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This issue contains the first part of the collection of papers sent to the Eurasian Mathematical Journal dedicated to the 70th birthday of Professor R. Oinarov.

The second part of the collection will be published in Volume 8, Number 2.

RYSKUL OINAROV

(to the 70th birthday)

On February 26, 2017 was the 70th birthday of Ryskul Oinarov, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, doctor of physical and mathematical sciences (1994), professor (1997), honoured worker of education of the Republic of Kazakhstan (2007), corresponding member of the National Academy of Sciences of the Republic of Kazakhstan (2012). In 2005 he was awarded the breastplate "For the merits in the development of science in the Republic of Kazakhstan", in 2007 and 2014 the state grant "The best university teacher", in 2016 the Order "Kurmet" (= "Honour").

R. Oinarov was born in the village Kul'Aryk, Kazalinsk district, Kyzylorda region. In 1969 he graduated from the S.M. Kirov Kazakh State University (Almaty). Starting with 1972 he worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (senior engineer, junior researcher, senior researcher, head of a laboratory). In 1981 he defended of the candidate of sciences thesis "Continuity and Lipschitzness of nonlinear integral operators of Uryson's type" at the Tashkent State University of the Uzbek SSR and in 1994 the doctor of sciences thesis "Weighted estimates of integral and differential operators" at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

Starting from 2000 he has been working as a professor at the L.N. Gumilyov Eurasian National University

Scientific works of R. Oinarov are devoted to investigation of linear and non-linear integral and discrete operators in weighted spaces; to studying problems of the well-posedness of differential equations; to weighted inequalities; to embedding theorems for the weighted function spaces of Sobolev type and their applications to the qualitative theory of linear and quasilinear differential equations. A certain class of integral operators is named after him - integral operators with *Oinarov's kernels* or *Oinarov condition.* On the whole, the results obtained by R. Oinarov have laid the groundwork for new perspective directions in the theory of function spaces and its applications to the theory of differential equations.

R. Oinarov has published more than 100 scientific papers. The list of his most important publications may be seen on the web-page

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Under his supervision 26 theses have been defended: 1 doctor of sciences thesis, 15 candidate of sciences theses and 10 PhD theses. The Editorial Board of the Eurasian Mathematical Journal congratulates Ryskul Oinarov on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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INVERSE PROBLEM FOR THE DIFFUSION OPERATOR WITH SYMMETRIC FUNCTIONS AND GENERAL BOUNDARY CONDITIONS

A.M. Akhtyamov, V.A. Sadovnichy, Ya.T. Sultanaev

Communicated by M. Otelbaev

Dedicated to the 70^{th} birthday of Professor Ryskul Oinarov

Key words: inverse eigenvalue problem, diffusion operator, nonseparated boundary conditions.

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. For the inverse problem of reconstructing the nonself-adjoint diffusion operator with symmetric functions and general boundary conditions a uniqueness theorem is proved. As spectral data only one spectrum and six eigenvalues are used. Earlier this inverse problem was not considered. The inverse problem of reconstructing the self-adjoint diffusion operator with nonseparated boundary conditions was considered. To uniquely reconstruct this operator two spectra, some sequence of signs, and some complex number were used as spectral data. We show that in the symmetric case to uniquely reconstruct the self-adjoint diffusion operator one can use even less spectral data as compared with the reconstruction of a self-adjoint problem in earlier papers; more precisely, we need one spectrum and, in addition, five eigenvalues. The special cases of these general inverse problems are considered too. In these special cases less spectral data are used. Algorithms of reconstructing diffusion operator are given. Moreover, we show that results obtained in the present paper generalize the results for the inverse problem of reconstructing the diffusion operator with separated boundary conditions.

1 Introduction

By L we denote the following nonself-ajoint problem for a diffusion equation with nonseparated boundary conditions

Problem L.

$$ly = y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0, \tag{1.1}$$

$$U_i(y) = a_{i1}\, y(0) + a_{i2}\, y'(0) + a_{i3}\, y(\pi) + a_{i4}\, y'(\pi) = 0, \qquad i = 1, 2, \tag{1.2}$$

where $p(x) \in W_2^1(0,\pi)$, $p(x) = p(\pi - x)$; $q(x) \in L_2(0,\pi)$ is a real-alued function such that $q(x) = q(\pi - x)$ and the a_{ij} with $i = 1$, 2 and $j = 1$, 2, 3, 4 are complex constants.

Note that if general Problem L is self-adjoint, then Problem L can be reduced to one of the following Problems G_1 and G_2:

Problem G_1.

$$ly = y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0,$$

$$V_1(y) = a_{11}\, y(0) + y'(0) + a_{13}\, y(\pi) = 0, \tag{1.3}$$

$$V_2(y) = a_{21}\, y(0) + a_{23}\, y(\pi) + y'(\pi) = 0, \tag{1.4}$$

where a_{11} and a_{23} are real numbers, $a_{13} \neq 0$ is a complex number, and $a_{21} = -\overline{a_{13}}$.

Problem G_2.

$$ly = y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0,$$

$$P_1(y) = y(0) + \omega\, y(\pi) = 0 \tag{1.5}$$

$$P_2(y) = \overline{\omega}\, y'(0) + y'(\pi) + \alpha\, y(\pi) = 0, \tag{1.6}$$

where $\omega \neq 0$ is a complex number and α is a real number.

For the inverse problem of reconstructing L in which all coefficients a_{ij} with $i = 1, 2$ and $j = 1, 2, 3, 4$ are arbitrary, no uniqueness theorems have been proved.

Problems G_1 and G_2 are the special cases of problem L which have been earlier studied (for details, see [8]). In particular, in [8] to uniquely reconstruct problem G_1 two spectra, some sequence of signs, and some complex number were used as spectral data. We show that, if $p(x) = p(\pi - x)$ and $q(x) = q(\pi - x)$, then to uniquely reconstruct problem G_1 one can use even less spectral data as compared with the reconstruction of a self-adjoint problem in [8]; more precisely, we need one spectrum and, in addition, five eigenvalues. Moreover, in this paper, we prove a theorem on the unique reconstruction of problem L with a symmetric functions $p(x)$ and $q(x)$ and general boundary conditions (1.2), which may be nonself-adjoint. As spectral data only one spectrum and six eigenvalues are used.

The special case of inverse problem of reconstructing G_1 and G_2 in which $p(x) = 0$ (inverse Sturm–Liouville problem) was considered in numerous papers (for details, see [1–30]). The analysis of the inverse nonself-adjoint problem Sturm–Liouville with nonseparated boundary conditions was initiated in [21]. It was shown there that three spectra and two sets of weight numbers and residues of certain functions are sufficient for the unique reconstruction of a nonself-adjoint Sturm–Liouville problem with nonseparated boundary conditions. Moreover, these spectral data were used essentially [22]. Later, there were attempts to choose the problem to be reconstructed or auxiliary problems so as to use less spectral data for the reconstruction [5, 19, 20, 25, 30]. In particular, in [19, 20] a nonself-adjoint problem was replaced by a self-adjoint one, and it was shown that, for its unique reconstruction, as spectral data it suffices to use three spectra, some sequence of signs, and some real number. In [5], an auxiliary problem was chosen so as to reduce the number of spectral data required for the reconstruction of a self-adjoint problem by one spectrum; i.e., only two spectra, some sequence of signs, and some real number were used as spectral data. The uniqueness theorems for an inverse nonselfadjoint Sturm-Liouville problem with symmetric potential and general boundary conditions are proved in [25]. The spectral data used for unique reconstruction of Sturm-Liouville problems are a spectrum and six eigenvalues. The results obtained in the present paper generalize the results for an inverse nonself-adjoint Sturm-Liouville problem with a symmetric potential and general boundary conditions proved in [25].

2 A unique determination of a nonself-adjoint diffusion operator L by a spectrum and six eigenvalues

In [16] I.M. Nabiev and A.Sh. Shukyurov proved a uniqueness theorem for the following inverse Sturm-Liouville problem:

Problem N.

$$ly = y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

Uniqueness theorem [16]. *Problem N is uniquely determined by its spectrum if* $p(x) = p(\pi - x)$ *and* $q(x) = q(\pi - x)$.

In this paper, we generalize this theorem to the case of general boundary conditions (1.2).

In what follows, we denote a problem of type L, but with different coefficients in the equation and different parameters in the boundary forms, by $\widetilde{L}$. Throughout the paper, we assume that if some symbol denotes an object from Problem L then the same symbol with the tilde $\tilde{}$ denotes the corresponding object from Problem $\widetilde{L}$.

Let A be the matrix composed of coefficients a_{lk} of the boundary conditions (1.2), i.e.,

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{matrix} \right\|, \tag{2.1}$$

and let M_{ij} be its minors composed of ith and jth columns:

$$M_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}, \qquad i, j = 1, 2, 3, 4.$$

Vectors are denoted by boldface letters. The symbol ^{T} denotes transposition. Column vectors are represented by rows with this superscript. For the rank of the matrix A we use the notation $\operatorname{rank} A$.

Together with Problems L and N we consider the following Problem L_1.

Problem L_1.

$$\begin{aligned} ly &= y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0, \\ U_{1,1}(y) &= y(0) - b(\lambda)\, y'(0) = 0, \\ U_{2,1}(y) &= y(\pi) = 0, \end{aligned}$$

where the function $b(\lambda)$ is a polynomial of the form

$$b(\lambda) = M_{12} + (1 - M_{13})\,\lambda + (M_{14} - M_{32})\,\lambda^2 + M_{42}\,\lambda^3 + M_{34}\,\lambda^4.$$

Theorem 2.1. *If* λ_0 *is an eigenvalue of Problem* L, λ_i $(i = 1, 2, 3, 4, 5)$ *are any pairwise distinct eigenvalues of Problem* L_1, $y_1(\pi, \lambda_i) \neq 0$, $i = 0, 1, 2, 3, 4, 5$; $p(x) = p(\pi - x)$ *and* $q(x) = q(\pi - x)$, *then Problems* L, N, *and* L_1 *are uniquely determined by the spectrum of Problem* N *and* λ_i, $i = 0, 1, 2, 3, 4, 5$, *i.e. the function* $q(x)$ *is uniquely determined and the matrix* $(a_{ij})_{2\times 4}$ *is determined up to a linear transformation of the rows.*

Proof. Applying the uniqueness theorem [16] for the inverse problem N, we see that the functions $p(x)$ and $q(x)$ in (1.1) are uniquely determined by the spectrum of Problem N.

To prove the theorem, it remains to find the boundary conditions (1.2).

Since the functions $p(x)$ and $q(x)$ are reconstructed we see that the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of equation (1.1) satisfying the conditions

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1 \tag{2.2}$$

are known.

The eigenvalues of Problem L are the roots of the entire function ([15, pp. 33–36], [17, p. 29])

$$\Delta(\lambda) = M_{12} + M_{34} + M_{32}\, y_1(\pi,\, \lambda) + M_{42}\, y_1'(\pi,\, \lambda) + \\ + M_{13}\, y_2(\pi,\, \lambda) + M_{14}\, y_2'(\pi,\, \lambda),$$

and the eigenvalues of Problem L_1 are the roots of the entire function

$$\Delta_1(\lambda) = y_2(\pi,\, \lambda) - a(\lambda)\, y_1(\pi,\, \lambda).$$

Since $y_1(\pi,\, \lambda_0) = y_2'(\pi,\, \lambda_0)$ if and only if $p(x) = p(\pi - x)$ and $q(x) = q(\pi - x)$[16, Lemma 3], it follows that

$$\Delta(\lambda) = M_{12} + M_{34} + (M_{32} + M_{14})\, y_1(\pi,\, \lambda) + M_{42}\, y_1'(\pi,\, \lambda) + M_{13}\, y_2(\pi,\, \lambda), \tag{2.3}$$

The numbers λ_i $(i = 1, 2, 3, 4, 5)$ are eigenvalues of Problem L_1, so $\Delta_1(\lambda_i) = 0$, $i = 1, 2, 3, 4, 5$. It now follows that

$$M_{12} + (1 - M_{13})\, \lambda_i + (M_{14} - M_{32})\, \lambda_i^2 + M_{42}\, \lambda_i^3 + M_{34}\, \lambda_i^4 = \frac{y_2(\pi,\, \lambda_i)}{y_1(\pi,\, \lambda_i)}. \tag{2.4}$$

The determinant of system (2.4) with respect to the unknowns M_{12}, $(1 - M_{13})$, $(M_{14} - M_{32})$, M_{42}, M_{34} is the fifth-order Vandermonde determinant equal to $\prod_{k_1 > k_2}(\lambda_{k_1} - \lambda_{k_2})$. Therefore, system (2.4) has a unique solution, which can be found by the Cramer formulas.

Since λ_0 is an eigenvalue of Problem L and $y_1(\pi,\, \lambda_0) \neq 0$, it follows from (2.3) that

$$(M_{32} + M_{14}) = \Big(- (M_{12} + M_{34}) - M_{42}\, y_1'(\pi,\, \lambda_0) - M_{13}\, y_2(\pi,\, \lambda_0)\Big)\, y_1^{-1}(\pi,\, \lambda_0). \tag{2.5}$$

Combining (2.4) and (2.5), we see that the unknowns M_{12}, M_{13}, M_{14} M_{32}, M_{42}, M_{34} are uniquely determined. It follows (see [2, p. 32]) that the matrix $(a_{ij})_{2\times 4}$ is determined up to a linear transformation of the rows. Hence Problems G_1, N, and L_1 are uniquely determined by the spectrum of Problem N and λ_i, $i = 0, 1, 2, 3, 4, 5$. □

Remark 1. *If L=N, then $M_{13} = 1$ and all other minors M_{ij} of the matrix A are equal to zero. So L=N=L_1. Therefore, uniqueness theorem [16] is a special case of Theorem 2.1 proved above, and Theorem 2.1 is a generalization of the uniqueness theorem [16].*

On the basis of the proof of Theorem 2.1, one can construct the following

algorithm for the unique identification of Problems L, $\mathbf{L_1}$, and N:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y(0) = y(\pi) = 0$ (see [16]), we find the functions functions $p(x)$ and $q(x)$.
Step 2. By using functions $p(x)$ and $q(x)$, we find the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1.1), satisfying conditions (2.2).
Step 3. By using numbers λ_i, $i = 0, 1, 2, 3, 4, 5$ satisfying the conditions of Theorem 2.1, we find the solution of systems (2.4) and (2.5) (M_{12}, M_{13}, M_{14} M_{32}, M_{42}, M_{34}).
Step 4. By using determinants M_{12}, M_{13}, M_{14} M_{32}, M_{42}, M_{34}, we find the matrix $(a_{ij})_{2\times 4}$ is determined up to a linear transformation of the rows. The matrix $(a_{ij})_{2\times 4}$ determined by the matrix identification methods (see [2, pp. 33-34])). We thereby completely reconstruct Problems L, L_1, and N.

3 A unique determination of the self-adjoint diffusion operator G_1 by a spectrum and five eigenvalues

I.M. Guseinov, I.M. Nabiev showed in [8] that Problem G_1 (with possibly nonsimmetric functions $p(x)$ and $q(x)$) can be uniquely reconstructed from two spectrum, a sequence of signs and the number a_{13}.

In what follows, we show that if the functions $p(x)$ and $q(x)$ of Problem G_1 are symmetric, then Problem G_1 can be reconstructed from one spectrum and five eigenvalues (a sequence of signs, an infinite sequence of eigenvalues, and the number a_{13} are not needed in this case).

Together with Problems L and N we consider the following Problem L_2.
Problem L_2.

$$\begin{aligned} ly &= y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0, \\ U_{1,1}(y) &= y(0) - b_1(\lambda)\, y'(0) = 0, \\ U_{2,1}(y) &= y(\pi) = 0, \end{aligned}$$

where the function $b(\lambda)$ is a polynomial of the form

$$b_1(\lambda) = M_{12} + (1 - M_{13})\,\lambda + (M_{14} - M_{32})\,\lambda^2 + M_{34}\,\lambda^3.$$

Theorem 3.1. *If λ_0 is an eigenvalue of Problem G_1, λ_i $(i = 1, 2, 3, 4)$ are any pairwise distinct eigenvalues of Problem L_2, $y_1(\pi,\, \lambda_i) \neq 0$, $i = 0, 1, 2, 3, 4$; $p(x) = p(\pi - x)$ and $q(x) = q(\pi - x)$, then Problems G_1, N, and L_2 are uniquely determined by the spectrum of Problem N and λ_i, $i = 0, 1, 2, 3, 4$.*

Proof. Applying uniqueness theorem [16] for the inverse problem N, we see that the functions $p(x)$ and $q(x)$ in (1.1) are uniquely determined by the spectrum of Problem N.

To prove the theorem, it remains to find the coefficients a_{11}, a_{13}, a_{21}, and a_{23}.

Let $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ be linearly independent solutions of equation (1.1) satisfying the conditions

$$y_1(0,\, \lambda) = 1, \quad y_1'(0,\, \lambda) = 0, \quad y_2(0,\, \lambda) = 0, \quad y_2'(0,\, \lambda) = 1. \tag{3.1}$$

The eigenvalues of Problem G_1 are the roots of the entire function (2.3) with $M_{42} = -1$.

The numbers λ_i $(i = 1, 2, 3, 4)$ are eigenvalues of Problem L_2 , so

$$M_{12} + (1 - M_{13})\,\lambda_i + (M_{14} - M_{32})\,\lambda_i^2 + M_{34}\,\lambda_i^3 = \frac{y_2(\pi,\, \lambda_i)}{y_1(\pi,\, \lambda_i)}. \tag{3.2}$$

The determinant of system (3.2) with respect to the unknowns M_{12}, $(1 - M_{13})$, $(M_{14} - M_{32})$, M_{34} is the fourth-order Vandermonde determinant equal to $\prod_{k_1 > k_2} (\lambda_{k_1} - \lambda_{k_2})$. Therefore, system (3.2) has a unique solution, which can be found by the Cramer formulas.

Since λ_0 is an eigenvalue of Problem G_1 and $y_1(\pi,\, \lambda_0) \neq 0$, it follows from (2.3) that

$$(M_{32} + M_{14}) = \Big(- (M_{12} + M_{34}) + y_1'(\pi,\, \lambda_0) - M_{13}\, y_2(\pi,\, \lambda_0)\Big)\, y_1^{-1}(\pi,\, \lambda_0). \tag{3.3}$$

Combining (3.2) and (3.3), we see that the unknowns $M_{12} = \overline{a_{13}}$, $M_{13} = a_{11}\, a_{23} + |a_{13}|^2$, $M_{14} = a_{11}$, $M_{32} = -a_{23}$, $M_{34} = a_{13}$ are uniquely determined. It follows from this that the coefficints a_{11}, a_{13}, a_{23}, and $a_{21} = -\overline{a_{21}}$ are uniquely determined. Hence Problems G_1, N, and L_2 are uniquely determined by the spectrum of Problem N and λ_i, $i = 0, 1, 2, 3, 4, 5$. □

Remark 2. *If G_1=N, then G_1=N=L_2. Therefore, uniqueness theorem [16] is a special case of Theorem 3.1 proved above, and Theorem 3.1 is a generalization of the uniqueness theorem [16].*

On the basis of the proof of Theorem 2.1, one can construct the following

algorithm for the unique identification of Problems $\mathbf{G_1}$, $\mathbf{L_2}$, and N:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y(0) = y(\pi) = 0$ (see[16]), we find the functions functions $p(x)$ and $q(x)$.
Step 2. By using the functions $p(x)$ and $q(x)$, we find the linearly independent solutions $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ of equation (1.1), satisfying conditions (2.2).
Step 3. By using the numbers λ_i, $i = 0, 1, 2, 3, 4$ satisfying the conditions of Theorem 3.1, we find the solution of systems (3.2) and (3.3) (M_{12}, M_{13}, M_{14} M_{32}, M_{34}).
Step 4. By using the minors $M_{12} = \overline{a_{13}}$, $M_{13} = a_{11}\,a_{23} + |a_{13}|^2$, $M_{14} = a_{11}$, $M_{32} = -a_{23}$, $M_{34} = a_{13}$ we find the coefficints a_{11}, a_{13}, a_{23}, and $a_{21} = -\overline{a_{21}}$. We thereby completely reconstruct Problems G_1, L_2, and N.

4 A unique determination of the self-adjoint diffusion operator G_2 by a spectrum and two eigenvalues

I.M. Guseinov, I.M. Nabiev showed in [8] that Problem G_2 (with possibly nonsimmetric functions $p(x)$ and $q(x)$) can be uniquely reconstructed from two spectrum, a sequence of signs and the number a_{13}.

In what follows, we show that if the functions $p(x)$ and $q(x)$ of Problem G_2 are symmetric, then Problem G_1 can be reconstructed from one spectrum and two eigenvalues (a sequence of signs, an infinite sequence of eigenvalues, and the number a_{13} are not needed in this case).

Together with Problems L and N we consider the following Problem L_3.
Problem L_3.

$$\begin{aligned} ly &= y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0, \\ U_{1,1}(y) &= y(0) - \omega\, y'(0) = 0, \\ U_{2,1}(y) &= y(\pi) = 0. \end{aligned}$$

Theorem 4.1. *If $p(x) = p(\pi - x)$ and $q(x) = q(\pi - x)$, λ_0 is an eigenvalue of Problem G_2, λ_1 is eigenvalues of Problem L_2, such that the following condition*

$$y_2(\pi,\,\lambda_0) \neq 0, \quad y_1(\pi,\,\lambda_1) \neq 0; \tag{4.1}$$

holds, then Problems G_2, N, and L_3 are uniquely determined by the spectrum of Problem N and λ_i, $i = 0, 1$.

Proof. Applying the uniqueness theorem [16] for the inverse problem N, we see that the functions $p(x)$ and $q(x)$ in (1.1) is uniquely determined by the spectrum of Problem N.

To prove the theorem, it remains to find the coefficients ω and α.

Let $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ be linearly independent solutions of equation (1.1) satisfying the conditions (2.2).

The eigenvalue λ_0 of Problem G_2 is the root of the following entire function

$$\Delta_3(\lambda) = \omega + \overline{\omega} + \omega \cdot \overline{\omega}\, y_1(\pi, \lambda) + \alpha\, y_2'(\pi, \lambda) \tag{4.2}$$

The number λ_1 is eigenvalue of Problem L_2 , so

$$\omega = \frac{y_2(\pi,\, \lambda_1)}{y_1(\pi,\, \lambda_1)}. \tag{4.3}$$

Since λ_0 is an eigenvalue of Problem G_2 and $y_2(\pi,\, \lambda_0) \neq 0$, it follows from (4.2) that

$$\alpha = -\frac{\omega + \overline{\omega} + \omega \cdot \overline{\omega}\, y_1(\pi, \lambda)}{y_2'(\pi,\, \lambda_0)}. \tag{4.4}$$

Combining (4.3) and (4.4), we see that the coefficients ω and α are uniquely determined. Hence Problems G_2, N, and L_3 are uniquely determined by the spectrum of Problem N and λ_i, $i = 0, 1$. □

On the basis of the proof of Theorem 4.1, one can construct

an algorithm for the unique identification of Problems G_2, L_3, and N:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y(0) = y(\pi) = 0$ (see[16]), we find the functions functions $p(x)$ and $q(x)$ from spectrum of Problem N.

Step 2. By the functions $p(x)$ and $q(x)$, we find the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1.1), satisfying conditions (2.2).

Step 3. By the numbers λ_i, $i = 0, 1$ satisfying the conditions of Theorem 4.1 and formulae (4.3) and (4.4) we find the coefficients ω and α of the boundary conditions of Problem G_2. We thereby completely reconstruct Problems G_2, L_3, and N.

5 A unique determination of the self-adjoint diffusion operator G_1 by a spectrum and three (or two) eigenvalues in special cases

Consider the following spectral problem.

Problem Y:

$$\begin{aligned} ly &= y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0,\ y, \\ U_{1,1}(y) &= a_{11}\, y(0) + y'(0) + a_{13}\, y(\pi) = 0, \\ U_{2,1}(y) &= -a_{13}\, y(0) + a_{23}\, y(\pi) + y'(\pi) = 0, \qquad a_{11}, a_{13}, a_{23} \in \mathbb{R}. \end{aligned}$$

Problem Y coincides with Problem G_1, where a_{13} are any real numbers and $a_{21} = -a_{13}$. V.A. Yurko showed in [30] that Problem Y in the case $p(x) \equiv 0$ can be uniquely reconstructed from two spectra and a sequence of signs, namely, from the spectrum of Problem Y (with $p(x) \equiv 0$), the spectrum $\{z_n\}$ of the problem for the Sturm-Liouville equation and the boundary conditions $y'(0) + a_{11}\, y(0) = y(\pi) = 0$, and the sequence of signs $\omega_n = \operatorname{sign}\left(|\theta'(\pi, z_n)| - |a_{13}|\right)$, where $\theta(x, \lambda)$ is the solution of the Sturm-Liouville equation under the boundary conditions $\theta(0, \lambda) = 1$, $\theta'(0, \lambda) = -a_{11}$.

In what follows, we show that if the functions $p(x)$ and $q(x)$ are symmetric and the boundary conditions hold, then Problem Y can be reconstructed from one spectrum and three (or two) eihenvalues (a sequence of signs and infinite set of eigenvalues of Problem Y are not needed in this case).

In [16] is considered the following problem N_1.
Problem N_1:

$$ly = y'' + \big(\lambda^2 - 2\,\lambda\, p(x) - q(x)\big)\, y = 0, \quad y'(0) - h\, y(0) = 0, \quad y'(\pi) + h\, y(\pi) = 0, \qquad h \in \mathbb{R}.$$

Remark [16]. *Problem N_1 is uniquely determined by its spectrum if $p(x) = p(x-\pi)$ and $q(x) = q(x-\pi)$.*

This section contains generalizations of this assertion to the case of nonseparated boundary conditions.

Theorem 5.1. *Suppose λ_1, λ_2 and λ_3 are eigenvalues of Problem Y and satisfy the following condition:*

$$\begin{vmatrix} 1 & y_1(\pi,\lambda_1) & y_2(\pi,\lambda_1) \\ 1 & y_1(\pi,\lambda_2) & y_2(\pi,\lambda_2) \\ 1 & y_1(\pi,\lambda_3) & y_2(\pi,\lambda_3) \end{vmatrix} \neq 0. \tag{5.1}$$

Then Problem Y (the functions $p(x)$, $q(x)$, and coefficients a_{11}, a_{13} and a_{23}) are uniquely determined by the spectrum of Problem N_1 and λ_i, $i = 1, 2, 3$.

Proof. Applying the method for reconstructing the diffusion operator (see [16]) to problem N_1 with $h = -a_{11}$, we see that the functions $p(x)$ and $q(x)$ and the coefficient $a_{11} = -h$ are uniquely determined by the spectrum of Problem N_1.

To prove the theorem, it remains to find the coefficients a_{21}, a_{13} and a_{23}.

The characteristic determinant $\Delta_2(\lambda)$ of Problem G_1 as follows

$$\begin{aligned} \Delta_2(\lambda) = 2\, a_{13} - a_{23}\, y_1(\pi,\, \lambda) - y_1'(\pi,\, \lambda) + \\ + (a_{11}\, a_{23} + a_{13}^2)\, y_2(\pi,\, \lambda) + a_{11}\, y_2'(\pi,\, \lambda). \end{aligned} \tag{5.2}$$

Since the functions $p(x)$ and $q(x)$ are reconstructed we see that the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1.1) under conditions (2.2) are known, and the eigenvalues of Problem G_1 are the roots of characteristic determinant (5.2). It now follows that

$$\begin{aligned} -2\, a_{13} + a_{23}\, \big(a_{11}\, y_2(\pi, \lambda_i) - y_1(\pi, \lambda_i)\big) + a_{13}^2\, y_2(\pi,\, \lambda) = \\ = -y_1'(\pi,\, \lambda_i) + a_{11}\, y_2'(\pi,\, \lambda_i), \qquad i = 1, 2, 3. \end{aligned} \tag{5.3}$$

It follows from (5.1) that the determinant

$$\begin{vmatrix} 1 & a_{11}\, y_2(\pi,\lambda_1) - y_1(\pi,\lambda_1) & y_2(\pi,\lambda_1) \\ 1 & a_{11}\, y_2(\pi,\lambda_2) - y_1(\pi,\lambda_2) & y_2(\pi,\lambda_2) \\ 1 & a_{11}\, y_2(\pi,\lambda_3) - y_1(\pi,\lambda_3) & y_2(\pi,\lambda_3) \end{vmatrix}$$

of system of equations (5.3) with respect to the three unknowns $2\, a_{13}$, a_{23}, a_{13}^2 is not equal to zero. Therefore, system (5.3) has a unique solution, which can be found by the Cramer formulas. The coefficient a_{13} is uniquely determined from $2\, a_{13}$ and a_{13}^2. So the coefficients a_{13}, $a_{21} = -a_{13}$ and a_{23} are uniquely determined by three eigenvalues λ_i $(i = 1, 2, 3)$ of Problem Y such that condition(5.1) holds. □

If condition (5.1) holds on the basis of the proof of Theorem 5.1, one can construct the following

algorithm for the unique identification of Problem Y:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y'(0) + a_{11}\, y(0) = y'(\pi) - a_{11}\, y(\pi) = 0$ (see[16]), we find the functions functions $p(x)$, $q(x)$, and coefficient a_{11}.
Step 2. By using the function $q(x)$, we find the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1.1), satisfying conditions (2.2).
Step 3. By using the eigenvalues λ_i, $i = 1, 2, 3$ of Problem Y satisfying condition (5.1), we find the solution of system (5.3) (the numbers a_{13}, a_{23}, a_{13}^2). We thereby completely reconstruct Problem Y.

Theorem 5.2. *Suppose the coefficients a_{11}, a_{21} and a_{23} are real, λ_1 and λ_2 are eigenvalues of Problem Y and satisfy the following condition:*

$$y_2(\pi, \lambda_1) = y_2(\pi, \lambda_2) = 0, \quad y_1(\pi, \lambda_2) - y_1(\pi, \lambda_1) \neq 0. \tag{5.4}$$

Then Problem Y (the functions $p(x)$, $q(x)$ and coefficients a_{11}, a_{21} and a_{23}) are uniquely determined by the spectrum of Problem N_1 with $h = -a_{11}$ and eigenvalues λ_i, $i = 1, 2$.

Proof. Applying the method for reconstructing the diffusion operator (see [16]) to problem N_1 with $h = -a_{11}$, we see that the functions $p(x)$ and $q(x)$ and the coefficient $a_{11} = -h$ is uniquely determined by the spectrum of Problem N_1. To prove the theorem, it remains to find the coefficients a_{21} and a_{23}.

Since the functions $p(x)$ and $q(x)$ are reconstructed we see that the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1.1) under conditions (2.2) are known. So the eigenvalues of Problem Y are the roots of the entire function

$$\begin{gathered} \Delta_2(\lambda) = -2\, a_{21} - a_{23}\, y_1(\pi,\, \lambda) - y_1'(\pi,\, \lambda) + \\ + (a_{11}\, a_{23} + a_{21}^2)\, y_2(\pi,\, \lambda) + a_{11}\, y_2'(\pi,\, \lambda). \end{gathered} \tag{5.5}$$

It now follows that

$$2\, a_{21} + a_{23}\, y_1(\pi,\, \lambda_i) = -y_1'(\pi,\, \lambda_i) + a_{11}\, y_2'(\pi,\, \lambda_i), \qquad i = 1, 2. \tag{5.6}$$

The determinant of system (5.6) with respect to the unknowns $2\, a_{21}$ and a_{23} is equal to $\big(y_1(\pi, \lambda_2) - y_1(\pi, \lambda_1)\big) \neq 0$. Therefore, system (5.6) has a unique solution, which can be found by the Cramer formulas.

Hence Problems Y and N_1 are uniquely determined by the spectrum of Problem N_1 and two eigenvalues of Problem Y. □

If condition (5.4) holds on the basis of the proof of Theorem 5.2, one can construct the following

algorithm for the unique identification of problems Y and $\mathbf{N}_1$:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y'(0) - a_{11}\, y(0) = y'(\pi) + a_{11}\, y(\pi) = 0$ (see [16]), we find the functions functions $p(x)$, $q(x)$, and coefficient a_{11}.

Step 2. By using the function $q(x)$, we find the linearly independent solutions $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ of equation (1.1), satisfying conditions (2.2).
Step 3. By using the eigenvalues λ_i, $i = 1, 2$ (or $i = 1, 2, 3$) of Problem Y_1 satisfying (5.4), we find the solution of system (5.6) (the coefficients a_{21} and a_{23}). We thereby completely reconstruct Problem Y and N_1.

6 A unique determination of the self-adjoint diffusion operators G_1 and G_2 by a spectrum and one eigenvalue in special cases

Theorem 6.1. *Suppose the coefficients a_{11}, $a_{21}$are real, and $a_{23} = -a_{11}$; the number λ_0 an eigenvalue of Problem Y. Then Problem Y (the functions $p(x)$, $q(x)$ and coefficients a_{11} and a_{21}) are uniquely determined by the spectrum of Problem N_1 with $h = -a_{11} = a_{23}$ and one eigenvalue λ_0.*

Proof. Applying the method for reconstructing the diffusion operator (see [16]) to problem N_1 with $h = -a_{11} = a_{23}$, we see that the functions $p(x)$ and $q(x)$ and the coefficients $a_{11} = -h$ and $a_{23} = h$ are uniquely determined by the spectrum of Problem N_1. To prove the theorem, it remains to find the coefficient a_{21}.

Since the functions $p(x)$ and $q(x)$ are reconstructed we see that the linearly independent solutions $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ of equation (1.1) under conditions (2.2) are known. So the eigenvalue of Problem Y is the root of the entire function (5.2) and we obtain the following equation:

$$a_{21} = \tfrac{1}{2}\Big(a_{11}\left(y_2'(\pi,\,\lambda_0) + y_1(\pi,\,\lambda_0)\right) - y_1'(\pi,\,\lambda_0)\Big). \tag{6.1}$$

Therefore, the coefficient a_{21} is uniquely determined by formula (6.1).

Hence Problems Y and N_1 are uniquely determined by the spectrum of Problem N_1 and an eigenvalue of Problem Y. □

Remark 3. *If $a_{21} = 0$, then Y=N_1. Therefore, results [16] on unique reconstruction of Problem N_1 is a special case of Theorem 5.2 proved above, and Theorem 5.2 is a generalization of the result [16].*

On the basis of the proof of Theorem 5.2, one can construct the following

algorithm for the unique identification of Problems Y and N_1:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y'(0) - a_{11}\,y(0) = y'(\pi) + a_{11}\,y(\pi) = 0$ (see [16]), we find the functions functions $p(x)$, $q(x)$, and coefficients a_{11} and $a_{23} = -a_{11}$.
Step 2. By using the function $q(x)$, we find the linearly independent solutions $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ of equation (1.1), satisfying conditions (2.2).
Step 3. Substituting an eigenvalue of Problem Y for λ_1 in (6.1), we get a_{21}. We thereby completely reconstruct Problems Y and N_1.

Theorem 6.2. *Suppose that the coefficient ω is real, the number λ_0 is an eigenvalue of Problem G_2 such that $y_2(\pi,\,\lambda_0) \neq 0$. Then Problem Y (the functions $p(x)$, $q(x)$ and coefficients a_{11} and a_{21}) are uniquely determined by the spectrum of Problem N_1 with $h = \omega$ and one eigenvalue λ_0.*

Proof. Applying the method for reconstructing the diffusion operator (see [16]) to problem N_1 with $h = \omega$, we see that the functions $p(x)$ and $q(x)$ and the coefficient $\omega = h$ are uniquely determined by the spectrum of Problem N_1. To prove the theorem, it remains to find the coefficient α.

Since the functions $p(x)$ and $q(x)$ are reconstructed we see that the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of equation (1.1) under conditions (2.2) are known. So the eigenvalue of Problem G_2 is the root of the entire function (4.2) with $\omega = \overline{\omega}$ and we obtain the following equation:

$$\alpha = -\frac{2\,\omega + \omega^2\,y_1(\pi, \lambda)}{y_2'(\pi,\, \lambda_0)}. \tag{6.2}$$

Hence Problems G_2 with real ω and N_1 are uniquely determined by the spectrum of Problem N_1 and one eigenvalue of Problem G_2. □

On the basis of the proof of Theorem 6.2, one can construct the following

algorithm for the unique identification Problems G_2 and N_1:

Step 1. By the method for reconstructing the diffusion operator with the boundary conditions $y(0) = y(\pi) = 0$ (see[16]), we find the functions functions $p(x)$ and $q(x)$ and the real number ω from spectrum of Problem N_1.
Step 2. By using the functions $p(x)$ and $q(x)$, we find the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of equation (1.1), satisfying conditions (2.2).
Step 3. By using the number λ_0 satisfying the condition $y_2(\pi, \lambda_0) \neq 0$ and formula (6.2) we find the coefficient α of boundary conditions of problem G_2. We thereby completely reconstruct Problems G_2 and N_1.

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