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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

https://scholar.google.com/citations?user=Zay4fxkAAAAJ&hl=ru&authuser=1

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

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ABSTRACT LINEAR VOLTERRA SECOND-ORDER INTEGRO-DIFFERENTIAL EQUATIONS

D.A. Zakora

Communicated by K.N. Ospanov

Key words: Volterra integro-differential equation, evolution equation, stable family of operators, C_0-semigroup, Cauchy problem.

AMS Mathematics Subject Classification: 39B42, 39B99.

Abstract. We study a class of time-dependent linear second-order integro-differential equations with the evolution equation approach. These equations arise naturally in the study of viscoelasticity. Existence theorems for strong solutions for three classes of complete integro-differential second-order equations are obtained.

1 Introduction

The purpose of this paper is to study the Cauchy problem for the abstract linear Volterra second-order integro-differential equation

$$\frac{d^2u}{dt^2} = A(t)\frac{du}{dt} + B(t)u + \int_0^t G(t,s)u(s)\,ds + f(t), \quad u(0) = u^0, \;\; u'(0) = u^1 \tag{1.1}$$

in a Banach space E or in a Hilbert space H. Such Cauchy problems arise naturally in the study of viscoelasticity (see [7], [12] and the references given there).

Operators $A(t)$ and $B(t)$ are compared by their domains of definition. We consider such equations which have a unique so-called main operator; it has the narrowest domain of definition compared with the other operators. Our purpose is to study three types of equations.

This paper consists of Introduction, Sections 2 and 3. Introduction contains a brief summary of the theory of evolutionary equations. In Section 2 we prove Theorem 2.1 on strong solvability of the Cauchy problem for the abstract linear Volterra first-order integro-differential equation (2.1). Such equations were studied, for example, in [6], [1], [5]. In [11] problem (2.1) was studied under the assumption that the operator coefficient $\mathcal{A}(t)$ is a generator of a holomorphic semigroup at any fixed time. Our Theorem 2.1 in Section 2 is quite close to the one which follows from [6].

We mention also monograph [14] where Cauchy problems are investigated for integro-differential and functional equations.

Section 3 is devoted to the study of Cauchy problem (1.1). Our method is based on the following simple idea. We reduce the second-order integro-differential equation to the first-order one and then apply Theorem 2.1. This method is similar to the one which is used in [9].

Now we introduce some notations and terminology. Let E be a Banach space and let $\mathcal{L}(E)$ denote the Banach space of all bounded linear operators acting in E.

The notation $A \in \mathcal{J}(M,\omega)$ ($M \geqslant 1$, $\omega \in \mathbb{R}$) means that the operator A is a generator of a strongly continuous (one-parameter) semigroup (or C_0-semigroup) $U(t)$ ($t \geqslant 0$) of bounded linear operators on a Banach space E. The family $U(t)$ satisfies the following inequality: $\|U(t)\|_{\mathcal{L}(E)} \leqslant Me^{\omega t}$ for all $t \geqslant 0$.

Let $A(t)$ be a generator of C_0-semigroup in E for all $t \in [0,T]$. The following definition and the theorems are in agreement with [13].

Definition 1 (see [13], p. 93). A family of linear operators $A(t)$ is called *stable* on $[0,T]$ if there exist real numbers $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\Big\| \prod_{k=n\searrow 1} \big(A(t_k) - \lambda\big)^{-1} \Big\|_{\mathcal{L}(E)} \equiv \Big\| \big(A(t_n) - \lambda\big)^{-1} \cdots \big(A(t_1) - \lambda\big)^{-1} \Big\|_{\mathcal{L}(E)} \leqslant \frac{M}{(\lambda - \omega)^n}$$

for all $\lambda > \omega$, $n \in \mathbb{N}$, and $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant T$.

Remark 1. If $A(t) \in \mathcal{J}(1,\omega)$ for each $t \in [0,T]$ then the family $A(t)$ is clearly stable with the stability constants $M = 1$ and ω.

Theorem 1.1 (see [13], p. 94). *Assume that $A(t)$ is stable with the stability constants M and ω. If $B(t) \in \mathcal{L}(E)$ for each $t \in [0,T]$ and $\|B(t)\|_{\mathcal{L}(E)} \leqslant K < +\infty$, then $A(t) + B(t)$ is stable with the stability constants M and $\omega + MK$.*

Theorem 1.2 (see [13], p. 102). *Suppose that $A(t)$ is stable, its domain $\mathcal{D}(A(t)) \equiv \mathcal{D}$ is independent of t and $A(t)u$ for each $u \in \mathcal{D}$ is strongly continuously differentiable on $[0,T]$. Then there exists a unique function $U(t,s) \in \mathcal{L}(E)$ such that*

1) The operator $U(t,s)$ is strongly continuous in t, s, $U(s,s) = I$ and $\|U(t,s)\|_{\mathcal{L}(E)} \leqslant Me^{\omega(t-s)}$ $(t \geqslant s)$;

2) $U(t,s) = U(t,r)U(r,s)$ $(s \leqslant r \leqslant t)$;

3) $U(t,s)$ maps $\mathcal{D}$ into $\mathcal{D}$ $(U(t,s)\mathcal{D} \subset \mathcal{D})$, $U(t,s)u$ for each $u \in \mathcal{D}$ is strongly continuously differentiable in t, s and

$$\frac{\partial}{\partial t} U(t,s)u = A(t)U(t,s)u, \quad \frac{\partial}{\partial s} U(t,s)u = -U(t,s)A(s)u.$$

Both sides of these equations are strongly continuous on $0 \leqslant s \leqslant t \leqslant T$.

Remark 2. If there exists an inverse operator $A^{-1}(t) \in \mathcal{L}(E)$ for each $t \in [0,T]$ then the operator $V(t,s) := A(t)U(t,s)A^{-1}(s)$ is bounded and strongly continuous on $0 \leqslant s \leqslant t \leqslant T$. This can be easily seen from Property 3 of Theorem 1.2, [10, p. 220, Lemma 1.5], and the representation $V(t,s) = \big(A(t)U(t,s)A^{-1}(0)\big)\big(A(0)A^{-1}(s)\big)$.

Let us consider the initial-value problem

$$\frac{du}{dt} = A(t)u + f(t), \quad u(0) = u^0 \tag{1.2}$$

in a Banach space E. For a fixed $t \in [0,T]$, $A(t)$ is assumed to be a closed operator. The domain $\mathcal{D}(A(t)) \equiv \mathcal{D}$ is independent of t and is dense in E.

Definition 2. We say that a function u is a *strong solution* to Cauchy problem (1.2) on the interval $[0,T]$ if $u(t) \in \mathcal{D}$ for all $t \in [0,T]$, $Au \in C([0,T];E)$, $u \in C^1([0,T];E)$, and $u(t)$ satisfies (1.2) for all $t \in [0,T]$.

Theorem 1.3 (see [13], p. 105). *Let the assumptions of Theorem 1.2 be satisfied. Then for any $u^0 \in \mathcal{D}$ and $f \in C^1([0,T];E)$ Cauchy problem (1.2) has a unique strong solution. This solution has the following form*

$$u(t) = U(t,0)u^0 + \int_0^t U(t,s)f(s)\,ds, \quad t \in [0,T].$$

Remark 3. For simplicity of the following notation we write $A \in SC^n_{\mathcal{D}}([0,T];E)$ ($\mathcal{D} \subset E$, $n \in \mathbb{N} \cup \{0\}$) if $Au \in C^n([0,T];E)$ for each $u \in \mathcal{D}$. Note that if $A \in SC_E([0,T];E)$ then the Banach-Steinhaus Theorem implies that $A(t) \in \mathcal{L}(E)$ for all $t \in [0,T]$ and $\sup_{t\in[0,T]} \|A(t)\|_{\mathcal{L}(E)} < +\infty$.

2 First-order integro-differential equation in a Banach space

Let us consider the Cauchy problem for the abstract integro-differential equation

$$\frac{dz}{dt} = \mathcal{A}(t)z + \int_0^t \mathcal{G}(t,s)z(s)\,ds + f(t), \quad z(0) = z^0. \tag{2.1}$$

For a fixed $t \in [0,T]$, $\mathcal{A}(t)$ is assumed to be a closed operator. The domain $\mathcal{D}(\mathcal{A}(t)) \equiv \mathcal{D}$ is independent of t and is dense in a Banach space $\mathcal{E}$. Let us also make the assumption that $\mathcal{D} \subset \mathcal{D}(\mathcal{G}(t,s))$ for all $t,s \in T_\Delta := \{0 \leqslant s \leqslant t \leqslant T\}$.

Theorem 2.1. *Let $\mathcal{C}$ be a closed operator such that $\mathcal{D}(\mathcal{C}) = \mathcal{D}$, $\mathcal{C}^{-1} \in \mathcal{L}(\mathcal{E})$. Let the following conditions be satisfied:*

1) $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}$, $\mathcal{A}(t) \in SC^1_{\mathcal{D}}([0,T];\mathcal{E})$ and is stable on $[0,T]$;

2) $\mathcal{G}(t,s)\mathcal{C}^{-1}$, $\big(\mathcal{G}(t,s)\mathcal{C}^{-1}\big)'_t \in SC_{\mathcal{E}}(T_\Delta;\mathcal{E})$.

Then for any $z^0 \in \mathcal{D}$ and $f \in C^1([0,T];\mathcal{E})$ Cauchy problem (2.1) has a unique strong solution.

Proof. Our proof starts with the observation that one may suppose that $\mathcal{A}^{-1}(t) \in \mathcal{L}(\mathcal{E})$ for all $t \in [0,T]$. Otherwise we carry out the substitution $z(t) = e^{at}v(t)$ $(a > \omega)$ in (2.1). The function v satisfies the Cauchy problem

$$\frac{dv}{dt} = \big(\mathcal{A}(t) - a\big)v + \int_0^t e^{-a(t-s)}\mathcal{G}(t,s)v(s)\,ds + e^{-at}f(t), \quad v(0) = z^0,$$

where $\big(\mathcal{A}(t) - a\big)^{-1} \in \mathcal{L}(\mathcal{E})$ for all $t \in [0,T]$.

Let us suppose that Cauchy problem (2.1) has a strong solution $z(t)$. It follows that

$$\begin{aligned} z(t) &= U(t,0)z^0 + \int_0^t U(t,s)\left[\int_0^s \mathcal{G}(s,\tau)z(\tau)\,d\tau + f(s)\right]ds \\ &= z_0(t) + \int_0^t \left[\int_\tau^t U(t,s)\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau)\,ds\right]d\tau, \end{aligned} \tag{2.2}$$

$$z_0(t) := U(t,0)z^0 + \int_0^t U(t,s)f(s)\,ds, \quad \mathcal{G}_0(s,\tau) := \mathcal{G}(s,\tau)\mathcal{C}^{-1}. \tag{2.3}$$

We transform the inner integral in (2.2). By the assumption, $z(\tau) \in \mathcal{D} = \mathcal{D}(\mathcal{C})$. It follows that there exists the partial derivative (see Theorem 1.2, Property 3)

$$\frac{\partial}{\partial s}\Big[-U(t,s)\mathcal{A}^{-1}(s)\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau)\Big] = U(t,s)\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau)+ \\ + U(t,s)\mathcal{A}^{-1}(s)\Big[\mathcal{A}'(s)\mathcal{A}^{-1}(s)\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau) - \frac{\partial}{\partial s}\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau)\Big], \quad (2.4)$$

which is strongly continuous in s on $[\tau, t]$. Here we use the fact that if $\mathcal{A}(s)z$ for each $z \in \mathcal{D}$ is strongly continuously differentiable in s and $\mathcal{A}^{-1}(s) \in \mathcal{L}(E)$ for all s, then the operator $\mathcal{A}'(s)\mathcal{A}^{-1}(s)$ is strongly continuous in s (see [10, p. 220, Lemma 1.5]).

Define $\mathcal{V}(t,s) := \mathcal{A}(t)U(t,s)\mathcal{A}^{-1}(s)$. Integrating (2.4) with respect to s leads to

$$\int_\tau^t U(t,s)\mathcal{G}(s,\tau)z(\tau)\,ds \equiv \int_\tau^t U(t,s)\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau)\,ds \\ = \mathcal{A}^{-1}(t)\Big\{-\mathcal{G}_0(t,\tau)\mathcal{C}z(\tau) + \mathcal{V}(t,\tau)\mathcal{G}_0(\tau,\tau)\mathcal{C}z(\tau) \\ - \int_\tau^t \mathcal{V}(t,s)\Big[\mathcal{A}'(s)\mathcal{A}^{-1}(s)\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau) - \frac{\partial}{\partial s}\mathcal{G}_0(s,\tau)\mathcal{C}z(\tau)\Big]ds\Big\}. \quad (2.5)$$

From (2.2) and (2.5) we conclude that

$$z(t) = z_0(t) + \int_0^t \mathcal{A}^{-1}(t)\mathcal{R}(t,\tau)\mathcal{C}z(\tau)\,d\tau, \quad (2.6)$$

$$\mathcal{R}(t,\tau) := -\mathcal{G}_0(t,\tau) + \mathcal{V}(t,\tau)\mathcal{G}_0(\tau,\tau)- \\ - \int_\tau^t \mathcal{V}(t,s)\Big[\mathcal{A}'(s)\mathcal{A}^{-1}(s)\mathcal{G}_0(s,\tau) - \frac{\partial}{\partial s}\mathcal{G}_0(s,\tau)\Big]\cdot ds. \quad (2.7)$$

Here z_0 (see (2.3)) is a strong solution to Cauchy problem (2.1) without integral term. Therefore $z_0 \in C^1([0,T];\mathcal{E})$, $\mathcal{A}z_0 \in C([0,T];\mathcal{E})$ (see Theorem 1.3). By the assumptions of this theorem and Remark 2, the operator $\mathcal{R}(t,s)$ is strongly continuous on T_Δ.

From the above it follows that a strong solution to Cauchy problem (2.1) is a solution of Volterra integral equation (2.6). Let us show that equation (2.6) has a unique solution and this solution is a strong solution to Cauchy problem (2.1).

We define $\mathcal{E}(\mathcal{C})$ to be $\mathcal{D}(\mathcal{C})$ endowed with the norm $\|z\|_{\mathcal{E}(\mathcal{C})} := \|\mathcal{C}z\|$. We conclude from (2.7) and [10, p 220, Lemma 1.5] that the function

$$\mathcal{A}^{-1}(t)\mathcal{R}(t,s)\mathcal{C}z = \mathcal{C}^{-1}\big(\mathcal{C}\mathcal{A}^{-1}(0)\big)\big(\mathcal{A}(0)\mathcal{A}^{-1}(t)\big)\mathcal{R}(t,s)\mathcal{C}z$$

for each $z \in \mathcal{E}(\mathcal{C})$ is strongly continuous in $t, s \in T_\Delta$. Therefore equation (2.6) is a Volterra integral equation in the Banach space $\mathcal{E}(\mathcal{C})$ and its kernel is strongly continuous.

From $\mathcal{A}z_0 \in C([0,T];\mathcal{E})$ we see that $z_0 \in C([0,T];\mathcal{E}(\mathcal{C}))$. Hence equation (2.6) has a unique solution $z \in C([0,T];\mathcal{E}(\mathcal{C}) = \mathcal{D})$. Let us show that the function z is a unique strong solution to Cauchy problem (2.1).

By (2.6), $z \in C([0,T];\mathcal{E}(\mathcal{C}) = \mathcal{D})$, and $\mathcal{A}(t)z_0(t) = z_0'(t) - f(t)$ we obtain

$$\mathcal{A}(t)z(t) = z_0'(t) - f(t) + \int_0^t \mathcal{R}(t,\tau)\mathcal{C}z(\tau)\,d\tau \in C([0,T];\mathcal{E}).$$

From $\mathcal{R}(t,t) \equiv 0$, (2.6), and $z_0 \in C^1([0,T];\mathcal{E})$ we have

$$z'(t) = z_0'(t) + \int_0^t \big(\mathcal{R}(t,\tau) + \mathcal{G}_0(t,\tau)\big)\mathcal{C}z(\tau)\,d\tau \in C([0,T];\mathcal{E}).$$

From the above and (2.3) it follows that the function z is a unique strong solution to Cauchy problem (2.1). □

Remark 4. If $\mathcal{A}(t) \equiv \mathcal{A}$ is a generator of a holomorphic semigroup, then it suffices to assume that $f \in C^\alpha([0,T];E)$ $(0 < \alpha \leqslant 1)$. This notation means that there exists a real number $K > 0$ such that

$$\|f(t) - f(s)\|_E \leqslant K|t-s|^\alpha \quad \forall\ 0 \leqslant s,t \leqslant T.$$

In this case we need to use [4, p. 130, Theorem 1.4] instead of Theorem 1.3.

3 Second-order integro-differential equation

In this section we study complete second-order integro-differential equation (1.1) in a Banach space or in a Hilbert space. We suppose that this equation has a unique so-called main operator; it has the narrowest domain compared with the other operators.

3.1 Second-order integro-differential equation in a Banach space. The main operator acts on a function

Let us consider the Cauchy problem for the integro-differential equation

$$\frac{d^2u}{dt^2} = A(t)\frac{du}{dt} + B(t)u + \int_0^t G(t,s)u(s)\,ds + f(t), \quad u(0) = u^0,\ \ u'(0) = u^1, \tag{3.1}$$

where $B(t) := B_0^2(t) + Q_0(t)$. For a fixed $t \in [0,T]$, $B_0(t)$ is assumed to be a closed operator. The domain $\mathcal{D}(B_0(t)) \equiv \mathcal{D}$ is independent of t and is dense in a Banach space E, $A(t) \in \mathcal{L}(E)$. The operators $Q_0(t)$ and $G(t,s)$ are relatively bounded with respect to the operators $B_0(t)$ and $B_0^2(t)$.

Definition 3. We say that a function u is a *strong solution* to Cauchy problem (3.1) on the interval $[0,T]$ if $u \in C^2([0,T];E)$, $u(t) \in \mathcal{D}(B_0^2(t))$, $u'(t) \in \mathcal{D}$ for all $t \in [0,T]$, B_0^2u, $B_0u' \in C([0,T];E)$, and $u(t)$ satisfies (3.1) for all $t \in [0,T]$.

Theorem 3.1. *Let the following conditions be satisfied:*

1) $\mathcal{D}(B_0(t)) = \mathcal{D}$, $\pm B_0(t) \in SC^1_{\mathcal{D}}([0,T];E)$ *and are stable on* $[0,T]$ *with the stability constants* $M_\pm$, $\omega_\pm$;

2) $Q_0B_1^{-1}$, A, $B_1'B_1^{-1} \in SC_E([0,T];E) \cap SC^1_{\mathcal{D}}([0,T];E)$, *where* $\lambda_0 > \omega_+$ *and* $B_1(t) := B_0(t) - \lambda_0$;

3) $G(t,s)B_1^{-1}(s)B_1^{-1}(0)$, $\big(G(t,s)B_1^{-1}(s)B_1^{-1}(0)\big)'_t \in SC_E(T_\Delta;E)$.

Then for any $u^0 \in \mathcal{D}(B_0^2(0))$, $u^1 \in \mathcal{D}$, *and* $f \in C^1([0,T];E)$ *Cauchy problem* (3.1) *has a unique strong solution.*

Proof. Step 1. Fix $\lambda_0 > \omega_+$ and rewrite the operator $B(t)$ in the following way

$$B(t) = B_0^2(t) + Q_0(t) = \big(B_1(t) + 2\lambda_0 I + \lambda_0^2 B_1^{-1}(t) + Q_0(t)B_1^{-1}(t)\big)B_1(t)$$
$$=: \big(B_1(t) + Q_1(t)\big)B_1(t), \quad B_1(t) := B_0(t) - \lambda_0, \tag{3.2}$$

where $B_1^{-1}(t) \in \mathcal{L}(E)$ for all $t \in [0,T]$, $Q_1 \in SC_E([0,T];E) \cap SC^1_{\mathcal{D}}([0,T];E)$.

Let us suppose that Cauchy problem (3.1) has a strong solution u (see Definition 3). Set $v(t) := u'(t)$, $w(t) := B_1(t)u(t)$. Definition 3 and the assumptions of the theorem imply that v, $w \in C^1([0,T];E)$. From (3.1), (3.2) we conclude that the functions v and w satisfy the system of equations

$$\begin{cases} \dfrac{dv}{dt} = A(t)v + B_1(t)w + Q_1(t)w + \displaystyle\int_0^t G(t,s)B_1^{-1}(s)w(s)\,ds + f(t), \\ \dfrac{dw}{dt} = B_1(t)v + B_1'(t)B_1^{-1}(t)w, \qquad v(0) = u^1, \;\; w(0) = B_1(0)u^0. \end{cases} \tag{3.3}$$

We rewrite system of equations (3.3) as the following Cauchy problem

$$\frac{dz}{dt} = \mathcal{B}(t)z + \mathcal{Q}(t)z + \int_0^t \mathcal{G}(t,s)z(s)\,ds + \mathcal{F}(t), \quad z(0) = z^0 \tag{3.4}$$

in the Banach space $\mathcal{E} := E \times E = \{z = (v;w)^\tau \,|\, v,w \in E\}$ (the symbol τ means the transposition operation). The following notation is used in (3.4):

$$\mathcal{B}(t) := \begin{pmatrix} 0 & B_1(t) \\ B_1(t) & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{B}(t)) = \mathcal{D} \times \mathcal{D}, \;\; \overline{\mathcal{D}(\mathcal{B}(t))} = \mathcal{E},$$
$$\mathcal{Q}(t) := \begin{pmatrix} A(t) & Q_1(t) \\ 0 & B_1'(t)B_1^{-1}(t) \end{pmatrix}, \quad \mathcal{Q}(t) \in \mathcal{L}(\mathcal{E}),$$
$$\mathcal{G}(t,s) := \begin{pmatrix} 0 & G(t,s)B_1^{-1}(s) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{G}(t,s)) = E \times \mathcal{D} \supset \mathcal{D}(\mathcal{B}(t)),$$
$$\mathcal{F}(t) := (f(t);0)^\tau, \quad z^0 := (u^1; B_1(0)u^0)^\tau.$$

The proof of the theorem is based on applying Theorem 2.1 to Cauchy problem (3.4).

Step 2. Let us prove that the family of operators $\mathcal{B}(t)$ is stable on $[0,T]$. Note first that the following factorization of the operator $\mathcal{B}(t)$ takes place:

$$\mathcal{B}(t) = \mathcal{T} \cdot \begin{pmatrix} B_1(t) & 0 \\ 0 & -B_1(t) \end{pmatrix} \cdot \mathcal{T}^{-1}, \quad \mathcal{T} := \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \mathcal{T}^{-1}. \tag{3.5}$$

From factorization (3.5) we see that the densely defined operator $\mathcal{B}(t)$ is closed and

$$\Big\| \prod_{k=n\searrow 1} \big(\mathcal{B}(t_k) - \lambda\big)^{-1} \Big\|_{\mathcal{L}(\mathcal{E})}$$
$$= \Big\| \mathcal{T} \cdot \mathrm{diag}\Big(\prod_{k=n\searrow 1} \big(B_1(t_k) - \lambda\big)^{-1}, \prod_{k=n\searrow 1} \big(-B_1(t_k) - \lambda\big)^{-1} \Big) \cdot \mathcal{T}^{-1} \Big\|_{\mathcal{L}(\mathcal{E})}$$
$$\leqslant \|\mathcal{T}\|^2_{\mathcal{L}(\mathcal{E})} \max\Big\{ \Big\| \prod_{k=n\searrow 1} \big(B_0(t_k) - (\lambda + \lambda_0)\big)^{-1} \Big\|_{\mathcal{L}(E)}, \Big\| \prod_{k=n\searrow 1} \big(-B_0(t_k) - (\lambda - \lambda_0)\big)^{-1} \Big\|_{\mathcal{L}(E)} \Big\}$$
$$\leqslant \|\mathcal{T}\|^2_{\mathcal{L}(\mathcal{E})} \max\Big\{ \frac{M_+}{(\lambda + \lambda_0 - \omega_+)^n}, \frac{M_-}{(\lambda - \lambda_0 - \omega_-)^n} \Big\} \leqslant \frac{\|\mathcal{T}\|^2_{\mathcal{L}(\mathcal{E})} \max\{M_+, M_-\}}{(\lambda - \omega_0)^n}$$

for all $\lambda > \omega_0 := \max\{\omega_+ - \lambda_0, \omega_- + \lambda_0\}$, $n \in \mathbb{N}$, and $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant T$. Hence $\mathcal{B}(t)$ is stable on $[0,T]$ (see Definition 1).

Step 3. The assumptions of the theorem imply that $\mathcal{Q}(t) \in \mathcal{L}(\mathcal{E})$, $\|\mathcal{Q}(t)\|_{\mathcal{L}(\mathcal{E})} \leqslant K < +\infty$ for all $t \in [0,T]$, where $K > 0$ is independent of t. By Theorem 1.1, the family of operators $\mathcal{B}(t) + \mathcal{Q}(t)$ is stable on $[0,T]$. Moreover, it follows that $\mathcal{B} + \mathcal{Q} \in SC^1_{\mathcal{D}\times\mathcal{D}}([0,T];\mathcal{E})$.

Let $\mathcal{C} := \operatorname{diag}\big(B_1(0), B_1(0)\big)$, $\mathcal{D}(\mathcal{C}) = \mathcal{D} \times \mathcal{D}$, then $\mathcal{C}^{-1} \in \mathcal{L}(\mathcal{E})$. The assumptions of the theorem imply that $\mathcal{G}(t,s)\mathcal{C}^{-1}$, $\big(\mathcal{G}(t,s)\mathcal{C}^{-1}\big)'_t \in SC_{\mathcal{E}}(T_\Delta;\mathcal{E})$.

The assumptions on the initial conditions and the function f imply also that

$$z^0 = (u^1; B_1(0)u^0)^\tau \in \mathcal{D} \times \mathcal{D} = \mathcal{D}(\mathcal{B}(t) + \mathcal{Q}(t)), \quad \mathcal{F}(t) = (f(t); 0)^\tau \in C^1([0,T];\mathcal{E}).$$

By Theorem 2.1, Cauchy problem (3.4) has a unique strong solution. In other words, there exists a function z such that $z \in C^1([0,T];\mathcal{E})$, $z(t) \in \mathcal{D} \times \mathcal{D}$ for all $t \in [0,T]$, $(\mathcal{B} + \mathcal{Q})z \in C([0,T];\mathcal{E})$, and $z(t)$ satisfies (3.4) for all $t \in [0,T]$.

Step 4. Let $z(t) = (v(t); w(t))^\tau$ be the unique strong solution to Cauchy problem (3.4). Define $u(t) := B_1^{-1}(t)w(t)$.

From $w(t) \in \mathcal{D}$ for all $t \in [0,T]$, $B_1 w \in C([0,T];E)$ it follows that $u(t) \in \mathcal{D}(B_1^2(t))$ for all $t \in [0,T]$, $B_1^2 u \in C([0,T];E)$.

The second equation in system (3.3) implies that

$$\begin{aligned} u'(t) &= -B_1^{-1}(t)B_1'(t)B_1^{-1}(t)w(t) + B_1^{-1}(t)w'(t) = \\ &= B_1^{-1}(t)\big(-B_1'(t)B_1^{-1}(t)w(t) + w'(t)\big) = v(t) \in C^1([0,T];E). \end{aligned}$$

Hence $u \in C^2([0,T];E)$ and $u'(t) \in \mathcal{D}$ for all $t \in [0,T]$, $B_1 u' \in C([0,T];E)$.

The first equation in system (3.3) and (3.2) imply that

$$\begin{aligned} u''(t) = v'(t) &= A(t)u'(t) + \big(B_1(t) + Q_1(t)\big)B_1(t)u(t) + \int_0^t G(t,s)u(s)\,ds + f(t) \\ &= A(t)\frac{du}{dt} + \big(B_0^2(t) + Q_0(t)\big)u + \int_0^t G(t,s)u(s)\,ds + f(t), \end{aligned}$$

i.e. the function u satisfies equation (3.1). It is easily seen that the function u satisfies also the initial conditions in (3.1). By Definition 3, the function u is a strong solution to Cauchy problem (3.1). □

3.2 Second-order integro-differential equation in a Hilbert space. The main operator acts on a function

Let us consider the Cauchy problem for the integro-differential equation

$$\frac{d^2u}{dt^2} = A(t)\frac{du}{dt} + B(t)u + \int_0^t G(t,s)u(s)\,ds + f(t), \quad u(0) = u^0, \ \ u'(0) = u^1, \tag{3.6}$$

where $B(t) := B_0^2(t) + Q_0(t)$. For a fixed $t \in [0,T]$, $B_0(t)$ is assumed to be a closed operator. The domain $\mathcal{D}(B_0(t)) \equiv \mathcal{D}$ is independent of t and is dense in a Hilbert space H. The operators $Q_0(t)$, $A(t)$, and $G(t,s)$ are relatively bounded with respect to the operators $B_0(t)$ and $B_0^2(t)$.

Function u is a strong solution to Cauchy problem (3.6) if it satisfies Definition 3.

Theorem 3.2. *Let the following conditions be satisfied:*

1) $\mathcal{D}(B_0(t)) = \mathcal{D}$, $\pm B_0(t) \in \mathcal{J}(1, \omega_\pm)$ *for all* $t \in [0,T]$, $B_0 \in SC^1_{\mathcal{D}}([0,T];H)$*;*

2) $Q_0 B_1^{-1}$, $B_1' B_1^{-1} \in SC_H([0,T];H) \cap SC^1_{\mathcal{D}}([0,T];H)$, *where* $\lambda_0 > \omega_+$ *and* $B_1(t) := B_0(t) - \lambda_0$*;*

3) $A(t) \in \mathcal{J}(1, \omega_A)$ *for all* $t \in [0,T]$, $A \in SC^1_{\mathcal{D}}([0,T];H)$*;*

4) $G(t,s)B_1^{-1}(s)B_1^{-1}(0)$, $\big(G(t,s)B_1^{-1}(s)B_1^{-1}(0)\big)'_t \in SC_H(T_\Delta;H)$.

Assume that the operators $A(t)$ *and* $B_0(t)$ *satisfy one of the following conditions:*

5a) $\exists\, b \geqslant 0 : \|A(t)u\|_H \leqslant \|B_0(t)u\|_H + b\|u\|_H \ \forall\, u \in \mathcal{D}$, $t \in [0,T]$*;*

5b) $A(t)B_1^{-1}(t) \in \mathfrak{S}_\infty(H)$*;*

5c) $B_0(t) = -B_0^*(t)$, $B_0^{-1}(t) \in \mathcal{L}(H)$ *for all* $t \in [0,T]$, $A(t)B_0^{-1}(t) \in \mathcal{L}(H)$.

Then for any $u^0 \in \mathcal{D}(B_0^2(0))$, $u^1 \in \mathcal{D}$, *and* $f \in C^1([0,T];H)$ *Cauchy problem* (3.6) *has a unique strong solution.*

Proof. Step 1. Let us suppose that Cauchy problem (3.6) has a strong solution u (see Definition 3). Set $v(t) := u'(t)$, $w(t) := B_1(t)u(t)$. Definition 3 and the assumptions of the theorem imply that v, $w \in C^1([0,T];E)$.As in the proof of Theorem 3.1, equation (3.6) and formulae (3.2) give that the functions v and w satisfy system of equations (3.3) (we set $\lambda_0 = 0$ in (3.2) in the case 5c).

Let us consider the cases 5a и 5b. Rewrite system (3.3) as a Cauchy problem

$$\frac{dz}{dt} = \mathcal{B}(t)z + \mathcal{A}(t)z + \mathcal{Q}(t)z + \int_0^t \mathcal{G}(t,s)z(s)\,ds + \mathcal{F}(t), \quad z(0) = z^0 \tag{3.7}$$

in the Hilbert space $\mathcal{H} := H \oplus H = \{z = (v;w)^\tau \mid v,w \in H\}$.

The following notation is used in (3.7):

$$\mathcal{B}(t) := \begin{pmatrix} 0 & B_1(t) \\ B_1(t) & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{B}(t)) = \mathcal{D} \oplus \mathcal{D}, \ \ \overline{\mathcal{D}(\mathcal{B}(t))} = \mathcal{H},$$

$$\mathcal{A}(t) := \begin{pmatrix} A(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(A(t)) \oplus H \supset \mathcal{D}(\mathcal{B}(t)), \ \ \overline{\mathcal{D}(\mathcal{A}(t))} = \mathcal{H},$$

$$\mathcal{Q}(t) := \begin{pmatrix} 0 & Q_1(t) \\ 0 & B_1'(t)B_1^{-1}(t) \end{pmatrix}, \quad \mathcal{Q}(t) \in \mathcal{L}(\mathcal{H}),$$

$$\mathcal{G}(t,s) := \begin{pmatrix} 0 & G(t,s)B_1^{-1}(s) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{G}(t,s)) = H \oplus \mathcal{D} \supset \mathcal{D}(\mathcal{B}(t)),$$

$$\mathcal{F}(t) := (f(t);0)^\tau, \quad z^0 := (u^1; B_1(0)u^0)^\tau.$$

The proof of the theorem is based on applying Theorem 2.1 to Cauchy problem (3.7).

Step 2. Let us prove that

$$\mathcal{B}(t) \in \mathcal{J}(1, \omega_0), \ \ \omega_0 := \max\{\omega_+ - \lambda_0, \omega_- + \lambda_0\}, \quad \mathcal{A}(t) \in \mathcal{J}(1, \max\{0, \omega_A\}) \tag{3.8}$$

for all $t \in [0,T]$.

From factorization (3.5), $\mathcal{T} = \mathcal{T}^{-1} = \mathcal{T}^*$, $\|\mathcal{T}\|_{\mathcal{L}(\mathcal{H})} = 1$ we see that the densely defined operator $\mathcal{B}(t)$ is closed and

$$\begin{aligned}\big\|\big(\mathcal{B}(t) - \lambda\big)^{-1}\big\|_{\mathcal{L}(\mathcal{H})} &= \big\|\mathcal{T} \cdot \operatorname{diag}\Big(\big(B_1(t) - \lambda\big)^{-1}, \big(-B_1(t) - \lambda\big)^{-1}\Big) \cdot \mathcal{T}^{-1}\big\|_{\mathcal{L}(\mathcal{H})} \\ &\leqslant \max\Big\{(\lambda + \lambda_0 - \omega_+)^{-1}, (\lambda - \lambda_0 - \omega_-)^{-1}\Big\} \leqslant (\lambda - \omega_0)^{-1}\end{aligned}$$

for all $\lambda > \omega_0$ and $t \in [0,T]$. Hence $\mathcal{B}(t) \in \mathcal{J}(1,\omega_0)$ for all $t \in [0,T]$.

Also the densely defined operator $\mathcal{A}(t)$ is closed and

$$\left\|\left(\mathcal{A}(t)-\lambda\right)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \left\|\operatorname{diag}\Big(\left(A(t)-\lambda\right)^{-1}, \lambda^{-1}I\Big)\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \max\Big\{(\lambda-\omega_A)^{-1}, \lambda^{-1}\Big\} \leqslant (\lambda - \max\{0,\omega_A\})^{-1}$$

for all $\lambda > \max\{0,\omega_A\}$, $t \in [0,T]$. Hence $\mathcal{A}(t) \in \mathcal{J}(1,\max\{0,\omega_A\})$ for all $t \in [0,T]$.

Step 3. From (3.8) we conclude that $\mathcal{B}(t)-\omega_0$, $\mathcal{A}(t)-\max\{0,\omega_A\} \in \mathcal{J}(1,0)$. Let us consider the case 5a. For each $z = (v;w)^\tau \in \mathcal{D}(\mathcal{B}(t)) = \mathcal{D} \oplus \mathcal{D}$ we have

$$\begin{aligned}\|(\mathcal{A}(t)-\max\{0,\omega_A\})z\|_{\mathcal{H}} &\leqslant \|A(t)v\|_H + \max\{0,\omega_A\}\|z\|_{\mathcal{H}} \\ &\leqslant \|B_0(t)v\|_H + b\|v\|_H + \max\{0,\omega_A\}\|z\|_{\mathcal{H}} \\ &\leqslant \|B_1(t)v\|_H + (b+|\lambda_0|)\|v\|_H + \max\{0,\omega_A\}\|z\|_{\mathcal{H}} \\ &\leqslant \|\mathcal{B}(t)z\|_{\mathcal{H}} + (b+|\lambda_0|+\max\{0,\omega_A\})\|z\|_{\mathcal{H}} \\ &\leqslant \|(\mathcal{B}(t)-\omega_0)z\|_{\mathcal{H}} + (b+|\lambda_0|+|\omega_0|+\max\{0,\omega_A\})\|z\|_{\mathcal{H}}\end{aligned}$$

for all $t \in [0,T]$. Hence the closure $\overline{(\mathcal{B}(t)-\omega_0)+(\mathcal{A}(t)-\max\{0,\omega_A\})}$ is a generator of a contractive C_0-semigroup (see [4, p. 65, Theorem 6.2]).

Let us prove that the operator $(\mathcal{B}(t)-\omega_0)+(\mathcal{A}(t)-\max\{0,\omega_A\})$ with the domain $\mathcal{D}\oplus\mathcal{D}$ is closed. It is sufficient to show that the operator $\mathcal{B}(t)+\mathcal{A}(t)$ with the domain $\mathcal{D}\oplus\mathcal{D}$ is closed. From 5a we obtain $A(t)B_1^{-1}(t) \in \mathcal{L}(H)$ for all $t \in [0,T]$. Hence

$$\mathcal{B}(t)+\mathcal{A}(t) = \begin{pmatrix} A(t)B_1^{-1}(t) & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} B_1(t) & 0 \\ 0 & -B_1(t) \end{pmatrix}, \tag{3.9}$$

$$\begin{pmatrix} A(t)B_1^{-1}(t) & -I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ -I & A(t)B_1^{-1}(t) \end{pmatrix} \in \mathcal{L}(\mathcal{H}).$$

From this it follows that the operator $\mathcal{B}(t)+\mathcal{A}(t)$ with the domain $\mathcal{D}\oplus\mathcal{D}$ is closed for all $t \in [0,T]$. Hence $\mathcal{B}(t)+\mathcal{A}(t) \in \mathcal{J}(1,\omega_0+\max\{0,\omega_A\})$. Remark 1 implies that the family of operators $\mathcal{B}(t)+\mathcal{A}(t)$ is stable on $[0,T]$.

The assumptions of the theorem imply that $\mathcal{Q}(t) \in \mathcal{L}(\mathcal{H})$, $\|\mathcal{Q}(t)\|_{\mathcal{L}(\mathcal{H})} \leqslant K < +\infty$ for all $t \in [0,T]$. By Theorem 1.1, the family of operators $\mathcal{B}(t)+\mathcal{A}(t)+\mathcal{Q}(t)$ is stable on $[0,T]$. Moreover, it follows that $\mathcal{B}+\mathcal{A}+\mathcal{Q} \in SC^1_{\mathcal{D}\oplus\mathcal{D}}([0,T];\mathcal{H})$.

Let $\mathcal{C} := \operatorname{diag}\big(B_1(0),B_1(0)\big)$, $\mathcal{D}(\mathcal{C}) = \mathcal{D}\oplus\mathcal{D}$, then $\mathcal{C}^{-1} \in \mathcal{L}(\mathcal{H})$. The assumptions of the theorem imply that $\mathcal{G}(t,s)\mathcal{C}^{-1}$, $\big(\mathcal{G}(t,s)\mathcal{C}^{-1}\big)'_t \in SC_{\mathcal{H}}(T_\Delta;\mathcal{H})$.

The assumptions on the initial conditions and the function f imply also that

$$z^0 = (u^1; B_1(0)u^0)^\tau \in \mathcal{D}\oplus\mathcal{D} = \mathcal{D}(\mathcal{B}(t)+\mathcal{A}(t)+\mathcal{Q}(t)), \quad \mathcal{F}(t) = (f(t);0)^\tau \in C^1([0,T];\mathcal{H}).$$

By Theorem 2.1, Cauchy problem (3.7) has a unique strong solution. In other words, there exists a function z such that $z \in C^1([0,T];\mathcal{H})$, $z(t) \in \mathcal{D}\oplus\mathcal{D}$ for all $t \in [0,T]$, $\big(\mathcal{B}+\mathcal{A}+\mathcal{Q}\big)z \in C([0,T];\mathcal{H})$, and $z(t)$ satisfies (3.7) for all $t \in [0,T]$.

Analysis similar to that in the proof of Theorem 3.1 (Step 4) completes the proof of Theorem 3.2 in the case 5a.

Step 4. Let us consider the case 5b. From [3, p. 179, Lemma 2.16] it follows that the relative bound of the operator $A(t)$, with respect to the operator $B_1(t)$, is equal to 0. This means that for any fix $\varepsilon > 0$ there exists a constant $b_\varepsilon \geqslant 0$ such that

$$\|A(t)u\|_H \leqslant \varepsilon\|B_1(t)u\|_H + b_\varepsilon\|u\|_H$$

for all $u \in \mathcal{D}$ and $t \in [0,T]$.

Hence for each $z = (v;w)^\tau \in \mathcal{D}(\mathcal{B}(t)) = \mathcal{D} \oplus \mathcal{D}$ we have

$$\begin{aligned}\|(\mathcal{A}(t) - \max\{0,\omega_A\})z\|_{\mathcal{H}} &\leqslant \|A(t)v\|_H + \max\{0,\omega_A\}\|z\|_{\mathcal{H}} \\ &\leqslant \varepsilon\|B_1(t)v\|_H + b_\varepsilon\|v\|_H + \max\{0,\omega_A\}\|z\|_{\mathcal{H}} \leqslant \varepsilon\|\mathcal{B}(t)z\|_{\mathcal{H}} + (b_\varepsilon + \max\{0,\omega_A\})\|z\|_{\mathcal{H}} \\ &\leqslant \varepsilon\|(\mathcal{B}(t) - \omega_0)z\|_{\mathcal{H}} + (b_\varepsilon + \varepsilon|\omega_0| + \max\{0,\omega_A\})\|z\|_{\mathcal{H}}\end{aligned}$$

for all $t \in [0,T]$.

Because we can choose $\varepsilon < 1$, from the above it follows that $(\mathcal{B}(t) - \omega_0) + (\mathcal{A}(t) - \max\{0,\omega_A\})$ is a generator of a contractive C_0-semigroup (see [3, p. 173, Theorem 2.7]). Hence $\mathcal{B}(t) + \mathcal{A}(t) \in \mathcal{J}(1, \omega_0 + \max\{0,\omega_A\})$. Remark 1 implies that the family of operators $\mathcal{B}(t) + \mathcal{A}(t)$ is stable on $[0,T]$.

The rest of the proof in the case 5b runs as in Step 3.

Step 5. Let us consider the case 5c. Rewrite system (3.3) as a Cauchy problem

$$\frac{dz}{dt} = \mathcal{C}(t)z + \mathcal{Q}(t)z + \int_0^t \mathcal{G}(t,s)z(s)\,ds + \mathcal{F}(t), \quad z(0) = z^0$$

in the Hilbert space $\mathcal{H}$, where we use the notations

$$\mathcal{C}(t) := \begin{pmatrix} A(t) & B_0(t) \\ B_0(t) & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{C}(t)) = \mathcal{D} \oplus \mathcal{D}, \ \overline{\mathcal{D}(\mathcal{C}(t))} = \mathcal{H},$$

$$\mathcal{Q}(t) := \begin{pmatrix} 0 & Q_0(t)B_0^{-1}(t) \\ 0 & B_0'(t)B_0^{-1}(t) \end{pmatrix}, \quad \mathcal{Q}(t) \in \mathcal{L}(\mathcal{H}),$$

$$\mathcal{G}(t,s) := \begin{pmatrix} 0 & G(t,s)B_1^{-1}(s) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{G}(t,s)) = H \oplus \mathcal{D} \supset \mathcal{D}(\mathcal{C}(t)),$$

$$\mathcal{F}(t) := (f(t);0)^\tau, \quad z^0 := (u^1; B_0(0)u^0)^\tau.$$

If we show that $\mathcal{C}(t) \in \mathcal{J}(1, \max\{0,\omega_A\})$ for all $t \in [0,T]$ then the rest of the proof will run as in Step 3. Let us prove this fact.

From (3.9) (under the condition $\lambda_0 = 0$) it follows that densely defined operator $\mathcal{C}(t)$ is closed. For each $z = (v;w)^\tau \in \mathcal{D}(\mathcal{C}(t)) = \mathcal{D} \oplus \mathcal{D}$ we have

$$\operatorname{Re}\big(\mathcal{C}(t)z, z\big)_{\mathcal{H}} = \operatorname{Re}\big(A(t)v, v\big)_H \leqslant \omega_A\|v\|_H^2 \leqslant \max\{0,\omega_A\}\|z\|_{\mathcal{H}}^2$$

for all $t \in [0,T]$. Hence $\mathcal{C}(t) - \max\{0,\omega_A\}$ is dissipative. To prove the operator $\mathcal{C}(t) - \max\{0,\omega_A\}$ is maximal dissipative we need to show that $\mathcal{C}(t)$ has a resolvent for some $\lambda > \max\{0,\omega_A\}$ and for all $t \in [0,T]$.

Let $\lambda \geqslant \max\{0,\omega_A\}$. We have

$$\begin{aligned}&\|\big(I + \lambda B_0^{-1}(t)(A(t) - \lambda)B_0^{-1}(t)\big)w_1\|_H\|w_1\|_H \\ &\qquad\geqslant \operatorname{Re}\big(\big(I + \lambda B_0^{-1}(t)(A(t) - \lambda)B_0^{-1}(t)\big)w_1, w_1\big)_H \\ &\qquad= \|w_1\|_H^2 - \lambda\operatorname{Re}\big(\big(A(t)B_0^{-1}(t)w_1, B_0^{-1}(t)w_1\big)_H + \lambda^2\|B_0^{-1}(t)w_1\|_H^2 \\ &\qquad\geqslant \|w_1\|_H^2 + \lambda(\lambda - \omega_A)\|B_0^{-1}(t)w_1\|_H^2 \geqslant \|w_1\|_H^2,\end{aligned}$$

$$
\begin{aligned}
&\|\big(I+\lambda B_0^{-1}(t)(A(t)-\lambda)B_0^{-1}(t)\big)^*w_2\|_H\|w_2\|_H\\
&\qquad\geqslant \mathrm{Re}\Big(\big(I+\lambda B_0^{-1}(t)(A(t)-\lambda)B_0^{-1}(t)\big)^*w_2,w_2\Big)_H\\
&\qquad= \|w_2\|_H^2+\lambda\mathrm{Re}\Big(\big((A(t)-\lambda)B_0^{-1}(t)\big)^*\big(B_0^{-1}(t)\big)^*w_2,w_2\Big)_H\\
&\qquad= \|w_2\|_H^2-\lambda\mathrm{Re}\big(B_0^{-1}(t)w_2,(A(t)-\lambda)B_0^{-1}(t)w_2\big)_H\\
&\qquad= \|w_2\|_H^2-\lambda\mathrm{Re}\big(\big(A(t)B_0^{-1}(t)w_2,B_0^{-1}(t)w_2\big)_H+\lambda^2\|B_0^{-1}(t)w_2\|_H^2\\
&\qquad\geqslant \|w_2\|_H^2+\lambda(\lambda-\omega_A)\|B_0^{-1}(t)w_2\|_H^2\geqslant\|w_2\|_H^2
\end{aligned}
$$

for all w_1, $w_2\in H$ and $t\in[0,T]$. Here we used the property $(B_0^{-1}(t))^*=(B_0^*(t))^{-1}$ (see [8, p. 214, Theorem 5.30]). From the above it follows that there exists an operator

$$R(t):=\big(I+\lambda B_0^{-1}(t)(A(t)-\lambda)B_0^{-1}(t)\big)^{-1}\in\mathcal{L}(H)$$

for all $t\in[0,T]$.

Let $\lambda\geqslant\max\{0,\omega_A\}$ and $t\in[0,T]$. Define

$$\mathcal{N}_\lambda(t):=\begin{pmatrix}\lambda B_0^{-1}(t)R(t)B_0^{-1}(t) & B_0^{-1}(t)R(t)\\ R(t)B_0^{-1}(t) & -R(t)B_0^{-1}(t)(A(t)-\lambda)B_0^{-1}(t)\end{pmatrix}\in\mathcal{L}(\mathcal{H}).$$

Let us show that $R(t)\mathcal{D}\subset\mathcal{D}$. Let $w_1\in\mathcal{D}$ and $R(t)w_1=w_2$. Then $w_1=R^{-1}(t)w_2=w_2+\lambda B_0^{-1}(t)(A(t)-\lambda)B_0^{-1}(t)w_2$. Hence $w_2=w_1-\lambda B_0^{-1}(t)(A(t)-\lambda)B_0^{-1}(t)w_2\in\mathcal{D}$. From the above it follows that $\mathcal{N}_\lambda(t)(\mathcal{D}\oplus\mathcal{D})\subset(\mathcal{D}\oplus\mathcal{D})$.

Direct calculations show that

$$
\begin{aligned}
&\big(\mathcal{C}(t)-\lambda\big)\mathcal{N}_\lambda(t)z=z\quad\forall\ z=(v;w)^\tau\in\mathcal{H}=H\oplus H,\\
&\mathcal{N}_\lambda(t)\big(\mathcal{C}(t)-\lambda\big)z=z\quad\forall\ z=(v;w)^\tau\in\mathcal{D}(\mathcal{C}(t))=\mathcal{D}\oplus\mathcal{D}.
\end{aligned}
$$

Consequently, $\mathcal{C}(t)$ has a resolvent $\big(\mathcal{C}(t)-\lambda\big)^{-1}=\mathcal{N}_\lambda(t)$ for all $\lambda\geqslant\max\{0,\omega_A\}$ and $t\in[0,T]$. Hence $\mathcal{C}(t)\in\mathcal{J}(1,\max\{0,\omega_A\})$ for all $t\in[0,T]$. □

3.3 Second-order integro-differential equation in a Banach space. The main operator acts on the derivative of a function

Let us consider the Cauchy problem for the integro-differential equation

$$\frac{d^2u}{dt^2}=A(t)\frac{du}{dt}+B(t)u+\int_0^t G(t,s)u(s)\,ds+f(t),\quad u(0)=u^0,\ \ u'(0)=u^1 \tag{3.10}$$

in a Banach space E. For a fixed $t\in[0,T]$, $A(t)$ is assumed to be a closed operator. The domain $\mathcal{D}(A(t))\equiv\mathcal{D}$ is independent of t and is dense in E. The operators $B(t)$ and $G(t,s)$ are relatively bounded with respect to the operator $A(t)$.

Definition 4. We say that a function u is a *strong solution* to Cauchy problem (3.10) on the interval $[0,T]$ if $u\in C^2([0,T];E)$, $u(t)\in\mathcal{D}(B(t))$, $u'(t)\in\mathcal{D}$ for all $t\in[0,T]$, Bu, $Au'\in C([0,T];E)$, and $u(t)$ satisfies (3.10) for all $t\in[0,T]$.

Theorem 3.3. *Let the following conditions be satisfied:*

1) $\mathcal{D}(A(t)) = \mathcal{D}$, $A(t) \in SC^1_{\mathcal{D}}([0,T];E)$ *and is stable on* $[0,T]$ *with the constants* M_A *and* ω_A*;*

2) $BA_0^{-1}(0) \in SC^1_E([0,T];E)$, *where* $\lambda_0 > \omega_A$ *and* $A_0(t) := A(t) - \lambda_0$ *;*

3) $G(t,s)A_0^{-1}(0)$, $\big(G(t,s)A_0^{-1}(0)\big)'_t \in SC_E(T_\Delta;E)$.

Then for any $u^0, u^1 \in \mathcal{D}$, *and* $f \in C^1([0,T];E)$ *Cauchy problem* (3.10) *has a unique strong solution.*

Proof. Step 1. Let us suppose that Cauchy problem (3.10) has a strong solution u (see Definition 4). Set $v(t) := u'(t)$, $w(t) := A_0(0)u(t)$. Definition 4 and the assumptions of the theorem imply that v, $w \in C^1([0,T];E)$. From (3.10) we conclude that the functions v and w satisfy the system of equations

$$\begin{cases} \dfrac{dv}{dt} = A_0(t)v + \lambda_0 v + B(t)A_0^{-1}(0)w + \displaystyle\int_0^t G(t,s)A_0^{-1}(0)w(s)\,ds + f(t), \\ \dfrac{dw}{dt} = A_0(0)v, \qquad v(0) = u^1, \;\; w(0) = A_0(0)u^0. \end{cases} \tag{3.11}$$

We rewrite system of equations (3.11) as the following Cauchy problem

$$\frac{dz}{dt} = \mathcal{A}(t)z + \mathcal{Q}(t)z + \int_0^t \mathcal{G}(t,s)z(s)\,ds + \mathcal{F}(t), \quad z(0) = z^0 \tag{3.12}$$

in the Banach space $\mathcal{E} := E \times E = \{z = (v;w)^\tau \,|\, v,w \in E\}$. The following notation is used in (3.12):

$$\mathcal{A}(t) := \begin{pmatrix} A_0(t) & 0 \\ A_0(0) & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}(t)) = \mathcal{D} \times E, \;\; \overline{\mathcal{D}(\mathcal{A}(t))} = \mathcal{E},$$

$$\mathcal{Q}(t) := \begin{pmatrix} \lambda_0 & B(t)A_0^{-1}(0) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}(t) \in \mathcal{L}(\mathcal{E}),$$

$$\mathcal{G}(t,s) := \begin{pmatrix} 0 & G(t,s)A_0^{-1}(0) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{G}(t,s) \in \mathcal{L}(\mathcal{E}),$$

$$\mathcal{F}(t) := (f(t);0)^\tau, \quad z^0 := (u^1; A_0(0)u^0)^\tau.$$

The proof of the theorem is based on applying Theorem 2.1 to Cauchy problem (3.12).

Step 2. Let us proof that the family of operators $\mathcal{A}(t)$ is stable on $[0,T]$. Note that the following factorization of the operator $\mathcal{A}(t)$ takes place:

$$\mathcal{A}(t) = \mathcal{N}(t) \cdot \begin{pmatrix} A_0(t) & 0 \\ 0 & 0 \end{pmatrix} \cdot \mathcal{N}^{-1}(t), \quad \mathcal{N}(t) := \begin{pmatrix} I & 0 \\ A_0(0)A_0^{-1}(t) & I \end{pmatrix} \in \mathcal{L}(\mathcal{E}). \tag{3.13}$$

Factorization (3.13) and direct calculations show that

$$\prod_{k=n\searrow 1} \big(\mathcal{A}(t_k) - \lambda\big)^{-1} = \mathcal{N}(t_n) \cdot \begin{pmatrix} R_\lambda & 0 \\ S_\lambda & (-1)^n\lambda^{-n} \end{pmatrix} \cdot \mathcal{N}^{-1}(t_1), \tag{3.14}$$

$$R_\lambda := \prod_{k=n\searrow 1} \big(A_0(t_k) - \lambda\big)^{-1}, \quad S_\lambda := \sum_{l=2}^n \frac{(-1)^{n-l}}{\lambda^{n-l+1}} C_l \prod_{k=l-1\searrow 1} \big(A_0(t_k) - \lambda\big)^{-1},$$

$$C_l := A_0(0)A_0^{-1}(t_l) - A_0(0)A_0^{-1}(t_{l-1}) \;\; (l = \overline{2,n})$$

for all $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant T$.

By the assumption, $A_0(t)u$ for each $u \in \mathcal{D}$ is strongly continuously differentiable on $[0,T]$. Moreover, it follows that $A_0^{-1}(t) \in \mathcal{L}(E)$ for all $t \in [0,T]$. Consequently (see [10, p. 220, Lemma 1.5]), $A_0(s)A_0^{-1}(t)$ is norm-continuous in t, s on T_Δ. Moreover, there exist constants L, $L_0 > 0$ such that

$$\begin{gathered} \|A_0(0)A_0^{-1}(t) - A_0(0)A_0^{-1}(s)\|_{\mathcal{L}(E)} \leqslant L|t-s| \quad \forall\, t,s \in [0,T], \\ \|A_0(0)A_0^{-1}(t)\|_{\mathcal{L}(E)} \leqslant L_0 \quad \forall\, t \in [0,T]. \end{gathered} \tag{3.15}$$

Note that the family of operators $A_0(t)$ is stable on $[0,T]$ with the stability constants M_A and $\omega_A - \lambda_0$. From this and (3.15) we obtain

$$\begin{aligned} &\|\mathcal{N}(t)\|_{\mathcal{L}(\mathcal{E})}, \|\mathcal{N}^{-1}(t)\|_{\mathcal{L}(\mathcal{E})} \leqslant \max\{\sqrt{1+2L_0^2}, \sqrt{2}\} \quad \forall\, t \in [0,T], \\ &\|R_\lambda\|_{\mathcal{L}(E)} = \Big\| \prod_{k=n\searrow 1} \big(A_0(t_k) - \lambda\big)^{-1} \Big\|_{\mathcal{L}(E)} \leqslant \frac{M_A}{(\lambda - (\omega_A - \lambda_0))^n} \quad \forall\, \lambda \geqslant 0 \quad (\lambda_0 > \omega_A), \\ &\|S_\lambda\|_{\mathcal{L}(E)} = \Big\| \sum_{l=2}^{n} \frac{(-1)^{n-l}}{\lambda^{n-l+1}}\, C_l \prod_{k=l-1\searrow 1} \big(A_0(t_k) - \lambda\big)^{-1} \Big\|_{\mathcal{L}(E)} \\ &\qquad \leqslant \sum_{l=2}^{n} \frac{L(t_l - t_{l-1})M_A}{\lambda^{n-l+1}(\lambda - (\omega_A - \lambda_0))^{l-1}} \leqslant \frac{LM_A}{\lambda^n} \sum_{l=2}^{n} (t_l - t_{l-1}) \leqslant \frac{LM_A T}{\lambda^n} \quad \forall\, \lambda > 0. \end{aligned}$$

From this and (3.14) it follows that

$$\begin{aligned} &\Big\| \prod_{k=n\searrow 1} \big(\mathcal{A}(t_k) - \lambda\big)^{-1} \Big\|_{\mathcal{L}(\mathcal{E})} \\ &\qquad \leqslant \max\{1+2L_0^2, 2\} \cdot \max\Big\{ \sqrt{\|R_\lambda\|_{\mathcal{L}(E)}^2 + 2\|S_\lambda\|_{\mathcal{L}(E)}^2}, \sqrt{2}\lambda^{-n} \Big\} \\ &\qquad \leqslant \max\{1+2L_0^2, 2\} \cdot \max\Big\{ \sqrt{\frac{M_A^2}{(\lambda - (\omega_A - \lambda_0))^{2n}} + \frac{2L^2M_A^2T^2}{\lambda^{2n}}}, \frac{\sqrt{2}}{\lambda^n} \Big\} \\ &\qquad \leqslant \frac{\max\{1+2L_0^2, 2\} \cdot \max\big\{ M_A\sqrt{1+2L^2T^2}, \sqrt{2} \big\}}{\lambda^n} \end{aligned}$$

for all $\lambda > 0$, $n \in \mathbb{N}$, and $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant T$. Hence the family of operators $\mathcal{A}(t)$ is stable on $[0,T]$ (see Definition 1).

Step 3. The assumptions of the theorem imply that $\mathcal{Q}(t) \in \mathcal{L}(\mathcal{E})$, $\|\mathcal{Q}(t)\|_{\mathcal{L}(\mathcal{E})} \leqslant K < +\infty$ for all $t \in [0,T]$. By Theorem 1.1 the family of operators $\mathcal{A}(t) + \mathcal{Q}(t)$ is stable on $[0,T]$. Moreover, it follows that $\mathcal{A} + \mathcal{Q} \in SC^1_{\mathcal{D}\times E}([0,T];\mathcal{E})$.

Let $\mathcal{C} := \operatorname{diag}\big(A_0(0), I\big)$, $\mathcal{D}(\mathcal{C}) = \mathcal{D} \times E$, then $\mathcal{C}^{-1} \in \mathcal{L}(\mathcal{E})$. The assumptions of the theorem imply that $\mathcal{G}(t,s)\mathcal{C}^{-1}$, $\big(\mathcal{G}(t,s)\mathcal{C}^{-1}\big)'_t \in SC_{\mathcal{E}}(T_\Delta;\mathcal{E})$.

The assumptions on the initial conditions and the function f imply that

$$z^0 = (u^1; A_0(0)u^0)^\tau \in \mathcal{D} \times E = \mathcal{D}(\mathcal{A}(t) + \mathcal{Q}(t)), \quad \mathcal{F}(t) = (f(t); 0)^\tau \in C^1([0,T];\mathcal{E}).$$

By Theorem 2.1, Cauchy problem (3.12) has a unique strong solution. In other words, there exists a function z such that $z \in C^1([0,T];\mathcal{E})$, $z(t) \in \mathcal{D} \times E$ for all $t \in [0,T]$, $\big(\mathcal{A} + \mathcal{Q}\big)z \in C([0,T];\mathcal{E})$, and $z(t)$ satisfies (3.12) for all $t \in [0,T]$.

Step 4. Let $z(t) = (v(t); w(t))^\tau$ be the unique strong solution of Cauchy problem (3.12). Define $u(t) := A_0^{-1}(0)w(t)$.

From $w \in C([0,T];E)$ it follows that $u(t) \in \mathcal{D} = \mathcal{D}(A_0(t)) \subset \mathcal{D}(B(t))$ for all $t \in [0,T]$, $Bu = BA_0^{-1}(0)w \in C([0,T];E)$.

The second equation in system (3.11) implies that

$$u'(t) = A_0^{-1}(t)w'(t) = v(t) \in C^1([0,T];E).$$

Hence $u \in C^2([0,T];E)$ and $u'(t) \in \mathcal{D}$ for all $t \in [0,T]$, $Au' = \big(A_0 + \lambda_0\big)u' \in C([0,T];E)$.

The first equation in system (3.11) implies that

$$u''(t) = v'(t) = A(t)u'(t) + B(t)u(t) + \int_0^t G(t,s)u(s)\,ds + f(t),$$

i.e. the function u satisfies equation (3.10). It can be easily seen that the function u satisfies also the initial condition in (3.10). By Definition 4, the function u is a strong solution to Cauchy problem (3.10). □

3.4 Second-order integro-differential equation in a Banach space. Parabolic case

Let us consider the Cauchy problem for the integro-differential equation

$$\frac{d^2u}{dt^2} = A\frac{du}{dt} + B_0^2 u + \int_0^t G(t,s)u(s)\,ds + f(t), \quad u(0) = u^0, \;\; u'(0) = u^1 \tag{3.16}$$

in a Banach space E. A is assumed to be a closed and densely defined operator, $\mathcal{D}(A) \subset \mathcal{D}(B_0)$, $A^{-1}, B_0^{-1} \in \mathcal{L}(E)$. The operator $G(t,s)$ is relatively bounded with respect to the operator B_0.

Definition 5. We say that a function u is a *strong solution* to Cauchy problem (3.16) on the interval $[0,T]$ if $u \in C^2([0,T];E)$, $u(t) \in \mathcal{D}(B_0^2)$, $u'(t) \in \mathcal{D}(A)$ for all $t \in [0,T]$, B_0^2u, $Au' \in C([0,T];E)$, and $u(t)$ satisfies (3.16) for all $t \in [0,T]$.

Assume that the operator $A^{-1}B_0$ is closable and $C := \overline{A^{-1}B_0} \in \mathcal{L}(E)$. Let us consider the Cauchy problem

$$\frac{d^2u}{dt^2} = A\Big(\frac{du}{dt} + CB_0u\Big) + \int_0^t G(t,s)u(s)\,ds + f(t), \quad u(0) = u^0, \;\; u'(0) = u^1. \tag{3.17}$$

Definition 6. We say that a function u is a *strong solution* to Cauchy problem (3.17) on the interval $[0,T]$ if $u \in C^2([0,T];E)$, $u(t), u'(t) \in \mathcal{D}(B_0)$, $u'(t) + CB_0u(t) \in \mathcal{D}(A)$ for all $t \in [0,T]$, B_0u, B_0u', $A\big(u' + CB_0u\big) \in C([0,T];E)$, and $u(t)$ satisfies (3.17) for all $t \in [0,T]$.

Remark 5. If the function u is a strong solution to Cauchy problem (3.16) (see Definition 5), then u is a strong solution to Cauchy problem (3.17) (see Definition 6). The inverse is not true.

However, if u is a strong solution to Cauchy problem (3.17) and $u(t) \in \mathcal{D}(B_0^2)$ for all $t \in [0,T]$, $B_0^2u \in C([0,T];E)$, then $u'(t) \in \mathcal{D}(A)$, $Au' \in C([0,T];E)$, and u is a strong solution to Cauchy problem (3.16).

Theorem 3.4. *Let the following conditions be satisfied:*

1) A, B_0 are closed operators, $A^{-1}, B_0^{-1} \in \mathcal{L}(E)$;

2) $B_0A^{-1}, C = \overline{A^{-1}B_0} \in \mathfrak{S}_\infty(E)$;

3) $A, -B_0C$ are generators of holomorphic semigroups;

4) $G(t,s)B_0^{-1}$, $\big(G(t,s)B_0^{-1}\big)'_t \in SC_E(T_\Delta; E)$.

Then for any $u^0, u^1 \in \mathcal{D}(B_0)$, $u^1 + CB_0u^0 \in \mathcal{D}(A)$ (in particular $u^0 \in \mathcal{D}(B_0^2)$, $u^1 \in \mathcal{D}(A)$), and $f \in C^\alpha([0,T];E)$ $(0 < \alpha \leqslant 1)$ (see Remark 4) Cauchy problem (3.17) *has a unique strong solution.*

Proof. Step 1. Let us suppose that Cauchy problem (3.17) has a strong solution u (see Definition 6). Set $v(t) := u'(t)$, $w(t) := B_0u(t)$. Definition 6 implies that $v, w \in C^1([0,T];E)$. From (3.17) we conclude that the functions v and w satisfy the system of equations

$$\begin{cases} \dfrac{dv}{dt} = A(v + Cw) + \displaystyle\int_0^t G(t,s)B_0^{-1}w(s)\,ds + f(t), \\ \dfrac{dw}{dt} = B_0v, \qquad v(0) = u^1, \quad w(0) = B_0u^0. \end{cases} \tag{3.18}$$

We rewrite system of equations (3.18) as the following Cauchy problem

$$\frac{dz}{dt} = \mathcal{A}z + \int_0^t \mathcal{G}(t,s)z(s)\,ds + \mathcal{F}(t), \quad z(0) = z^0 \tag{3.19}$$

in the Banach space $\mathcal{E} := E \times E = \{z = (v;w)^\tau \,|\, v, w \in E\}$. The following notation is used in (3.19):

$$\mathcal{A}z := \begin{pmatrix} A(v + Cw) \\ B_0v \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) := \{z = (v;w)^\tau \,|\, v \in \mathcal{D}(B_0), \ \ v + Cw \in \mathcal{D}(A)\},$$

$$\mathcal{G}(t,s) := \begin{pmatrix} 0 & G(t,s)B_0^{-1} \\ 0 & 0 \end{pmatrix}, \ \ \mathcal{G}(t,s) \in \mathcal{L}(\mathcal{E}), \ \ \mathcal{F}(t) := (f(t);0)^\tau, \ \ z^0 := (u^1; B_0u^0)^\tau.$$

Step 2. Let us first show that $\mathcal{A}$ is a densely defined and closed operator. Set

$$\mathcal{A}_0 := \operatorname{diag}\big(A, -B_0C\big), \quad \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \times \mathcal{D}(-B_0C). \tag{3.20}$$

By the assumption, A, $-B_0C$ are the generators of holomorphic semigroups in E. Hence $\mathcal{A}_0$ is a generator of a holomorphic semigroup in $\mathcal{E}$. Consequently, $\mathcal{A}_0$ is a densely defined and closed operator.

Define

$$\mathcal{B} := (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_0(\mathcal{I} + \mathcal{S}_2), \quad \mathcal{S}_1 := \begin{pmatrix} 0 & 0 \\ B_0A^{-1} & 0 \end{pmatrix}, \quad \mathcal{S}_2 := \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}. \tag{3.21}$$

Since $\mathcal{A}_0$ is a densely defined operator and $(\mathcal{I} + \mathcal{S}_2)^{-1} = (\mathcal{I} - \mathcal{S}_2) \in \mathcal{L}(\mathcal{E})$, the operator $\mathcal{B}$ is densely defined. Since $(\mathcal{I} + \mathcal{S}_1)$, $(\mathcal{I} + \mathcal{S}_1)^{-1} = (\mathcal{I} - \mathcal{S}_1) \in \mathcal{L}(\mathcal{E})$ and the operator $\mathcal{A}_0$ is closed, the operator $\mathcal{B}$ with the domain

$$\mathcal{D}(\mathcal{B}) = \{z = (v;w)^\tau \,|\, (\mathcal{I} + \mathcal{S}_2)z \in \mathcal{D}(\mathcal{A}_0)\} = \{v + Cw \in \mathcal{D}(A), \ w \in \mathcal{D}(-B_0C)\} =$$
$$= \{v + Cw \in \mathcal{D}(A) \subset \mathcal{D}(B_0), \ Cw \in \mathcal{D}(B_0)\} = \{v \in \mathcal{D}(B_0), \ v + Cw \in \mathcal{D}(A)\} = \mathcal{D}(\mathcal{A})$$

is closed. It is easy to check also that $\mathcal{B}z = \mathcal{A}z$ for all $z \in \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A})$.

From the above it follows that the operator $\mathcal{A}$ is densely defined, closed and has the Schur-Frobenius form: $\mathcal{A} = (\mathcal{I} + \mathcal{S}_1)\mathcal{A}_0(\mathcal{I} + \mathcal{S}_2)$.

Step 3. Taking into account (3.20), (3.21) we carry out the substitution $x(t) := (\mathcal{I} + \mathcal{S}_2)z(t)$ in (3.19). The function x satisfies the Cauchy problem

$$\frac{dx}{dt} = (\mathcal{I} + \mathcal{S}_2)(\mathcal{I} + \mathcal{S}_1)\mathcal{A}_0 x + \int_0^t (\mathcal{I} + \mathcal{S}_2)\mathcal{G}(t,s)(\mathcal{I} - \mathcal{S}_2)x(s)\,ds + (\mathcal{I} + \mathcal{S}_2)\mathcal{F}(t),$$
$$x(0) = (\mathcal{I} + \mathcal{S}_2)z^0. \tag{3.22}$$

The assumptions of the theorem imply that $(\mathcal{I} + \mathcal{S}_2)(\mathcal{I} + \mathcal{S}_1)\mathcal{A}_0 =: (\mathcal{I} + \mathcal{S})\mathcal{A}_0$, where $\mathcal{S} \in \mathfrak{S}_\infty(\mathcal{E})$. From [3] (see [3, p. 180, Corollary 2.17], see also [2]) we conclude that $(\mathcal{I} + \mathcal{S}_2)(\mathcal{I} + \mathcal{S}_1)\mathcal{A}_0$ is a generator of a holomorphic semigroup in $\mathcal{E}$.

The assumptions on the initial conditions imply that $z^0 = (u^1; B_0 u^0)^\tau \in \mathcal{D}(\mathcal{A})$. Hence $x(0) = (\mathcal{I} + \mathcal{S}_2)z^0 \in \mathcal{D}(\mathcal{A}_0)$.

By the assumption $f \in C^\alpha([0,T];E)$ $(0 < \alpha \leqslant 1)$. From this and the inequality

$$\|(\mathcal{I} + \mathcal{S}_2)\mathcal{F}(t) - (\mathcal{I} + \mathcal{S}_2)\mathcal{F}(s))\|_{\mathcal{E}} \leqslant \|\mathcal{I} + \mathcal{S}_2\|_{\mathcal{L}(\mathcal{E})}\|f(t) - f(s)\|_E, \quad 0 \leqslant s,t \leqslant T,$$

it follows that $(\mathcal{I} + \mathcal{S}_2)\mathcal{F} \in C^\alpha([0,T];\mathcal{E})$ $(0 < \alpha \leqslant 1)$.

By Theorem 2.1 (see Remark 4), Cauchy problem (3.22) has a unique strong solution. In other words, there exists a function x such that $x \in C^1([0,T];\mathcal{E})$, $x(t) \in \mathcal{D}(\mathcal{A}_0)$ for all $t \in [0,T]$, $\mathcal{A}_0 x \in C([0,T];\mathcal{E})$, and $x(t)$ satisfies (3.22) for all $t \in [0,T]$.

Hence Cauchy problem (3.19) has a unique strong solution. In other words, there exists a function z such that $z \in C^1([0,T];\mathcal{E})$, $z(t) \in \mathcal{D}(\mathcal{A})$ for all $t \in [0,T]$, $\mathcal{A}z \in C([0,T];\mathcal{E})$, and $z(t)$ satisfies (3.19) for all $t \in [0,T]$.

Step 4. Let $z(t) = (v(t); w(t))^\tau$ be the unique strong solution of Cauchy problem (3.19). Then we have

$$\begin{aligned} &v(t) \in C^1([0,T];E), \quad v(t) \in \mathcal{D}(B_0), \quad B_0 v(t) \in C([0,T];E), \\ &w(t) \in C^1([0,T];E), \quad v(t) + Cw(t) \in \mathcal{D}(A), \quad A(v(t) + Cw(t)) \in C([0,T];E). \end{aligned} \tag{3.23}$$

Define $u(t) := B_0^{-1}w(t)$. From the above it follows that $u(t) \in \mathcal{D}(B_0)$, $B_0 u = w \in C([0,T];E)$. From (3.18) it follows that $u'(t) = B_0^{-1}w'(t) = v(t)$. From (3.23) we obtain $u \in C^2([0,T];E)$, $u'(t) \in \mathcal{D}(B_0), u'(t) + CB_0 u(t) \in \mathcal{D}(A)$ for all $t \in [0,T]$, $B_0 u', A(u' + CB_0 u) \in C([0,T];E)$. The first equation in (3.18) implies

$$u''(t) = v'(t) = A\big(u'(t) + CB_0\big)u(t) + \int_0^t G(t,s)u(s)\,ds + f(t),$$

i.e. the function u satisfies equation (3.17). It is easily seen that the function u satisfies also the initial condition in (3.17). By Definition 6, the function u is a strong solution to Cauchy problem (3.17). □

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Dmitry Alexandrovich Zakora
Department of Mathematics and Informatics
Crimean Federal University
4 Academician Vernadsky Ave
295007 Simferopol, Russia
and
Voronezh State University
1 University Sq
394006 Voronezh, Russia
E-mails: dmitry.zkr@gmail.com, dmitry_@crimea.edu

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