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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate “For the merits in the development of science in the Republic of Kazakhstan”, in 2007 and 2014 the state grant “The best university teacher”, in 2016 the Order “Kurmet” (= “Honour”).

R. Oinarov was born in the village Kul’Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis “Continuity and Lipschitzness of nonlinear integral operators of Uryson’s type” at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis “Weighted estimates of integral and differential operators” at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov’s kernels* or *Oinarov condition*. On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user=NzXYMS4AAAAJhl=ruoi=ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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ON AN ILL-POSED PROBLEM FOR THE LAPLACE OPERATOR WITH NONLOCAL BOUNDARY CONDITION

T.Sh. Kal'menov, B.T. Torebek

Communicated by A.A. Shkalikov

Dedicated to the 70^{th} *birthday of Professor Ryskul Oinarov*

Key words: Laplace operator, nonlocal boundary value problem, differential operator, criterion of well-posedness.

AMS Mathematics Subject Classification: 31A30, 31B30, 35J40.

Abstract. In this paper a nonlocal problem for the Poisson equation in a rectangular is considered. It is shown that this problem is ill-posed as well as the Cauchy problem for the Laplace equation. The method of spectral expansion via eigenfunctions of the nonlocal problem for equations with deviating argument allows us to establish a criterion of the strong solvability of the considered nonlocal problem. It is shown that the ill-posedness of the nonlocal problem is equivalent to the existence of an isolated point of the continuous spectrum for a nonself-adjoint operator with the deviating argument.

1 Introduction and statement of the main result

As is known, Hadamard [1] constructed an example showing the instability of the solutions of the Cauchy problem for the Laplace equation. In [2, 3] and other papers, this Cauchy problem is reduced to integral equations of the first kind, various methods of regularization of the problem are used, and its conditional correctness is established. In contrast to the that results, in this paper a new criterion of well-posedness (ill-posedness) of nonlocal boundary value problem for the Poisson equation in a rectangular is proved. The principal distinction of our work from the works of other authors is the application of spectral problems for equations with deviating argument in the study of ill-posed nonlocal boundary value problems. This method was first used in [4] for the solution of the Cauchy problem for the two-dimensional Laplace equation. Further, this method was developed in [5, 6, 7]. Let $\Pi = (0,1) \times (0,1)$ be the square. In Π we consider the nonlocal problem for the Poisson equation

$$Lu \equiv -\Delta u\left(x,t\right) = f\left(x,t\right), (x,t) \in \Pi, \tag{1.1}$$

with the Dirichlet condition

$$u\left(0,t\right) = 0, u\left(1,t\right) = 0, t \in \left[0,1\right], \tag{1.2}$$

and with nonlocal conditions

$$u\left(x,0\right) - \alpha u\left(x,1\right) = 0, u_t\left(x,0\right) + \alpha u_t\left(x,1\right) = 0, x \in \left[0,1\right]. \tag{1.3}$$

Here $\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$ is the Laplace operator and α is a real number.

Definition. A function $u \in L_2(\Pi)$ will be called a strong solution of the nonlocal problem (1.1)-(1.3), if there exists a sequence of functions $u_n \in C^2\left(\bar{\Pi}\right)$ satisfying conditions (1.2) and (1.3), such that u_n and Lu_n converge in the norm $L_2(\Pi)$ to $u(x,t)$ and $f(x,t)$, respectively.

It is easy to show that, if $\alpha^2 = 1$, then the nonlocal problem (1.1)-(1.3) has infinite number of linearly independent solutions. Consequently, when $\alpha^2 = 1$ problem (1.1)-(1.3) is not Fredholm. Therefore, everywhere in what follows, we assume that $\alpha^2 \neq 1$.

We construct an example showing that the stability of the solution of problem (1.1)-(1.3) is disrupted. By direct calculation, it is not difficult to make sure that if $\alpha^2 \neq 1$, then the function

$$u_k(x,t) = \frac{\sinh k\pi t + \alpha \sinh k\pi(1-t)}{(1-\alpha^2)k^2} \sin k\pi x,$$

is a solution to the Laplace equation with the boundary conditions (2) and

$$u_k(x,0) - \alpha u_k(x,1) = 0, \frac{\partial u_k}{\partial t}(x,0) + \alpha \frac{\partial u_k}{\partial t}(x,1) = \frac{\sin k\pi x}{k}, x \in [0,1].$$

It is easy to see that the boundary data tends to zero as $k \to \infty$, but the solution $u_k(x,t)$ does not tend to zero in any norm. So the solution of problem (1.1)-(1.3) is unstable. Therefore, problem (1.1) - (1.3) is ill-posed in the Hadamard sense.

In the sequel, the following eigenvalue problem for an elliptic equation with deviating argument will play an important role.

Find numerical values of λ (eigenvalues), for which the problem for the differential equation with a deviating argument

$$Lu \equiv -\Delta u(x,t) = \lambda u(x,1-t), (x,t) \in \Pi, \tag{1.4}$$

has nonzero solutions (eigenfunctions) satisfying conditions (1.2) and (1.3). It is easy to show that eigenvalue problem (1.4), (1.2), (1.3) is nonself-adjoint. Obviously, the equivalent representation of equation (1.4) has the form

$$LPu = \lambda u, (t,x) \in \Pi,$$

where $Pu(x,t) = u(x,1-t)$ is a unitary operator.

We consider the spectral problem for the Sturm-Liouville operator

$$-u_k''(x) = \mu_k u_k(x), x \in (0,1), \tag{1.5}$$

$$u_k(0) = u_k(1) = 0. \tag{1.6}$$

It is known, that problem (1.5)-(1.6) is self-adjoint and non-negatively definite in $L_2(0,1)$. All eigenvalues of problem (1.5) - (1.6) have the form $\mu_k = (\pi k)^2$, and the system of eigenfunctions $\left\{\frac{1}{\sqrt{2}}\sin \pi k x\right\}_{k=1}^{\infty}$ forms a complete orthonormal system in $L_2(0,1)$.

Theorem 1.1. *If $\alpha^2 \neq 1$, then spectral problem (1.4), (1.2), (1.3) has a system of eigenvectors forming a Riesz basis*

$$u_{km}(x,t) = v_{km}(t) \sin k\pi x, \tag{1.7}$$

where $k, m \in N$, $v_{km}(t)$ are nonzero solutions of the problem

$$v_{km}''(t) - (\pi k)^2 v_{km}(t) = \lambda_{km} v_{km}(1-t), 0 < t < 1, \tag{1.8}$$

$$v_{km}(0) - \alpha v_{km}(1) = v'_{km}(0) + \alpha v'_{km}(1) = 0, \tag{1.9}$$

and λ_{km} *are eigenvalues of problem* (1.4), (1.2), (1.3). *In addition, for large* k *the smallest eigenvalue* λ_{k1} *has the asymptotic behavior*

$$\lambda_{k1} = 4(\pi k)^2 e^{-\pi k}(1 + o(1)). \tag{1.10}$$

For the other eigenvalues of problem (1.4), (1.2), (1.3) *there is a uniform estimate (separated from zero and goes to infinity).*

Theorem 1.2. *Let* $\alpha^2 \neq 1$. *A strong solution of nonlocal problem* (1.1) - (1.3) *exists if and only if* $f(x, t)$ *satisfies the inequality*

$$\sum_{k=1}^{\infty} \left| \frac{\tilde{f}_{k1}}{\lambda_{k1}} \right|^2 < \infty, \tag{1.11}$$

where $\tilde{f}_{km} = (f(x, 1-t), w_{km}(x, t))$, *the system* $w_{km}(x, t)$ *is orthogonal to* $u_{km}(x, t)$.

If condition (1.11) *holds, then a solution of* (1.1)-(1.3) *can be written as*

$$u(x,t) = \sum_{k=1}^{\infty} \frac{\tilde{f}_{k1}}{\lambda_{k1}} v_{k1}(t) \sin k\pi x + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} v_{km}(t) \sin k\pi x. \tag{1.12}$$

By $\tilde{L}_2(\Pi)$ we denote a subspace of $L_2(\Pi)$, spanned by the $\{v_{k1}(t) \sin k\pi x\}_{k=p+1}^{\infty}$, $p \in N$ and by $\hat{L}_2(\Pi)$ we denote its orthogonal complement $L_2(\Pi) = \tilde{L}_2(\Pi) \oplus \hat{L}_2(\Pi)$.

Theorem 1.3. *Let* $\alpha^2 \neq 1$. *Then for any* $f \in \hat{L}_2(\Pi)$ *a solution of problem* (1.1)-(1.3) *exists, is unique and belongs to* $\hat{L}_2(\Pi)$. *This solution is stable and has the form*

$$u(x,t) = \sum_{k=1}^{p} \frac{\tilde{f}_{k1}}{\lambda_{k1}} v_{k1}(t) \sin k\pi x + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} v_{km}(t) \sin k\pi x. \tag{1.13}$$

2 Some auxiliary statements

In this section we present some auxiliary results to prove the main results.

Lemma 2.1. *Let* $\alpha^2 \neq 1$. *For each fixed value of the index* k *a system of normalized eigenvectors* $v_{km}(t), m = 1, 2, \ldots$ *of eigenvalue problem* (1.8)-(1.9), *corresponding to the eigenvalues* λ_{km}, *forms a Riesz basis in* $L_2(0,1)$.

The eigenvalues λ_{km} *are roots of the equation*

$$\frac{\sqrt{(k\pi)^2 + \lambda}}{\sqrt{(k\pi)^2 - \lambda}} = \coth \frac{\sqrt{(k\pi)^2 + \lambda}}{2} \coth \frac{\sqrt{(k\pi)^2 - \lambda}}{2}. \tag{2.1}$$

For each k *the system of eigenvectors of problem* (1.8)-(1.9) *can be obtained from orthonormal basis via bounded invertible transformation* $\mathcal{A}$. *Here the operator* $\mathcal{A}$ *does not depend on the index* k.

Proof. Applying the operator $\frac{d^2}{dt^2} - (\pi k)^2$ to both sides of equation (1.8), taking into account nonlocal condition in (1.9), we obtain the following problem for the equation

$$\frac{d^4 v_{km}}{dt^4}(t) - 2(\pi k)^2 \frac{d^2 v_{km}}{dt^2}(t) = \left(\lambda_{km}^2 + (k\pi)^4\right) v_{km}(t), 0 < t < 1, \tag{2.2}$$

with nonlocal conditions

$$\begin{cases} u(0) = \alpha u(1), \ u'(0) = -\alpha u'(1), \\ u''(1) - \alpha u''(0) = (1-\alpha^2)(k\pi)^2 u(1), \\ u'''(1) + \alpha u'''(0) = (1-\alpha^2)(k\pi)^2 u'(1). \end{cases} \tag{2.3}$$

It is easy to verify that for $\alpha^2 \neq 1$ boundary conditions (2.3) are Birkhof regular, and even strongly regular [8]. Then the system of eigenvectors of spectral problem (2.2)-(2.3) forms a Riesz basis (see [9, 10, 11]). It is easy to notice that the eigenfunctions of problem (2.2)-(2.3) are also the eigenfunctions of problem (1.8). Therefore the system of eigenvectors of spectral problem (1.8)-(1.9) forms a Riesz basis in $L_2(0,1)$.

A system of elements is a Riesz basis if and only if, when this system can be obtained from orthonormal basis via bounded invertible transformation (see. [12]). Further we need the exact form of this transformation. Let us consider the operator acting in $L_2(0,1)$ according to the formula

$$\mathcal{A}\varphi(t) = \varphi(t) - \alpha\varphi(1-t).$$

It is easy to verify that the operator $\mathcal{A}$ is bounded and for $\alpha^2 \neq 1$ is bounded invertible. The inverse operator acts by the formula

$$\mathcal{A}^{-1}\varphi(t) = \frac{1}{1-\alpha^2}\left(\varphi(t) + \alpha\varphi(1-t)\right).$$

Let $v_{km}(t)$ be eigenvectors of problem (2.2)-(2.3). We denote $\hat{v}_{km}(t) = \mathcal{A}^{-1}v_{km}(t)$. By direct calculation is not difficult to make sure that for each k the system of functions $\hat{v}_{km}(t)$ are eigenfunctions of an operator defined by the differential expression

$$\ell_k \hat{v} = \frac{d^4\hat{v}}{dt^4}(t) - 2(\pi k)^2 \frac{d^2\hat{v}}{dt^2}(t), \ 0 < t < 1,$$

and boundary conditions

$$\begin{cases} u(0) = 0, \ u'(0) = 0, \\ u''(1) = (k\pi)^2 u(1), \ u'''(1) = (k\pi)^2 u'(1). \end{cases}$$

The operator ℓ_k is self-adjoint. Consequently, for each k the system of normalized eigenvectors of the operator ℓ_k forms an orthonormal basis in $L_2(0,1)$. Thus, for each k the system of eigenvectors of problem (2.2)-(2.3) can be obtained from orthonormal basis $\hat{v}_{km}(t)$ by bounded invertible transformation $\mathcal{A}$. At the same time, this operator $\mathcal{A}$ does not depend on k.

It is easy to show that the general solution of equation (1.8) has the form

$$v(t) = c_1 \cosh\sqrt{(k\pi)^2 + \lambda}\left(t - \frac{1}{2}\right) + c_2 \sinh\sqrt{(k\pi)^2 - \lambda}\left(t - \frac{1}{2}\right),$$

where c_1 and c_2 are some constants. Using the nonlocal conditions (1.8), we arrive at the system of linear homogeneous equations concerning these constants. As we know, this system has a nontrivial solution if the determinant of the system is equal to zero. Thus, for determining the parameter λ we get equation (2.1). □

Let

$$\varpi_k(\lambda) = \ln\coth\frac{\sqrt{(k\pi)^2+\lambda}}{2} + \ln\coth\frac{\sqrt{(k\pi)^2-\lambda}}{2} - \ln\sqrt{\frac{(k\pi)^2+\lambda}{(k\pi)^2-\lambda}} = 0. \qquad (2.4)$$

Lemma 2.2. *There exists a number λ_0 such that for all*

$$0 < \lambda < \lambda_0 < \frac{(k\pi)^2}{4(k\pi)^2+\theta},\ k \geq 1,\ \theta \in (0,\,1),$$

the following statements are true:

1) the function $\varpi_k'(\lambda)$ is of a fixed sign;

2) for the function $\varpi_k''(\lambda)$ we have the inequality $\left|\lambda (k\pi)^2 \varpi_k''(\lambda)\right| < 1,\ k > 1.$

Proof. By Lemma 2.1 we have the real eigenvalues of (1.8) - (1.6), that is, real roots λ_{km} of equation (2.1). It is easy to verify that $\lambda_{km} > 0$. Indeed, let us write the asymptotic behavior of the smallest eigenvalues λ_{km} at $k \to \infty$.

Assuming $|\lambda| < 1$ and taking the logarithm of both sides of (2.1), we obtain (2.4). By calculating the derivative, we get $\varpi_k'(0) = -\frac{1}{(k\pi)^2}$. Then the desired boundary of monotonicity of $\varpi_k(\lambda)$ can be found from the relation

$$\varpi_k'(\lambda_0) = \varpi_k'(0) + \varpi_k''(\theta\lambda_0)\,\lambda_0 < 0.$$

Here $0 < \lambda_0 < 1$ and $\theta \in (0,1)$ are arbitrary numbers. Thus, for determining λ_0, we have the condition

$$\lambda_0 (k\pi)^2 \varpi_k''(\theta\lambda_0) < 1. \qquad (2.5)$$

Then the inequality

$$\varpi_k''(\lambda_0\theta) \leq \frac{1}{\left((k\pi)^2-\lambda_0\theta\right)} \frac{2+\left(1-e^{-\sqrt{(k\pi)^2-\lambda_0\theta}}\right)^2}{\left(1-e^{-\sqrt{(k\pi)^2-\lambda_0\theta}}\right)^2}$$

is true. Hence

$$\varpi_k''(\lambda_0\theta) < \frac{1}{\left((k\pi)^2-\lambda_0\theta\right)} \frac{3-2e^{-\sqrt{(k\pi)^2-\lambda_0\theta}}+e^{-2\sqrt{(k\pi)^2-\lambda_0\theta}}}{\left(1-e^{-\sqrt{(k\pi)^2-\lambda_0\theta}}\right)^2} \qquad (2.6)$$

Further, for large values k from (2.6) we obtain the validity of the inequality

$$\varpi_k''(\lambda_0\theta) \leq \frac{4}{(k\pi)^2-\lambda_0\theta}.$$

Applying condition (2.5) to the last inequality, we obtain the desired estimate for λ_0 :

$$\lambda_0 < \frac{(k\pi)^2}{4(k\pi)^2+\theta},\ k > 1,\ 0 < \theta < 1.$$

□

Consider now the problem of an asymptotic behavior of the eigenvalues of problem (1.8)-(1.9) for large k.

Lemma 2.3. *An asymptotic behavior of eigenvalues of problem* (1.8)-(1.9), *not exceeding* λ_0, *for the large values of* k *has form* (1.10).

Proof. According to Lemma 2.2, the monotonic function $f_k(\lambda)$ in the interval $(0,\ \lambda_0)$ can have only one zero. By the Taylor formula we have

$$\varpi_k(\lambda) = \varpi_k(0) + \frac{\varpi_k'(0)}{1!}\lambda + \frac{\varpi_k''(\theta\lambda)}{2!}\lambda^2 < 0\,,\ 0 < \theta < 1\,.$$

Substituting the calculated values of the function ϖ_k and its derivative ϖ_k', we get

$$\varpi_k(\lambda) = 2\ln\left(\coth\frac{k\pi}{2}\right) - \frac{\lambda}{(k\pi)^2} + \varpi_k''(\theta\lambda)\frac{\lambda^2}{2}.$$

Then the zero of the linear part of the function

$$(k\pi)^2\varpi_k(\lambda) = 2(k\pi)^2\ln\left(\coth\frac{k\pi}{2}\right) - \lambda + \frac{(k\pi)^2\lambda^2}{2}\varpi_k''(\theta\lambda)$$

will be

$$\lambda_{k1} = 2(k\pi)^2\ln\left(\frac{1+e^{-k\pi}}{1-e^{-k\pi}}\right).$$

For sufficiently large values $k\in N$, considering the asymptotic formulas, λ_{k1} can be written as

$$\lambda_{k1} = 4(k\pi)^2 e^{-k\pi}(1+o(1))\ .$$

Taking into account the result of Lemma 2.2 on a circle $|\lambda| = 4(k\pi)^2 e^{-k\pi}(1+\varepsilon)\,,$ where ε is a very small positive number, for sufficiently large $k\geq k_0(\varepsilon)$ it is easy to check the validity of the inequality

$$\left|\varpi_k''(\theta\lambda)(k\pi\lambda)^2\right|_{|\lambda|=4(k\pi)^2e^{-k\pi}(1+\varepsilon)} \leq C\left|2(k\pi)^2\ln\left(\frac{1+e^{-k\pi}}{1-e^{-k\pi}}\right) - \lambda\right|_{|\lambda|=4(k\pi)^2e^{-k\pi}(1+\varepsilon)}.$$

Then, by Rouche's theorem [13] we have that the quantity of zeros of $(k\pi)^2\varpi_k(\lambda)$ and its linear part coincide and are inside the circle $|\lambda| = 4(k\pi)^2e^{-k\pi}(1+\varepsilon)\,.$ Consequently, the function $(k\pi)^2\varpi_k(\lambda)$ for $0<\lambda<\lambda_0$ has one zero, the asymptotic behavior is given by formula (1.10). □

3 Proof of the main results

Proof of Theorem 1.1. The system of eigenfunctions $\sin k\pi x, k\in N$ of Sturm-Liouville problem (1.5) forms a complete orthonormal system in $L_2(0,1)\,.$ By Lemma 2.1 for each fixed value of k and for $\alpha^2\neq 1$ the spectral problem (1.8) has the system of eigenvectors $v_{km}(t)\,, m=1,2,...$ forming a Riesz basis in $L_2(0,1)\,.$ Here the system of eigenvectors $v_{km}(t)$ of the eigenvalue problem (1.8)-(1.9) can be obtained from orthonormal basis $\hat{v}_{km}(t)$ by the bounded invertible transformation $\mathcal{A}$, which does not depend on the index k. Therefore, the system (1.7) also can be obtained from the orthonormal basis $\hat{v}_{km}\sin k\pi x$ via the bounded invertible transformation $\mathcal{A}$. Consequently, system (1.7) forms a Riesz basis in $L_2(\Pi)\,.$ □

Proof of Theorem 1.2. Let $u(x,t) \in C^2(\Pi)$ be a solution to problem (1.1) - (1.3). Then, by the basicity of the eigenfunctions $u_{km}(x,t)$ of problem (1.4), (1.2), (1.3), the function $u(x,t)$ in $L_2(\Pi)$ can be expanded in a series [8]

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_{km} u_{km}(x,t), \tag{3.1}$$

where a_{km} are Fourier coefficients via the system $u_{km}(x,t)$. Rewriting equation (1.1) in the form

$$LPu = P\left(u_{tt}(x,t) + u_{xx}(x,t)\right) = Pf(x,t), \tag{3.2}$$

and substituting the solution of form (3.1) in equation (3.2) according to representation

$$P\Delta u_{km}(x,t) = \lambda_{km} u_{km}(x,t),$$

we have $a_{km} = \frac{\tilde{f}_{km}}{\lambda_{km}}$, where $\tilde{f}_{km} = \left(f(x,\, 1-t),\, w_{km}(x,\, t)\right)$.

Thus for solutions $u(x,t)$ we obtain the following explicit representation

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x,t). \tag{3.3}$$

Note that representation (3.3) remains true for any strong solution of problem (1.1) - (1.3). We have obtained this representation under the assumption that the solution to of nonlocal problem (1.1) - (1.3) exists.

The question naturally arises, for what subset of the functions $f \in L_2(\Pi)$ there exists a strong solution? To answer this question, we represent formula (3.3) in form (1.12) from which, by Bessel's inequality, it follows that

$$\|\mathrm{u}\|^2 \le \sum_{k=1}^{\infty}\left|\frac{\tilde{f}_{k1}}{\lambda_{k1}}\right|^2 + \sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\left|\frac{\tilde{f}_{km}}{\lambda_{km}}\right|^2. \tag{3.4}$$

By Lemma 2.3 we have $\lambda_{km} \ge \frac{1}{4}$, $m > 1$. Therefore, the right-hand side of equality (3.4) is bounded only for those $f(x,\, t)$, for which weighted norm (1.11) is finite. This fact completes the proof. □

Proof of Theorem 1.3. Obviously, the operator L is invariant in $\hat{L}_2(\Pi)$. By Theorem 1.2 for any $f \in \hat{L}_2(\Pi)$ there exists a unique solution of problem (1.1)-(1.3) and it can be represented in form (1.13). Therefore, the determined infinite-dimensional space $\hat{L}_2(\Pi)$ is the space of well-posedness of nonlocal problem (1.1)-(1.3). □

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References

[1] J. Hadamard, *Lectures on the Cauchy problem in linear differential equations*, Yale University Press, New Haven, CT, 1923.

[2] M.M. Lavrent'ev, *On a Cauchy problem for the Poisson equation*, Izvestya Akad. Nauk USSR, Ser. Math. 20 (1955), no. 6, 819–842.

[3] A.N. Tikhonov, V.Ya. Arsenin, *Methods for solving ill-posed problems*, Moskow. Nauka. 1979. 142 pp.

[4] T.S. Kal'menov, U.A. Iskakova, *A criterion for the strong solvability of the mixed Cauchy problem for the Laplace equation*, Doklady Mathematics. 75 (2007), no. 3, 370–373.

[5] T.S. Kal'menov, U.A. Iskakova, *Criterion for the strong solvability of the mixed Cauchy problem for the Laplace equation*, Differential Equations. 45 (2009), no. 10, 1460–1466.

[6] B.T. Torebek, *A method for solving ill-posed Robin-Cauchy problems for second-order elliptic equations in multi-dimensional cylindrical domains*, Electronic Journal of Differential Equations. 2016 (2016), no. 252. 1–9.

[7] T.S. Kal'menov, M.A. Sadybekov and U.A. Iskakova, *On a criterion for the solvability of one ill-posed problem for the biharmonic equation*, J. Inverse Ill-Posed Probl. (2016), doi: 10.1515/jiip-2016-0003

[8] M.A. Naimark, *Linear differential operators*, Part II. Ungar, New York, 1968.

[9] V.P. Mihailov, *On Riesz bases in* $L_2(0,1)$, Dokl. Akad. Nauk USSR, 144 (1962), 981–984.

[10] G.M. Kesel'man, *On the unconditional convergence of eigenfunction expansions of certain differential operators*, Izv. Vys. Uchebn. Zaved. Matematika, 39 (1964), no. 2, 82–93.

[11] A.A. Shkalikov, *On the basis problem of the eigenfunctions of an ordinary differential operator*, Russian Mathematical Surveys, 34 (1979), no. 34, 249–250.

[12] A.A. Shkalikov, *Perturbations of self-adjoint and normal operators with discrete spectrum*, Russian Mathematical Surveys, 71 (2016), no. 5, 907–964.

[13] E. Titchmarsh, *The theory of functions*, Moscow, Nauka, 1980.

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