\left(y, \frac{1}{x}\right) dx$$

$$\leqslant \sup_{y \geqslant t} \frac{2}{\sqrt{2\pi}} \int_0^{\frac{1}{t}} f^*(x) dx + \sup_{y \geqslant t} \frac{4}{y\sqrt{2\pi}} \int_{\frac{1}{t}}^{\infty} \frac{f^*(x)}{x} dx$$

$$= \frac{2}{\sqrt{2\pi}}\left(\int_0^{\frac{1}{t}} f^*(x)dx + \frac{1}{t}\int_{\frac{1}{t}}^{\infty}\frac{f^*(x)}{x}dx\right).$$

Hence, we have the following estimate:

$$\sup_{y\geqslant t}\frac{1}{2y}\left|\int_{-y}^{y}\widehat{f}(s)ds\right| \leqslant \frac{2}{\sqrt{2\pi}}\left(\int_0^{\frac{1}{t}} f^*(x)dx + \frac{1}{t}\int_{\frac{1}{t}}^{\infty}\frac{f^*(x)}{x}dx\right).$$

Thus, we get that

$$\|\overline{\widehat{f}}\|_{\Lambda_q(\nu,\mathbb{R})} = \left(\int_0^{+\infty}\left(\overline{\widehat{f}}(t)\nu(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\leqslant \frac{2}{\sqrt{2\pi}}\left(\int_0^{+\infty}\left(\left(\int_0^{\frac{1}{t}} f^*(x)dx + \frac{1}{t}\int_{\frac{1}{t}}^{\infty}\frac{f^*(x)}{x}dx\right)\nu(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\leqslant \frac{2^{1+\left(\frac{1}{q}-1\right)_+}}{\sqrt{2\pi}}\left[\left(\int_0^{+\infty}\left(\nu(t)\int_0^{\frac{1}{t}} f^*(x)dx\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{+\infty}\left(\frac{\nu(t)}{t}\int_{\frac{1}{t}}^{+\infty} f^*(x)\frac{dx}{x}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\right]$$

$$= \frac{2^{1+\left(\frac{1}{q}-1\right)_+}}{\sqrt{2\pi}}\left[\left(\int_0^{+\infty}\left(\nu\left(\frac{1}{t}\right)\int_0^{t} f^*(x)dx\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{+\infty}\left(t\nu\left(\frac{1}{t}\right)\int_{t}^{+\infty} f^*(x)\frac{dx}{x}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\right].$$

By using the fact that

$$G^1_{pq}\left(\frac{\nu^q(\frac{1}{t})}{t},1,\frac{\mu^p(t)}{t};\mathfrak{M}^{\downarrow}\right)<\infty,\ \ G^2_{pq}\left(\frac{\left(t\nu(\frac{1}{t})\right)^q}{t},\frac{1}{t},\frac{\mu^p(t)}{t};\mathfrak{M}^{\downarrow}\right)<\infty$$

we obtain that

$$\|\overline{\widehat{f}}\|_{\Lambda_q(\nu,\mathbb{R})} \leqslant \frac{2^{1+\left(\frac{1}{q}-1\right)_+}}{\sqrt{2\pi}}\left(G^1_{pq}\left(\frac{\nu^q(\frac{1}{t})}{t},1,\frac{\mu^p(t)}{t};\mathfrak{M}^{\downarrow}\right)+\right.$$

$$\left.+G^2_{pq}\left(\frac{\left(t\nu(\frac{1}{t})\right)^q}{t},\frac{1}{t},\frac{\mu^p(t)}{t};\mathfrak{M}^{\downarrow}\right)\right)\|f\|_{\Lambda_p(\mu,\mathbb{R})} \leqslant c_1\|f\|_{\Lambda_p(\mu,\mathbb{R})}.$$

□

Proof of Theorem 3.2: Let $f \in \Lambda_p(\mu, \mathbb{R})$. Let $f = f_0 + f_1$ and for $t \in [0, \infty)$, by using the obvious inequalities

$$\|\widehat{f}\|_{L_{2,\infty}} \leqslant c_1 \|f\|_{L_{2,1}},$$

$$\|\widehat{f}\|_{L_\infty} \leqslant c_2 \|f\|_{L_1}$$

and $\widehat{f}^*(t) \leqslant \widehat{f_0}^*\left(\frac{t}{2}\right) + \widehat{f_1}^*\left(\frac{t}{2}\right)$, we can estimate $\widehat{f}^*(t)$ from above as follows:

$$\widehat{f}^*(t) \leqslant \widehat{f_0}^*\left(\frac{t}{2}\right) + \left(\frac{2}{t}\right)^{\frac{1}{2}} \left(\frac{t}{2}\right)^{\frac{1}{2}} \widehat{f_1}^*\left(\frac{t}{2}\right)$$

$$\leqslant c_3 \int_0^{+\infty} f_0^*(x)dx + \frac{c_3}{t^{\frac{1}{2}}} \int_0^{+\infty} x^{-\frac{1}{2}} f_1^*(x)dx.$$

Consider $E_t \subset \left\{x \in \mathbb{R} : \ |f(x)| \geqslant f^*\left(\frac{1}{t}\right)\right\}$ such that $|E_t| = t$. We define the functions f_0 and f_1 by

$$f_0(x) = \begin{cases} f(x) - f^*(\frac{1}{t}), & \text{if} \quad x \in E(t), \\ 0, & \text{if} \quad x \notin E(t), \end{cases} \tag{4.3}$$

$$f_1(x) = \begin{cases} f^*(\frac{1}{t}), & \text{if} \quad x \in E(t), \\ f(x), & \text{if} \quad x \notin E(t). \end{cases} \tag{4.4}$$

Now, by using (4.3) and (4.4) we obtain that

$$\int_0^{+\infty} f_0^*(x)dx = \int_0^{\frac{1}{t}} \left(f(x) - f^*\left(\frac{1}{t}\right)\right) dx \leqslant \int_0^{\frac{1}{t}} f^*(x)dx - \frac{f^*(\frac{1}{t})}{t}. \tag{4.5}$$

Similary, for the second integral we have that

$$\frac{1}{t^{\frac{1}{2}}} \int_0^{+\infty} x^{-\frac{1}{2}} f_1^*(x)dx = \frac{1}{t^{\frac{1}{2}}} \left(\int_0^{\frac{1}{t}} x^{-\frac{1}{2}} f^*\left(\frac{1}{t}\right) dx + \int_{\frac{1}{t}}^{+\infty} x^{-\frac{1}{2}} f^*(x)dx\right)$$

$$\leqslant \frac{2f^*(\frac{1}{t})}{t} + \frac{1}{t^{\frac{1}{2}}} \int_{\frac{1}{t}}^{+\infty} x^{-\frac{1}{2}} f^*(x)dx. \tag{4.6}$$

By combinig (4.5) and (4.6) we find that

$$\int_0^{\frac{1}{t}} f^*(x)dx - \frac{f^*(\frac{1}{t})}{t} + \frac{2f^*(\frac{1}{t})}{t} + \frac{1}{t^{\frac{1}{2}}} \int_{\frac{1}{t}}^{+\infty} x^{-\frac{1}{2}} f^*(x)dx$$

$$= \int_0^{\frac{1}{t}} f^*(x)dx + \frac{f^*(\frac{1}{t})}{t} + \frac{1}{t^{\frac{1}{2}}} \int_{\frac{1}{t}}^{+\infty} x^{-\frac{1}{2}} f^*(x)dx$$

$$\leqslant 2 \int_0^{\frac{1}{t}} f^*(x)dx + \frac{1}{t^{\frac{1}{2}}} \int_{\frac{1}{t}}^{+\infty} x^{-\frac{1}{2}} f^*(x)dx. \tag{4.7}$$

According to (4.7), we get that

$$\|\widehat{f}\|_{\Lambda_q(\nu,\mathbb{R})} \leqslant c_4 \left(\int_0^{+\infty} \left(\nu(t) \int_0^{\frac{1}{t}} f^*(x)dx + \nu(t) \frac{1}{t^{\frac{1}{2}}} \int_{\frac{1}{t}}^{+\infty} x^{-\frac{1}{2}} f^*(x)dx \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Hence, by using Minkowski's inequality and by making a change of variables in the outer integrals, we get that

$$\|\widehat{f}\|_{\Lambda_q(\nu,\mathbb{R})} \leqslant c_4 \left(\int_0^{+\infty} \left(\nu\left(\frac{1}{t}\right) \int_0^{t} f^*(x)dx \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$+ c_4 \left(\int_0^{+\infty} \left(t^{\frac{1}{2}} \nu\left(\frac{1}{t}\right) \int_t^{+\infty} f^*(x) \frac{dx}{x^{\frac{1}{2}}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

By using the fact that

$$G^1_{pq} \left(\frac{\nu^q\left(\frac{1}{t}\right)}{t}, 1, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow} \right) < \infty$$

and

$$G^2_{pq} \left(\frac{\left(t^{\frac{1}{2}} \nu\left(\frac{1}{t}\right)\right)^q}{t}, \frac{1}{t^{\frac{1}{2}}}, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow} \right) < \infty,$$

we have that

$$\|\widehat{f}\|_{\Lambda_q(\nu,\mathbb{R})} \leqslant c_5 \left(G^1_{pq} \left(\frac{\left(\nu\left(\frac{1}{t}\right)\right)^q}{t}, 1, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow} \right) \right.$$

$$\left. + G^2_{pq} \left(\frac{\left(t^{\frac{1}{2}} \nu\left(\frac{1}{t}\right)\right)^q}{t}, \frac{1}{t^{\frac{1}{2}}}, \frac{\mu^p(t)}{t}; \mathfrak{M}^{\downarrow} \right) \right) \|f\|_{\Lambda_p(\mu,\mathbb{R})}$$

$$\leqslant c_6 \|f\|_{\Lambda_p(\mu,\mathbb{R})}.$$

□

Proof of Corollaries: The proofs of Corollaries 3.1 and 3.2 follow directly from our Theorems 3.1 and 3.2 and the results in the papers [11] and [12]. □

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References

[1] L.S. Arendarenko, R. Oinarov, L.-E. Persson, *Some new Hardy-type integral inequalities on cones of monotone functions,* Advances in harmonic analysis and operator theory, 77–89, Oper. Theory Adv. Appl., 229, *Birkhäuser/Springer Basel AG, Basel*, 2013.

[2] J. Bergh, J.Löfström, *Interpolation spaces. An introduction,* Grundlehren der Mathematischen Wissenschaften, Springer Verlag, Berlin-New York, No. 223, 1976.

[3] A. Gogatishvili, V.D. Stepanov, *Reduction theorems for weighted integral inequalities on the cone of monotone functions,* Uspekhi Mat. Nauk, 68 (2013), no. 4 (412), 3–68 (in Russian). Translation in Russian Math. Surveys 68 (2013), no. 4, 597–664.

[4] M. Dyachenko, S. Tikhonov, *General monotone sequences and convergence of trigonometric series,* Topics in Classical Analysis and Applications in Honor of Daniel Waterman, World Scientific (2008), 88–101.

[5] G.B. Folland, *Fourier analysis and its applications,* Brooks/Cole Publishing, 1992.

[6] A. Kopezhanova, L.-E. Persson, *On summability of the Fourier coefficients in bounded orthonormal systems for functions from some Lorentz type spaces,* Eurasian Math. J. 1 (2010), no. 2, 76–85.

[7] A. Kufner, L. Maligranda, L.-E. Persson. *The Hardy inequality. About its history and some related results,* Vydavatelsky Servis, Plzen, 2007. 162 pp.

[8] A. Kufner, L.-E. Persson. *Weighted inequalities of Hardy type,* World Scientific, New Jersey-London-Singapore-Hong Kong, 2003.

[9] G.G. Lorentz, *Some new functional spaces,* Ann. Math. 51 (1950), 37 – 55.

[10] E.D. Nursultanov, *On the coefficients of multiple Fourier series from L_p-spaces,* Izv. Ross. Akad. Nauk, Ser. Mat. 64 (2000), no. 1, 95–122. (in Russian). Translation in Izv. Math. 64 (2000), no. 1, 93–120.

[11] E.D. Nursultanov, *Network space and Fourier transform,* Dokl. Ross. Akad. Nauk. 361 (1998), no. 5, 597–599 (in Russian). Translation in Acad. Sci. Dokl. Math. 58 (1998), no. 1, 105–107.

[12] E.D. Nursultanov, S. Tikhonov, *Net spaces and boundedness of integral operators,* J. Geom. Anal. 21 (2011), no. 4, 950–981.

[13] E.M. Stein, *Interpolation of linear operators,* Trans. Amer. Math. Soc. 83 (1956), 482–492.

[14] E.M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidian spaces,* Princeton University Press, 1971.

[15] A. Zygmund, *Trigonometric series (2nd ed.),* Cambridge University Press, 1959.

Aigerim Nurzhanovna Kopezhanova
Department of Engineering Sciences and Mathematics
Lulea University of Technology
SE 97187, Lulea, Sweden
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
Satpayev St 2
010008 Astana, Kazakhstan
E-mail: Kopezhanova@mail.ru

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