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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate “For the merits in the development of science in the Republic of Kazakhstan”, in 2007 and 2014 the state grant “The best university teacher”, in 2016 the Order “Kurmet” (= “Honour”).

R. Oinarov was born in the village Kul’Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis “Continuity and Lipschitzness of nonlinear integral operators of Uryson’s type” at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis “Weighted estimates of integral and differential operators” at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov’s kernels* or *Oinarov condition*. On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

$$https://scholar.google.com/citations?user=NzXYMS4AAAAJhl=ruoi=ao$$

Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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MODULAR AND NORM INEQUALITIES FOR OPERATORS ON THE CONE OF DECREASING FUNCTIONS IN ORLICZ SPACE

E.G. Bakhtigareeva, M.L. Goldman

Communicated by V.D. Stepanov

Dedicated to the 70^{th} birthday of Professor Ryskul Oinarov

Key words: weighted Orlicz spaces, modular and norm inequalities, cone of decreasing functions, reduction theorems.

AMS Mathematics Subject Classification: 46E30, 42A16.

Abstract. Modular and norm inequalities are considered on the cone of all nonnegative functions as well as on the cone Ω of all nonnegative decreasing functions in the weighted Orlicz space. Reduction theorems are proved for the norm of positively homogeneous operator on the cone Ω. We show that it is equivalent to the norm of a certain modified operator on the cone of all nonnegative functions in this space. Analogous results are established for modular inequalities

1 Introduction

We consider modular and norm inequalities for positively homogeneous operators on the cone of all nonnegative functions as well as on the cone Ω of all nonnegative decreasing functions in the weighted Orlicz space.

In Section 2 we show that modular inequality is equivalent to one-parametrical family of norm-type inequalities. Here we extend approaches developed in the papers of S. Bloom and R. Kerman [1, 2], and P. Drabek, A. Kufner, and H. P. Heinig [3].

In Section 3 a reduction theorem is established for the norm of positively homogeneous operator on the cone Ω. We show that it is equivalent to the norm of a certain modified operator on the cone of all nonnegative functions in this space. The result is based on the principle of duality that gives the description of the associate Orlicz norm for the cone Ω. First it was developed in the well-known paper of E. Sawyer [4] for the cone Ω in the weighted Lebesgue space. Then, H. P. Heinig and A. Kufner [5] have obtained the description of the associate norm for the cone Ω in the weighted Orlicz space under assumptions that N-function Φ determining the Orlicz space L_Φ as well as its complementary function Ψ both satisfy the well-known Δ_2-condition. Without the Δ_2 conditions the description of the associate norm for the cone Ω in the weighted Orlicz space was obtained first in [6]. In [7, 8] we develop this approach by including the case of arbitrary Young functions and general operators. Note that in general case the form of the answer differs essentially from the one given in [5].

Section 4 is devoted to a reduction theorem for modular inequalities. On the base of the results obtained in Sections 2 and 3 we show the equivalence of modular inequalities on the cone Ω and modified modular inequalities on the cone of all nonnegative functions in the Orlicz space.

2 The relations between modular and norm-type inequalities

Let (X, Σ, μ) and (Y, Ξ, ν) be the spaces with nonnegative σ-finite measures; let $M(X), M(Y)$ be the corresponding sets of all real-valued measurable functions, $M_+(X), M_+(Y)$ be the subsets of all functions that are nonnegative almost everywhere. Let $K \subset M_+(X)$ be a cone of functions, such that

$$f \in K, \alpha \geqslant 0 \quad \Rightarrow \quad \alpha f \in K; \tag{2.1}$$

and $T : K \to M_+(Y)$ be a positively homogenous operator, that is

$$f \in K, \alpha \geqslant 0 \quad \Rightarrow \quad T(\alpha f) = \quad \alpha T(f). \tag{2.2}$$

We introduce the class Θ_0 of functions Φ with the following properties

$$\Theta_0 = \left\{ \begin{array}{l} \Phi : [0, \infty) \to [0, \infty]; \quad \Phi \uparrow; \quad \Phi(t) = 0 \Leftrightarrow t = 0; \\ \exists t_\infty = t_\infty(\Phi) \in (0, \infty] : \quad \Phi \in C[0, t_\infty), \Phi(t_\infty - 0) = \infty \end{array} \right\}. \tag{2.3}$$

By Φ^{-1} we denote the left-continuous inverse function for $\Phi \in \Theta_0$, that is

$$\Phi^{-1}(\tau) = \inf\{t \in [0, \infty) : \ \Phi(t) \geqslant \tau\}, \quad \tau \in [0, \infty). \tag{2.4}$$

Introduce,

$$\sigma_\Phi(\varepsilon) = \Phi\left(\varepsilon^{-1}\right)^{-1}, \quad \Phi \in \Theta_0. \tag{2.5}$$

$$\|f\,|L_\Phi(X, d\mu)\| = \inf\{\lambda > 0 : J_\lambda(f; \Phi, X, d\mu) \leqslant 1\}; \tag{2.6}$$

$$J_\lambda(f; \Phi, X, d\mu) := \int_X \Phi\left(\lambda^{-1}|f|\right) d\mu, \quad f \in M(X). \tag{2.7}$$

In the case of Young functions considered in Sections 2 and 3 (2.6) is a norm.

Theorem 2.1. *Under the above notations and conditions let*

$$\Phi_1, \Phi_2 \in \Theta_0; \quad t_\infty(\Phi_1) = t_\infty(\Phi_2) := t_\infty; \quad u \in M_+(X), w \in M_+(Y).$$

Then, the modular inequality

$$\Phi_2^{-1}\left\{\int_Y \Phi_2(wTf)\, d\nu\right\} \leqslant \Phi_1^{-1}\left\{\int_X \Phi_1(Cuf)\, d\mu\right\} \tag{2.8}$$

holds for all $f \in K$ with a constant $C \in R_+$ independent of f if and only if the following norm-type inequalities hold for all $f \in K$ and $\varepsilon \in (t_\infty^{-1}, \infty)$

$$\|wTf\,|L_{\Phi_2}(Y, \sigma_{\Phi_2}(\varepsilon)\, d\nu)\| \leqslant C\,\|uf\,|L_{\Phi_1}(X, \sigma_{\Phi_1}(\varepsilon)\, d\mu)\|. \tag{2.9}$$

Remark 1. Theorem 2.1 is essentially more general then related Proposition 2.1 in [3]. Indeed, we deal here with more general modular functions $\Phi_1, \Phi_2 \in \Theta_0$, we work on the cone $K \subset M_+$ and we do not require the linearity of operator T. Moreover, in the case of $N-$functions Φ considered in [3] automatically follows that $t_\infty(\Phi) = \infty$, hence $t_\infty(\Phi_1) = t_\infty(\Phi_2) = \infty$. Nevertheless, the proof of Theorem 2.1 is similar to the proof in [3] and it is omitted here.

Corollary 2.1. *Under the assumptions of Theorem 2.1 let $\Phi_1 = \Phi_2 \equiv \Phi$. Then the integral inequality holds*

$$\int\limits_Y \Phi\left(wTf\right) d\nu \leqslant \int\limits_X \Phi\left(Cuf\right) d\mu \tag{2.10}$$

for all $f \in K$ with a constant $C \in R_+$ independent of f if and only if the norm-type inequality holds

$$\left\| wT\left(f\right) | L_\Phi\left(Y, \delta d\nu\right) \right\| \leqslant C \left\| uf \left| L_\Phi\left(X, \delta d\mu\right)\right.\right\|. \tag{2.11}$$

for all $f \in K$ and $\delta \in R_+$.

This conclusion generalizes Proposition 2.5 in [1].

3 The reduction theorem for norms of operators on the cone of monotonic functions in Orlicz space

3.1 Notations and preliminary results

We assume in this Section that $X = Y = R_+$, $M\left(X\right) = M\left(Y\right) = M\left(R_+\right)$ is the set of all Lebesgue-measurable functions, M_+ is the cone of all almost everywhere nonnegative functions in $M = M\left(R_+\right)$; μ, ν are weighted Lebesgue measures with weights in $\dot{M}_+$, that is

$$\dot{M}_+ = \left\{f \in M\left(R_+\right) : 0 < f < \infty\right\}, \quad d\mu = vdt, \quad d\nu = udt, \quad u, v \in \dot{M}_+; \tag{3.1}$$

Let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function, i.e.

$$\Phi\left(t\right) = \int\limits_0^t \varphi\left(\tau\right) d\tau, \tag{3.2}$$

where a function $\varphi : [0, \infty) \to [0, \infty]$ is increasing and left-continuous, $\varphi\left(0\right) = 0$, and φ is neither identically zero nor identically infinity on $(0, \infty)$. Denote

$$t_0 \equiv t_0\left(\Phi\right) = \sup\left\{t \in [0, \infty) : \ \Phi\left(t\right) = 0\right\}; \tag{3.3}$$

$$t_\infty \equiv t_\infty\left(\Phi\right) = \inf\left\{t \in R_+ : \ \Phi\left(t\right) = \infty\right\} \tag{3.4}$$

(if $\Phi\left(t\right) < \infty, t \in R_+$, we assume $t_\infty\left(\Phi\right) = \infty$). Then,

$$t_0 \in [0, \infty); \quad t_\infty \in (0, \infty]; \quad t_0 \leqslant t_\infty, \tag{3.5}$$

$$\Phi\left(t\right) = 0, t \in [0, t_0], \quad \Phi\left(t\right) = \infty, \quad t > t_\infty \tag{3.6}$$

(the last condition for $t_\infty < \infty$). We require that

$$t_0 t_\infty^{-1} = 0 \tag{3.7}$$

(i.e., at least one of the conditions is sutisfied: $t_0 = 0$ or $t_\infty = \infty$).

The function Φ is convex on $[t_0, t_\infty)$ because $0 \leqslant \varphi \uparrow$. Let Ψ be a complementary Young function for Φ, that is

$$\Psi(t) = \int_0^t \psi(\tau) d\tau, \quad t \in [0,\infty]; \quad \psi(\tau) = \inf\{\sigma: \quad \varphi(\sigma) \geqslant \tau\}, \quad \tau \in [0,\infty]. \tag{3.8}$$

Function ψ is left-inverse for the left-continuous increasing function φ. It has the same properties as φ, so that Ψ is the Young function. Moreover, $\varphi(\sigma) = \inf\{\tau: \quad \psi(\tau) \geqslant \sigma\}$, so that Φ is complementary Young function for Ψ(the complementary properties are mutual, see [9], page 271). It is known that

$$\Psi(t) = \sup_{s \geqslant 0} [st - \Phi(s)]; \tag{3.9}$$

$$st \leqslant \Phi(s) + \Psi(t), \quad s, t \in [0,\infty), \tag{3.10}$$

and equality in (3.10) takes place if and only if $\varphi(s) = t$ or $\psi(t) = s$ (see [9], pp. 271-273).

For the simplicity we use short notations

$$J_\lambda(f) := J_\lambda(f; \Phi, R_+, vdx) = \int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\, dx, \tag{3.11}$$

$$\|f\|_{\Phi,v} := \|f\,|L_\Phi(R_+, vdx)\| = \inf\{\lambda > 0 : J_\lambda(f) \leqslant 1\}, \tag{3.12}$$

where $\lambda > 0$, $f \in M$, $v \in \dot{M}_+$ (see (1.6), (1.7)).

Recall that Orlicz space $L_{\Phi,v}$ *is determined as the set of functions* $f \in M : \|f\|_{\Phi,v} < \infty$. It is well-known that in these conditions $L_{\Phi,v}$ is Banach function space with monotone norm and it possess the so called Fatou property (see for example [9], Section 8 in Chapter 4). Moreover, the following principle of duality is known (see for example [9], Section 8 in Chapter 4, or [5], p. 257). Let us introduce for $g \in M$ the associated Orlicz norm

$$\|g\|' = \sup\left\{\int_0^\infty |fg|\, dt : \ f \in L_{\Phi,v}; \quad \|f\|_{\Phi,v} \leqslant 1\right\} =$$

$$= \sup\left\{\int_0^\infty |fg|\, dt : \ f \in L_{\Phi,v}; \quad J_1(f) \leqslant 1\right\}. \tag{3.13}$$

Theorem 3.1. *Let* Φ, Ψ *be the complementary Young functions, let* $v \in \dot{M}_+$, *the condition (3.7) be fulfilled, and*

$$0 < V(t) := \int_0^t v d\tau < \infty, \quad \forall t \in R_+, V(+\infty) = \infty. \tag{3.14}$$

Then, Orlicz norm (3.13) is equivalent to the norm $\|v^{-1}g\|_{\Psi,v}$. *Namely,*

$$\left\|v^{-1}g\right\|_{\Psi,v} \leqslant \|g\|' \leqslant 2\left\|v^{-1}g\right\|_{\Psi,v}. \tag{3.15}$$

We consider the cone

$$\Omega = \{f \in L_{\Phi,v} : \ 0 \leqslant f \downarrow\}, \tag{3.16}$$

and introduce for $g \in M_+$ the associated norm on the cone Ω

$$\|g\|'_\Omega = \sup\left\{\int_0^\infty fg dt : \ f \in \Omega; \quad \|f\|_{\Phi,v} \leqslant 1\right\} = \\ = \sup\left\{\int_0^\infty fg dt : \ f \in \Omega; \quad J_1(f) \leqslant 1\right\}. \tag{3.17}$$

The following result gives the description of norm (3.17). The source of this description is contained in the paper [6]. Modern development of this approach for general Young functions and estimates of the norms of monotone operators is presented in papers [7, 8]. The result given below is contained in [8].

Theorem 3.2. *Let Φ, Ψ be complementary Young functions, let $v \in \dot{M}_+$, and conditions (3.7) and (3.14) be satisfied. For a fixed number $a \in (0,1)$ the following two-sided estimate holds for associate norm (3.17)*

$$\|g\|'_\Omega \cong \|\Re_a(g)\|_{\Psi,v} = \inf\left\{\lambda > 0 : \ \int_0^\infty \Psi\left(\lambda^{-1}|\Re_a(g;t)|\right) v(t)\, dt \leqslant 1\right\}, \tag{3.18}$$

where

$$\Re_a(g;t) := V(t)^{-1} \int_{\delta_a(t)}^t g(\tau) d\tau, \quad \delta_a(t) := V^{-1}(aV(t)), t \in R_+. \tag{3.19}$$

For different values $a \in (0,1)$ norms (3.18) are equivalent.

Here and below we use the notation

$$A \cong B \quad \Leftrightarrow \quad \exists c = c(a) \in [1,\infty) : \quad c^{-1} \leqslant A/B \leqslant c. \tag{3.20}$$

Remark 2. Let us assume additionally that the function Φ in Theorem 3.2 satisfies the Δ_2-condition, that is

$$\exists C \in (1,\infty) : \quad \Phi(2t) \leqslant C\Phi(t), \quad \forall t \in R_+ \tag{3.21}$$

Then we can set $a = 0$ in (3.18), and (3.19) so that

$$\|g\|'_\Omega \ \cong \left\|V(t)^{-1} \int_0^t g d\tau\right\|_{\Psi,v} \tag{3.22}$$

with constants depending only on the constant C in (3.21).

Description (3.22) was obtained in [5] under the restriction that both complementary functions Φ and Ψ are $N-$ functions that satisfy the Δ_2-condition. We claim that (3.22) is valid for any Young functions Φ and Ψ and the restriction $\Psi \in \Delta_2$ is superfluous and may be omitted here.

In the following considerations we will use the formula for the conjugate operator to operator (3.19):

$$\Re_a^*(f;\tau) = \int\limits_{\tau}^{\delta_{a^{-1}}(\tau)} \frac{f(t)}{V(t)}dt, \tau \in R_+. \tag{3.23}$$

It is based on equality

$$\int\limits_{R_+} f(t)\,\Re_a(g;t)dt = \int\limits_{R_+} \Re_a^*(f;\tau)\,g(\tau)\,d\tau, \quad f,g \in M_+$$

obtained by the change of variables in the double integral.

Now we formulate the main result of this Section which allows us to reduce the estimates of the norms of operators on the cone Ω (3.16) to the ones on the cone M_+.

Theorem 3.3. *Let T, T^* be positively homogeneous operators mapping M_+ in M_+ which are conjugate, i.e.,*

$$\int\limits_{R_+} gTfd\tau = \int\limits_{R_+} fT^*gd\tau, \quad f,g \in M_+. \tag{3.24}$$

Let Φ_1, Φ_2 be Young functions satisfying (3.7), and Ψ_1, Ψ_2 be their complementary functions; let $u, v, w \in M_+$, and condition (3.14) be satisfied. We fix $a \in (0,1)$ and define the operator $\Re_a$ by formula (3.19). Then the following three inequalities are equivalent:

$$\exists c_1 \in R_+ : \ \|wTf\|_{\Phi_2,u} \leqslant c_1 \|f\|_{\Phi_1,v}, \quad f \in \Omega; \tag{3.25}$$

$$\exists c_2 \in R_+ : \ \|\Re_a T^*(wg)\|_{\Psi_1,v} \leqslant c_2 \left\|gu^{-1}\right\|_{\Psi_2,u}, \quad g \in M_+; \tag{3.26}$$

$$\exists c_3 \in R_+ : \ \|wT\Re_a^*(vf)\|_{\Phi_2,u} \leqslant c_3 \|f\|_{\Phi_1,v}, \quad f \in M_+. \tag{3.27}$$

Remark 3. Constants c_2, c_3 in (3.26)–(3.27) depend on $a \in (0,1)$, besides $0 < e(a) \leqslant c_1 c_3^{-1} \leqslant E(a) < \infty$;

$$0 < d \leqslant c_2 c_3^{-1} \leqslant D < \infty;$$

where d, D do not depend of a.

Proof. **1.** Estimate (3.25) is equivalent to

$$A_1 = \sup\left\{\|wTf\|_{\Phi_2,u} : \ f \in \Omega; \|f\|_{\Phi_1,v} \leqslant 1\right\} < \infty. \tag{3.28}$$

Theorem 3.1 implies the following relation of duality

$$\|wTf\|_{\Phi_2,u} \cong \sup\left\{\int\limits_0^\infty w(Tf)\,gudt : \ g \in M_+; \quad \|g\|_{\Psi_2,u} \leqslant 1\right\} =$$

$$= \sup\left\{\int\limits_0^\infty fT^*(wgu)\,d\tau : \ g \in M_+; \quad \|g\|_{\Psi_2,u} \leqslant 1\right\}. \tag{3.29}$$

In the last equality we take into account definition (3.24). Then we have

$$A_1 \cong \sup\left\{ \sup\left\{ \int_0^\infty fT^*(wgu)\,d\tau :\ g \in M_+; \|g\|_{\Psi_2,u} \leqslant 1 \right\} :\ f \in \Omega; \|f\|_{\Phi_1,v} \leqslant 1 \right\}.$$

By changing the order of supremum we obtain

$$A_1 \cong \sup\left\{ \sup\left\{ \int_0^\infty fT^*(wgu)\,d\tau :\ f \in \Omega; \|f\|_{\Phi_1,v} \leqslant 1 \right\} : g \in M_+; \|g\|_{\Psi_2,u} \leqslant 1 \right\}.$$

Now, we apply the principle of duality of Theorem 3.2 to the inner supremum and obtain

$$\sup\left\{ \int_0^\infty fT^*(wug)\,d\tau :\ f \in \Omega; \|f\|_{\Phi_1,v} \leqslant 1 \right\} \cong \|\Re_a(T^*(wug))\|_{\Psi_1,v}.$$

Therefore,

$$A_1 \cong \sup\left\{ \|\Re_a T^*(wug)\|_{\Psi_1,v} :\ g \in M_+; \quad \|g\|_{\Psi_2,u} \leqslant 1 \right\} = \\ = \sup\left\{ \|\Re_a T^*(wg)\|_{\Psi_1,v} :\ g \in M_+; \quad \left\|gu^{-1}\right\|_{\Psi_2,u} \leqslant 1 \right\} =: A_2. \quad (3.30)$$

Thus, (3.25) $\Leftrightarrow A_1 < \infty \Leftrightarrow A_2 < \infty \Leftrightarrow$ (3.26).

2. To prove the equivalence (3.26) $\Leftrightarrow$ (3.27) we apply again the assertion of duality. According to Theorem 3.1, we have

$$\|\Re_a T^*(g)\|_{\Psi_1,v} \cong \sup\left\{ \int_0^\infty vf\Re_a T^*(wg)\,dt :\ f \in M_+; \|f\|_{\Phi_1,v} \leqslant 1 \right\}.$$

In the integral we get the conjugate operator $[\Re_a T^*]^* = T\Re_a^*$ and obtain

$$\|\Re_a T^*(g)\|_{\Psi_1,v} = \sup\left\{ \int_0^\infty wgT\Re_a^*(vf)\,d\tau : f \in M_+; \|f\|_{\Phi_1,v} \leqslant 1 \right\}.$$

We insert this equality in (3.30) and change the order of supremums. Then,

$$A_2 = \sup\left\{ \sup\left\{ \int_0^\infty gwT\Re_a^*(vf)\,d\tau : g \in M_+; \quad \left\|gu^{-1}\right\|_{\Psi_2,u} \leqslant 1 \right\} : f \in M_+; \quad \|f\|_{\Phi_1,v} \leqslant 1 \right\}$$

To the inner $\sup\{\ldots\}$ we apply Theorem 3.1. Then,

$$\sup\left\{ \int_0^\infty gwT\Re_a^*(vf)\,d\tau :\ g \in M_+; \left\|gu^{-1}\right\|_{\Psi_2,u} \leqslant 1 \right\} \cong \|wT\Re_a^*(vf)\|_{\Phi_2,u}.$$

As a result,

$$A_2 \cong \sup\left\{ \|wT\Re_a^*(vf)\|_{\Phi_2,u} : f \in M_+; \|f\|_{\Phi_1,v} \leqslant 1 \right\} =: A_3 \quad (3.31)$$

Thus, (3.26) $\Leftrightarrow A_2 < \infty \Leftrightarrow A_3 < \infty \Leftrightarrow$ (3.27). $\square$

Remark 4. Let us compare inequalities (3.25) and (3.27). Their structures and right-hand sides are the same, the only distriction we have is that in the left-hand side of (3.25) there is the Orlicz norm for the function

$$w(t)(Tf)(t), \quad f \in \Omega, \tag{3.32}$$

and in the left-hand side of (3.27) there is the norm of the function

$$w(t)(T\Re_a^*(vf))(t), \quad \Re_a^*(vf)(\tau) = \int_{\tau}^{\delta_{a^{-1}}(\tau)} \frac{fdV}{V}, \quad f \in M_+. \tag{3.33}$$

Thereby, in (3.27) the monotonicity condition is omitted for the function f, thanks to the passage from $f \in \Omega$ to the integral average $\Re_a^*(vf)$.

Remark 5. Additionally, let the Young function Φ_1 in Theorem 3.3 satisfy the Δ_2 - condition, that is

$$\exists C \in (1, \infty): \quad \Phi_1(2t) \leqslant C\Phi_1(t), \quad t \in R_+. \tag{3.34}$$

Then, according to Remark 2, the description (3.18), (3.19) remains true for $a = 0$, i. e., we have instead of (3.23)

$$\Re_0^*(f;\tau) = \int_{\tau}^{\infty} \frac{f(t)}{V(t)} dt, \tau \in R_+; \tag{3.35}$$

(3.33) is also valid with $a = 0$, and (3.25) is equivalent now to the inequality

$$\exists c_3 \in R_+ : \left\| wT\left(\int_{\tau}^{\infty} \frac{fdV}{V}\right)\right\|_{\Phi_2,u} \leqslant c_3 \|f\|_{\Phi_1,v}, \quad f \in M_+. \tag{3.36}$$

4 Reduction theorem for modular inequalities

We will apply the results of Sections 2 and 3, and we will assume that in the notations of Section 1 $X = Y = R_+$, $M(X) = M(Y) = M(R_+)$ is the set of all Lebesgue-measurable functions; M_+ is the cone of all almost everywhere nonnegative functions in $M = M(R_+)$; μ, ν are weighted Lebesgue measures with weights in $\dot{M}_+$, so that assertions (3.1) hold. We preserve also notations (3.2)–(3.12).

Theorem 4.1. *Under the assunptions of Theorem 3.3 we assume that*

$$t_0(\Phi_1) = t_0(\Phi_2) = 0, \quad t_\infty(\Phi_1) = t_\infty(\Phi_2) \equiv t_\infty \in (0, \infty]$$

(see (3.3), (3.4)). Then the following inequalities are equivalent

$$\exists c_1 \in R_+ : \quad \Phi_2^{-1}\left\{\int_{R_+} \Phi_2(wTf)\,udt\right\} \leqslant \Phi_1^{-1}\left\{\int_{R_+} \Phi_1(c_1 f)\,vdt\right\}, \quad f \in \Omega; \tag{4.1}$$

$$\exists c_3 \in R_+ : \Phi_2^{-1}\left\{\int_{R_+} \Phi_2(wT\Re_a^*(vf))\,udt\right\} \leqslant \Phi_1^{-1}\left\{\int_{R_+} \Phi_1(c_3 f)\,vdt\right\}, f \in M_+. \tag{4.2}$$

Moreover, c_3 is connected with c_1 as in Theorem 3.3; in particular the following estimate holds

$$0 < e(a) \leqslant c_1 c_3^{-1} \leqslant E(a) < \infty. \tag{4.3}$$

Proof. Under the assumptions of Theorem 4.1 we have $\Phi_1, \Phi_2 \in \Theta_0$, see (2.3). We apply Theorem 2.1 with $K = \Omega$ in (4.1) and obtain the equivalence (4.1) $\Leftrightarrow$ (4.4), where (4.4) is the family of the norm inequalities

$$\|wTf\|_{\Phi_2,u_\varepsilon} \leqslant c_1 \|f\|_{\Phi_1,v_\varepsilon}, \quad f \in \Omega; \varepsilon \in \left(t_\infty^{-1}, \infty\right). \tag{4.4}$$

Here, we use the notations

$$u_\varepsilon = \sigma_{\Phi_2}(\varepsilon)\, u; \quad v_\varepsilon = \sigma_{\Phi_1}(\varepsilon)\, v; \quad V_\varepsilon(t) = \int_0^t v_\varepsilon d\tau = \sigma_{\Phi_1}(\varepsilon)\, V(t). \tag{4.5}$$

Now, we apply Theorem 3.3 to inequalities (4.4) with $u, v \in \dot{M}_+$ replased by $u_\varepsilon, v_\varepsilon \in \dot{M}_+$ in (3.25) and in (3.27). Then,(4.4) $\Leftrightarrow$ (4.6), where (4.6)

$$\left\|wT\Re^*_{a,\varepsilon}(v_\varepsilon f)\right\|_{\Phi_2,u_\varepsilon} \leqslant c_3 \|f\|_{\Phi_1,v_\varepsilon}, \quad f \in M_+; \quad \varepsilon \in \left(t_\infty^{-1}, \infty\right). \tag{4.6}$$

Moreover estimate (4.3) holds. Here, we calculate $\Re^*_{a,\varepsilon}(v_\varepsilon f)$ by formula (3.33) with V t replaced by V_ε, and, respectively, $\delta_{a^{-1}}(\tau)$ through

$$\delta_{a^{-1},\varepsilon}(\tau) = V_\varepsilon^{-1}\left(a^{-1} V_\varepsilon(\tau)\right).$$

We see by (4.5) that

$$\delta_{a^{-1},\varepsilon}(\tau) = V^{-1}\left(a^{-1}\sigma_{\Phi_1}(\varepsilon)^{-1} V_\varepsilon(\tau)\right) = V^{-1}\left(a^{-1} V(\tau)\right) = \delta_{a^{-1}}(\tau).$$

As a result,

$$\Re^*_{a,\varepsilon}(v_\varepsilon f)(\tau) = \int_\tau^{\delta_{a^{-1},\varepsilon}(\tau)} \frac{f dV_\varepsilon}{V_\varepsilon} = \int_\tau^{\delta_{a^{-1}}(\tau)} \frac{f dV}{V} = \Re^*_a(vf)(\tau). \tag{4.7}$$

We insert (4.7) in (4.6), and obtain that (4.6) coincides with (4.8), which is

$$\|wT\Re^*_a(vf)\|_{\Phi_2,u_\varepsilon} \leqslant c_3 \|f\|_{\Phi_1,v_\varepsilon}, \quad f \in M_+; \varepsilon \in \left(t_\infty^{-1}, \infty\right). \tag{4.8}$$

To the family of inequalities (4.8) we apply Theorem 2.1 and see that (4.8) $\Leftrightarrow$ (4.2). As a result we have the chain of equivalences

$$(4.1) \Leftrightarrow (4.4) \Leftrightarrow (4.6) \Leftrightarrow (4.8) \Leftrightarrow (4.2),$$

which proves the theorem. □

Corollary 4.1. *Under the assumptions of Theorem 4.1 let us assume additionally that the Young function Φ_1 satisfies the Δ_2-condition. Then, (4.1) is equivalent to the following inequality*

$$\exists c_3 \in R_+ : \Phi_2^{-1}\left\{\int_{R_+} \Phi_2\left(wT\left(\int_\tau^\infty \frac{f dV}{V}\right)\right) u dt\right\} \leqslant \Phi_1^{-1}\left\{\int_{R_+} \Phi_1(c_3 f)\, v dt\right\}, f \in M_+.$$

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