ISSN 2077–9879

Eurasian Mathematical Journal

2016, Volume 7, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

https://scholar.google.com/citations?user=Zay4fxkAAAAJ&hl=ru&authuser=1

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 2 (2016), 100 – 105

CONSTRUCTION OF GREEN'S FUNCTION OF THE NEUMANN PROBLEM IN A BALL

M.A. Sadybekov, B.T. Torebek, B.Kh. Turmetov

Communicated by K.N. Ospanov

Key words: Green's function, Neumann problem, Poisson equation, Laplace operator, Neumann kernel.

AMS Mathematics Subject Classification: 35J05, 35J08, 35J25, 65N80.

Abstract. Representation of the Green's function of the classical Neumann problem for the Poisson equation in the unit ball of arbitrary dimension is given. In constructing this function we use the representation of the fundamental solution of the Laplace equation in the form of a series. It is shown that Green's function can be represented in terms of elementary functions and its explicit form can be written out. An explicit form of the Neumann kernel was constructed for $n = 4$ and $n = 5$.

1 Introduction

The Dirichlet problem for the Poisson equation

$$-\Delta u(x) \equiv -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u(x) = f(x),\ x \in D; \quad u = \varphi,\ x \in \partial D, \tag{1.1}$$

in the domain $D \subset \mathbb{R}^n,\ n \geq 2$ with the regular boundary ∂D is a classic and well-investigated problem. The solution of problem (1.1) exists, is unique and is represented by Green's function $G_D\left(x, y\right)$ in the form (see [1], p. 277):

$$u\left(x\right) = \int_D G_D\left(x, y\right) f\left(y\right) dy - \int_{\partial D} \frac{\partial G_D}{\partial n_y}\left(x, y\right) \varphi\left(y\right) dS_y. \tag{1.2}$$

Here and in the sequel $\frac{\partial}{\partial n}$ is the derivative in the direction of the outer normal to ∂D.

In the case of the unit ball $D = \{x \in \mathbb{R}^n : |x| < 1\}$ is a unit ball, the Green's function of the Dirichlet problem can be constructed by reflection method and has the form:

$$G_D\left(x, y\right) = \frac{1}{\omega_n}\left[\varepsilon\left(x - y\right) - \varepsilon\left(x\left|y\right| - \frac{y}{\left|y\right|}\right)\right], \tag{1.3}$$

where $\omega_n = 2\pi^{\frac{n}{2}} / \Gamma\left({}^{n}\!/_{2}\right)$ is the area of the unit sphere in $\mathbb{R}^n$, and $\varepsilon\left(x - y\right)$ is the fundamental solution of the Laplace equation:

$$\varepsilon\left(x - y\right) = \begin{cases} -\ln\left|x - y\right|, & n = 2; \\ \frac{1}{n-2}\left|x - y\right|^{2-n}, & n \geq 3. \end{cases} \tag{1.4}$$

Along with the Dirichlet problem, the Neumann problem for the Poisson equation is classic and well-investigated:

$$-\Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u(x) = f(x), \ x \in D; \quad \frac{\partial u}{\partial n} = \psi, \ x \in \partial D \tag{1.5}$$

It is well known that the solution of Neumann problem (1.5) is not unique, however it is unique up to a constant summand. The fulfillment of the following condition

$$\int_D f(y)\, dy + \int_{\partial D} \psi(y)\, dS_y = 0. \tag{1.6}$$

is necessary and sufficient for the existence of a solution to this problem.

If a solution to problem (1.5) exists then this solution can be represented in the integral form by means of Green's function of the Neumann problem $G_N(x, y)$ by the formula similar to representation (1.2) (see.[1], p. 280):

$$u(x) = \int_D G_N(x, y) f(y)\, dy + \int_{\partial D} G_N(x, y) \psi(y)\, dS_y + Const. \tag{1.7}$$

In the mathematical literature, it is recognized that finding of Green's function of the Neumann problem requires a rather complicated construction [2], [3], [4], [5].

Green's function of Neumann problem (1.5) is understood as a function (see [2], p. 286) having the representation

$$G_N(x, y) = \frac{1}{\omega_n} \left[\varepsilon(x - y) + g(x, y)\right], \tag{1.8}$$

where $g(x, y)$ is a harmonic function in the domain D. Also the boundary condition:

$$\frac{\partial G_N}{\partial n_y}(x, y) = -\frac{1}{\omega_n}, y \in \partial D, \tag{1.9}$$

must be held.

If such Green's function $G_N(x, y)$ exists then from (1.6) and (1.9) it easily follows that function (1.7) satisfies all the conditions of problem (1.5).

Various construction methods of Green's function of the Dirichlet problem (1.1) exist. Green's function was constructed in the explicit form for many types of the domain D. But for the Neumann problem (1.5) in the multidimensional spaces the construction of the Green's function is an open problem. Here we have only samples for the simplest domains - halfspaces, quarters of the space, a semicircle etc. For such domains, the Neumann problem is an outer boundary value problem and therefore is well-posed without fulfilment of solvability conditions of the form (1.6).

For the unit ball in $\mathbb{R}^n$ Green's function of the Neumann problem has been constructed in the explicit form only for the cases $n = 2$ and $n = 3$:

$$G_N(x, y) = \frac{1}{2\pi} \left[-\ln|x - y| - \ln\left|x\,|y| - \frac{y}{|y|}\right|\right], \ n = 2;$$

$$G_N(x,y)=\frac{1}{4\pi}\left[\frac{1}{|x-y|}+\frac{1}{\left|x\,|y|-\frac{y}{|y|}\right|}-\ln\left|1-(x,y)+\left|x\,|y|-\frac{y}{|y|}\right|\right|\right],n=3;\qquad(1.10)$$

where $(x,y)=x_1y_1+\ldots+x_ny_n$ is the scalar product of the vectors x and y in $\mathbb{R}^n$.

Note that the interest to the construction of Green's functions of classical problems in the explicit form was lately renewed. Green's functions of the classical biharmonic problems in the two-dimensional disc were constructed by means of Green's harmonic functions of classical problems in [6]. Similar results for the class of nonhomogeneous biharmonic and triharmonic functions in a sector were obtained in [7], [8]. Green's function of the Dirichlet problem for the polyharmonic equation in a multidimensional ball was constructed in the explicit form in [9], [10]. We also note that work [11] is devoted to the construction of Green's function of the Robin problem in the explicit form for a circle.

In the present paper we the give the representation of Green's function of the Neumann problem for unit ball of arbitrary dimension in the explicit form (in terms of elementary functions). It is shown that function can be represented in terms of elementary functions and its explicit form can be written out in particular cases.

2 Main results

Theorem. *For Green's function of Neumann problem* (1.5) *we have the following representation*

$$G_N(x,y)=\frac{1}{\omega_n}\left[\varepsilon(x-y)+\varepsilon\left(x\,|y|-\frac{y}{|y|}\right)+\varepsilon_1(x,y)\right]+Const,\qquad(2.1)$$

where

$$\varepsilon_1(x,y)=\int_0^1\left[(n-2)\,\varepsilon\left(sx\,|y|-\frac{y}{|y|}\right)-1\right]\frac{ds}{s}.\qquad(2.2)$$

Sketch of the proof.

By using the expansion of the function $(1-2\eta t+\eta^2)^{-\nu}$ in a power series in η, we easily prove

Lemma 1. *For fundamental solution* (1.4) *of the Laplace operator, we have the representation*

$$\varepsilon(x-y)=\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\frac{|x|^k}{|y|^{k+n-2}}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right),|x|<|y|,\qquad(2.3)$$

where $H_k^{(i)}(\cdot)$ is a complete system of homogeneous harmonic polynomials of degree k having the property of orthonormality [12], and *h_k is the number of these polynomials:* $h_k=[(2k+n-2)(k+n-3)!]/[k!(n-2)!]$.

Proof. We will search Green's function $G_N(x,y)$ in form (1.8). We apply the method used (see [2], p. 348), for constructing Green's function of the Neumann three-dimensional problem. We search the function $g(x,y)$ in the form

$$g(x,y)=\sum_{k=0}^{\infty}b_k\,|x|^k\,|y|^k\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right),\qquad(2.4)$$

where b_k are the unknown coefficients. This function satisfies boundary condition (1.9), where $\varepsilon(x-y)$ is defined by (2.3), if:

$$b_k = \frac{1}{2k+n-2} + \frac{n-2}{k\,(2k+n-2)}, k \geq 1.$$

Green's function (2.1) is defined up to an arbitrary constant. Therefore we can arbitrarily choose the coefficient b_0. We choose the coefficient $b_0 = 1/(n-2)$ for uniformity of further calculations.

Substituting the found coefficients in (2.4), we get

$$g(x,y) = \sum_{k=0}^{\infty} \frac{|x|^k |y|^k}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}\left(\frac{x}{|x|}\right) H_k^{(i)}\left(\frac{y}{|y|}\right) +$$

$$+ \sum_{k=1}^{\infty} \frac{(n-2)|x|^k |y|^k}{k\,(2k+n-2)} \sum_{i=1}^{h_k} H_k^{(i)}\left(\frac{x}{|x|}\right) H_k^{(i)}\left(\frac{y}{|y|}\right) \equiv g_1(x,y) + g_2(x,y).$$

The first sum gives the function $g_1(x,y) = \varepsilon\left(x\,|y| - \frac{y}{|y|}\right)$. And for the second sum, using the equality $\frac{1}{k} = \int_0^1 s^{k-1} ds, k \geq 1$, we get

$$g_2(x,y) = (n-2)\int_0^1 \left[\sum_{k=1}^{\infty} \frac{s^k |x|^k |y|^k}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}\left(\frac{x}{|x|}\right) H_k^{(i)}\left(\frac{y}{|y|}\right) - \frac{1}{n-2}\right] \frac{ds}{s},$$

hence, taking into account representation (2.3), we obtain (2.2). □

Thus, the construction of Green function of the Neumann problem is reduced to the calculation of the integral in the right-hand side of (2.2). Since $\left|sx\,|y| - \frac{y}{|y|}\right| = \sqrt{1 - 2(x,y)\,s + |x|^2 |y|^2 s^2}$, the problem of constructing Green's functions for $n \geq 3$ is reduced to the calculation of integrals of the form $\int_0^1 s^{-1}\left(R(s)^{\frac{2-n}{2}} - 1\right) ds$, where $R(s) = 1 - 2(x,y)\,s + |x|^2 |y|^2 s^2$. Integrals of this type can be calculated in terms of elementary functions using the formulas in [13].

Corollary 1. Green's function $G_N(x,y)$ *of Neumann problem* (1.5) *in the unit ball for* $n \geq 3$ *can always be represented as a finite sum of elementary functions.*

3 Concluding remarks

We demonstrate now formula (2.1) works for special cases. By direct calculation for $n = 4$, we have

$$\varepsilon_1(x,y) = \frac{(x,y)}{\sqrt{|x|^2 |y|^2 - (x,y)^2}} tan^{-1} \frac{\sqrt{|x|^2 |y|^2 - (x,y)^2}}{1-(x,y)} - \ln\left|x\,|y| - \frac{y}{|y|}\right|, \qquad (3.1)$$

and for $n = 5$, we obtain:

$$\varepsilon_1(x,y) = \frac{(x,y)}{|x|^2|y|^2-(x,y)^2}\left(\frac{|x|^2|y|^2-(x,y)}{|x|y|-y/|y||}+(x,y)\right)$$

$$+\left|x|y|-\frac{y}{|y|}\right|^{-1}-\ln\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|. \qquad (3.2)$$

In conclusion we note that the solution of Neumann problem (1.5) for the Laplace equation ($f \equiv 0$) is represented in the form:

$$u(x) = \frac{1}{\omega_n}\int_{\partial D} N(x,y)\,\psi(y)\,ds_y,$$

where $N(x,y)$ is the Neumann kernel found with the help of Green's function by the formula $N(x,y) = \omega_n G_N(x,y)$, $x \in D, y \in \partial D$.

The method of constructing the explicit form of the function $N(x,y)$ for the multidimensional unit sphere was considered in the paper of A.V. Bitsadze [5]. It was shown that the Neumann kernel could be expressed in elementary functions. The explicit form of the Neumann kernel at $n = 4$ was given.

From (2.1) and (3.1) for $|y| = 1$, we obtain

$$N(x,y) = |x-y|^{-2} - \ln|x-y| + \frac{(x,y)}{\sqrt{|x|^2-(x,y)^2}} tan^{-1}\frac{\sqrt{|x|^2-(x,y)^2}}{1-(x,y)}.$$

This equality coincides with formula (21) in [5]. To demonstrate the effectiveness of Corollary 1, we present the formula for the Neumann kernel $N(x,y)$ for $n = 5$ obtained from (2.1) and (3.2) for $|y| = 1$:

$$N(x,y) = \frac{2}{3}\frac{1}{|x-y|^3} + \frac{1}{|x-y|}$$

$$+\frac{(x,y)}{|x|^2-(x,y)^2}\left(\frac{|x|^2-(x,y)}{(x,y)}+(x,y)\right) - \ln|1-(x,y)+|x-y||.$$

Acknowledgments

This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 0824/GF4).

References

[1] N.S. Koshlyakov, E.B. Gliner, M.M. Smirnov, *Differential equations of mathematical physics*, Amsterdam, North-Holland Publ., 1964, 712 pp.

[2] G. Bouligand, G. Giraud, P. Delens, *Le problem de la derive oblique en theorie de potential*, P.: Hermann edit., 1935, 78 pp.

[3] O.D. Kellog, *Foundations of potential theory*, N.Y.: Frederick Ungar publ. Comp., 1970, 384 pp.

[4] S.L. Sobolev, *Partial differential equations of mathematical physics*, London. Pergamon Press. 1964, 427 pp.

[5] A.V. Bitsadze, *Singular integral equations of the 1st kind with Neumann kernels*, Differential Equations. 22 (1986), no. 5, 591–595.

[6] H. Begehr, *Biharmonic Green functions*, Le matematiche. 2006; LXI(II), 395–405.

[7] Ying Wang, Liuqing Ye, *Biharmonic Green function and biharmonic Neumann function in a sector*, Complex Variables and Elliptic Equations. 58 (2013), no. 1, 7-22.

[8] Ying Wang, *Tri-harmonic boundary value problems in a sector*, Complex Variables and Elliptic Equations. 59 (2014), no. 5, 732-749

[9] T.Sh. Kalmenov, B.D. Koshanov *Representation for the Green function of the Dirichlet problem for polyharmonic equations in a ball*, Siberian Mathematical Journal. 49 (2008), no. 3, 423-428.

[10] T.Sh. Kalmenov, B.D. Koshanov, M.Y. Nemchenko, *Green function representation in the Dirichlet problem for polyharmonic equations in a ball*, Doklady Mathematics. 78 (2008), no. 1, 528-530.

[11] M.A. Sadybekov, B.T. Torebek, B.Kh. Turmetov, *On an explicit form of the Green function of the Robin problem for the Laplace operator in a circle*, Advances in Pure and Applied Mathematics. 6 (2015), no. 3, 163-172.

[12] H. Bateman, *Higher transcendental functions*, Vol. 2. Mc-Graw-Hill Book Company Inc. 1953, 414 pp.

[13] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press is an imprint of Elsevier. 2007, 1108 pp.

Makhmud Abdysametovich Sadybekov
Institute of Mathematics and Mathematical Modeling
125 Pushkin St
050010 Almaty, Kazakhstan
E-mail: makhmud-s@mail.ru

Berikbol Tillabayuly Torebek
Institute of Mathematics and Mathematical Modeling
125 Pushkin St
050010 Almaty, Kazakhstan
E-mail: torebek@math.kz

Batirkhan Khudaybergenovich Turmetov
A. Yasawi International Kazakh-Turkish University
29 Sattarhanov St
161200 Turkestan, Kazahstan
E-mail: turmetovbh@mail.ru

Received: 07.07.2015