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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

$$https://scholar.google.com/citations?user=Zay4fxkAAAAJ\&hl=ru\&authuser=1$$

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

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CONTINUOUS DEPENDENCE OF SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS ON THE SCALING PARAMETER

L.E. Rossovskii

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Abstract. For a functional differential equation with rescaling, we establish the Gårding-type inequality uniform with respect to the scaling parameter p. This allows us to study the limit behaviour of solutions to the Dirichlet problem as $p \to 1$.

1 Introduction

The paper is devoted to the functional differential equation with contracted arguments (we also use the term *rescaling*)

$$-\sum_{i,j=1}^{n} \left(a_{ij}u_{x_i}(x) + b_{ij}u_{x_i}(x/p) + c_{ij}u_{x_i}(x/q)\right)_{x_j} = f(x) \quad (x \in \Omega), \tag{1.1}$$

supplemented by the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, containing the origin, $a_{ij}, b_{ij}, c_{ij} \in \mathbb{C}$ $(i, j = 1, \dots, n)$, and $f \in L_2(\Omega)$. Assuming one of the scaling parameters $q > 1$ fixed and letting the other one range over some segment $[1, p_0]$, we are interested in the unique solvability of problem (1.1), (1.2) (by a solution, we mean a generalized solution from the Sobolev space $\mathring{H}^1(\Omega)$, understood in the standard way) for all considered values of p and in the behaviour of the solution family $u = u_p$ as $p \to 1$.

Exact conditions for solvability in terms of the coefficients were earlier found in [10, 11] in the case where different scaling parameters were all integer powers, positive or negative, of one parameter. Now, the parameter p changes independently of the parameter q. On the other hand, the dependence of a solution to a functional differential equation on a parameter defining the argument transformation, is studied for the first time.

Equation (1.1) has a famous, albeit distant, prototype, *the pantograph equation*

$$\dot{y} = ay(\lambda t) + by(t)$$

emerging in such diverse areas as astrophysics [1], engineering [9], and biology [3]. For the first time, this equation was studied extensively in [6].

2 Uniform Gårding-type inequality

As a tool in the study of solvability and behaviour of solutions to problem (1.1), (1.2), we make use of the well-known Gårding-type inequality

$$\operatorname{Re}\sum_{i,j=1}^{n}\int_{\Omega}\left(a_{ij}u_{x_i}(x)+b_{ij}u_{x_i}(x/p)+c_{ij}u_{x_i}(x/q)\right)\bar{u}_{x_j}\,dx \geqslant \gamma\||\nabla u|\|^2_{L_2(\Omega)} \tag{2.1}$$

$$(\forall u\in C_0^\infty(\Omega)).$$

We will find in this section necessary and sufficient conditions guaranteeing the fulfillment of inequality (2.1) with the same constant $\gamma>0$ for all $u\in C_0^\infty(\Omega)$ and all $p\in[1,p_0]$ under a suitable value of $p_0>1$. These conditions are expressed explicitly via the coefficients of the equation.

First, we need to note that the Gårding-type inequality is usually written in a more general form,

$$\operatorname{Re}\sum_{i,j=1}^{n}\int_{\Omega}\left(a_{ij}u_{x_i}(x)+b_{ij}u_{x_i}(x/p)+c_{ij}u_{x_i}(x/q)\right)\bar{u}_{x_j}\,dx \geqslant \gamma\|u\|^2_{H^1(\Omega)}-\gamma_1\|u\|^2_{L_2(\Omega)}. \tag{2.2}$$

It is readily seen, however, that for equation (1.1) estimate (2.2) on the class $C_0^\infty(\Omega)$ implies stronger estimate (2.1) on the same class.

Indeed, rewriting (2.2) as

$$\operatorname{Re}\sum_{i,j=1}^{n}\int_{\Omega}\left(a_{ij}u_{x_i}(x)+b_{ij}u_{x_i}(x/p)+c_{ij}u_{x_i}(x/q)\right)\bar{u}_{x_j}\,dx$$

$$\geqslant \gamma\||\nabla u|\|^2_{L_2(\Omega)}-(\gamma_1-\gamma)\|u\|^2_{L_2(\Omega)}$$

and making the change of variables $y=tx$, $t>1$, we obtain the inequality

$$\operatorname{Re}\sum_{i,j=1}^{n}\int_{t\Omega}\left(a_{ij}v_{y_i}(y)+b_{ij}v_{y_i}(y/p)+c_{ij}v_{y_i}(y/q)\right)\bar{v}_{y_j}\,dy$$

$$\geqslant \gamma\||\nabla v|\|^2_{L_2(t\Omega)}-(\gamma_1-\gamma)t^{-2}\|v\|^2_{L_2(t\Omega)} \tag{2.3}$$

now valid for an arbitrary function $v\in C_0^\infty(t\Omega)$. Since Ω contains the origin, it follows from (2.3) that

$$\operatorname{Re}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\left(a_{ij}u_{x_i}(x)+b_{ij}u_{x_i}(x/p)+c_{ij}u_{x_i}(x/q)\right)\bar{u}_{x_j}\,dx \geqslant \gamma\||\nabla u|\|^2_{L_2(\mathbb{R}^n)}$$

for all $u\in C_0^\infty(\mathbb{R}^n)$. The last inequality becomes (2.1) if $\operatorname{supp} u\subset\Omega$.

We recall that in the case of a differential operator (put $b_{ij}=c_{ij}=0$ in (2.2)), inequality (2.2) is a synonym of strong ellipticity [15, 2]. So differential and functional differential operators satisfying the Gårding-type inequality are usually called *strongly elliptic*. The problem of finding an algebraic criterion, i.e. necessary and sufficient conditions for (2.1), expressed directly via the coefficients, is called the *coercivity problem*. The coercivity problem for differential-difference operators was studied in [13] (see also [14]), and for functional differential equations with contracted or expanded arguments in [10, 12]. In addition to such consequences as the Fredholm property, discreteness of the spectrum, and sectorial structure of the Dirichlet problem, the Garding-type inequality is of great importance in the study of the well-known T. Kato conjecture on the square root of an m-accretive operator [4, 7, 8].

If $p > 1$ is a rational power of the parameter q, $p = q^{M/N}$, then the previously known result can be applied. Denoting $q_1 = q^{1/N}$, we write $p = q_1^M$ and $q = q_1^N$, and make use of the following necessary and sufficient conditions for the Gårding-type inequality, established in [10]:

$$\operatorname{Re}\sum_{i,j=1}^{n}\left(a_{ij} + b_{ij}\lambda^M + c_{ij}\lambda^N\right)\xi_i\xi_j > 0 \quad (\lambda \in \mathbb{C}, |\lambda| = q_1^{n/2}, \xi \in \mathbb{R}^n, |\xi| = 1). \tag{2.4}$$

More precisely, the arguments in [10] show that the constant γ in inequality (2.1) can be chosen equal to the infimum of the expression herein (the real part of the symbol of the functional differential operator). Granting this, introducing the notations

$$a(\xi) = \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j, \quad b(\xi) = \sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j, \quad c(\xi) = \sum_{i,j=1}^{n} c_{ij}\xi_i\xi_j,$$

and substituting $q_1^{n/2}e^{i\theta}$ for λ in inequality (2.4), we write the latter in the form

$$\operatorname{Re}\left(a(\xi) + |b(\xi)|q_1^{nM/2}e^{i(M\theta+\varphi(\xi))} + |c(\xi)|q_1^{nN/2}e^{i(N\theta+\psi(\xi))}\right) \geqslant \gamma$$

or

$$\operatorname{Re} a(\xi) + |b(\xi)|q^{nM/(2N)}\cos(M\theta + \varphi(\xi)) + |c(\xi)|q^{n/2}\cos(N\theta + \psi(\xi)) \geqslant \gamma \quad (\theta \in \mathbb{R}, |\xi| = 1),$$

where $\varphi(\xi) = \arg b(\xi)$ and $\psi(\xi) = \arg c(\xi)$. It is convenient to put $\eta = M\theta + \varphi(\xi)$ and $\varepsilon = M/N$ to come to the inequality

$$\operatorname{Re} a(\xi) + |b(\xi)|q^{n\varepsilon/2}\cos\eta + |c(\xi)|q^{n/2}\cos\left(\frac{\eta - \varphi(\xi)}{\varepsilon} + \psi(\xi)\right) \geqslant \gamma \quad (\eta \in \mathbb{R}, |\xi| = 1) \tag{2.5}$$

equivalent to estimate (2.1) for a given rational number $\varepsilon = \log_q p$.

Lemma 2.1. *The algebraic inequality*

$$|b(\xi)| + q^{n/2}|c(\xi)| < \operatorname{Re} a(\xi) \quad (\xi \neq 0). \tag{2.6}$$

is necessary and sufficient for the existence of a number $p_0 > 1$ such that estimate (2.1) *holds with a constant $\gamma > 0$ independent of $p \in [1, p_0]$.*

Proof. Let us prove sufficiency first, assuming that the inequality in (2.6) holds on the unit sphere $|\xi| = 1$. Fixed ξ, one can choose a number $\varepsilon = \varepsilon_\xi > 0$ so small that $\operatorname{Re} a(\xi) - q^{n\varepsilon/2}|b(\xi)| - q^{n/2}|c(\xi)| > 0$ at this point ξ and in some neighborhood of this point as well.

As ξ runs through the sphere, such neighborhoods form an open covering of the sphere. Extracting a finite subcovering, we establish the existence of a number $\varepsilon_0 > 0$ such that

$$\operatorname{Re} a(\xi) - q^{n\varepsilon_0/2}|b(\xi)| - q^{n/2}|c(\xi)| > 0 \quad (|\xi| = 1).$$

Now inequality (2.5) obviously follows for all ε from the interval $(0, \varepsilon_0]$ with γ equal to the infimum of $\operatorname{Re} a(\xi) - q^{n\varepsilon_0/2}|b(\xi)| - q^{n/2}|c(\xi)|$ on the unit sphere. But as we have seen before, the latter is equivalent to (2.1) as soon as ε is rational. Thus, inequality (2.1) is obtained for all values of the parameter $p \in (1, p_0]$ being rational powers of the parameter q, where $p_0 = q^{\varepsilon_0}$. It remains to note that, fixed a function $u \in C_0^\infty(\Omega)$, the expression on the left-hand side in (2.1) depends continuously on the contraction parameter p due to the mean-square continuity of the derivatives u_{x_i}. Therefore, estimate (2.1) extends to the entire segment $[1, p_0]$.

Let us continue the proof of necessity. Now we know that the inequality

$$\operatorname{Re} a(\xi) + |b(\xi)|q^{n\varepsilon/2}\cos\eta + |c(\xi)|q^{n/2}\cos\left(\frac{\eta - \varphi(\xi)}{\varepsilon} + \psi(\xi)\right) \geqslant \gamma \tag{2.7}$$

is fulfilled as η runs through $\mathbb{R}$, ξ ranges within the unit sphere, and ε is any rational number from the interval $(0, \log_q p_0]$. It is also clear that this inequality extends to the whole interval specified. Choose natural numbers k and m to have

$$0 < \frac{\pi(2k+1) - \varphi(\xi)}{\pi(2m+1) - \psi(\xi)} \leqslant \log_q p_0 \quad (|\xi| = 1),$$

and set $\eta = \pi(2k+1)$, $\varepsilon = (\pi(2k+1) - \varphi(\xi))/(\pi(2m+1) - \psi(\xi))$ in inequality (2.7). Then we get

$$\operatorname{Re} a(\xi) - q^{n\varepsilon/2}|b(\xi)| - q^{n/2}|c(\xi)| \geqslant \gamma \quad (|\xi| = 1),$$

and (2.6) even more so. □

3 Solvability of boundary value problem and behaviour of solutions as $p \to 1$

Let $H^1(\Omega)$ denote the Sobolev space of all complex-valued functions in Ω belonging to $L_2(\Omega)$ together with all their first-order generalized derivatives, where the inner product is defined as

$$(u, v)_{H^1(\Omega)} = \int_\Omega \Big(u\bar{v} + \sum_{i=1}^n u_{x_i}\bar{v}_{x_i}\Big)dx,$$

and $\mathring{H}^1(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ of all smooth compactly supported functions in $H^1(\Omega)$,

$$(u, v)_{\mathring{H}^1(\Omega)} = \int_\Omega \sum_{i=1}^n u_{x_i}\bar{v}_{x_i}dx.$$

Introduce the family

$$\mathfrak{a}_p[u, v] = \sum_{i,j=1}^n \int_\Omega (a_{ij}u_{x_i}(x) + b_{ij}u_{x_i}(x/p) + c_{ij}u_{x_i}(x/q))\,\bar{v}_{x_j}\,dx$$

of continuous sesquilinear forms on the space $\mathring{H}^1(\Omega)$. This family depends on the parameter p running through some segment $[1, p_0]$, $p_0 > 1$.

By a generalized solution to problem (1.1), (1.2), we understand any function $u = u_p$ in the space $\mathring{H}^1(\Omega)$, satisfying the identity $\mathfrak{a}_p[u_p, v] = (f, v)_{L_2(\Omega)}$ as v runs over $\mathring{H}^1(\Omega)$.

Theorem 3.1. *Let condition* (2.6) *be satisfied. Then there exists a number* $p_0 > 1$ *such that problem* (1.1), (1.2) *has a unique generalized solution* $u = u_p \in \mathring{H}^1(\Omega)$ *for all values of* $p \in [1, p_0]$ *with* $u_p \to u_1$ *in* $\mathring{H}^1(\Omega)$ *as* $p \to 1$.

Proof. The existence of a unique generalized solution u_p to problem (1.1), (1.2) for all values of $p \in [1, p_0]$ follows from the estimate

$$\operatorname{Re} \mathfrak{a}_p[u, u] \geqslant \gamma \|u\|^2_{\mathring{H}^1(\Omega)} \quad (u \in \mathring{H}^1(\Omega),\ p \in [1, p_0]) \tag{3.1}$$

valid under the hypothesis of the theorem. The corresponding arguments are well-known (see, e.g., [5, 14]), we do not quote them here.

Consider the difference

$$\mathfrak{a}_p[u, v] - \mathfrak{a}_1[u, v] = \sum_{i,j=1}^{n} b_{ij} \int_{\Omega} (u_{x_i}(x/p) - u_{x_i}(x)) v_{x_j}\, dx,$$

where each of the integrals on the right-hand side is majorized by (we omit multiplication by a suitable constant) the expression

$$\|v_{x_j}\|_{L_2(\Omega)} \left(\int_{\Omega} |u_{x_i}(x/p) - u_{x_i}(x)|^2\, dx \right)^{1/2}.$$

Now the mean-square continuity of the derivatives u_{x_i} results in the estimate

$$|\mathfrak{a}_p[u, v] - \mathfrak{a}_1[u, v]| \leqslant h_p(u) \|v\|_{\mathring{H}^1(\Omega)} \tag{3.2}$$

with $h_p(u) \to 0$ as $p \to 1$ for any fixed function $u \in \mathring{H}^1(\Omega)$.

Further, it follows from the relations

$$\mathfrak{a}_p[u_p, v] = (f, v)_{L_2(\Omega)}, \quad \mathfrak{a}_1[u_1, v] = (f, v)_{L_2(\Omega)}$$

that $\mathfrak{a}_p[u_p, v] = \mathfrak{a}_1[u_1, v]$, and so

$$\mathfrak{a}_p[u_p - u_1, v] = \mathfrak{a}_1[u_1, v] - \mathfrak{a}_p[u_1, v]$$

for all $v \in \mathring{H}^1(\Omega)$. Setting $v = u_p - u_1$ in the last identity and passing to the real parts, we get

$$\begin{aligned} \gamma \|u_p - u_1\|^2_{\mathring{H}^1(\Omega)} &\leqslant \operatorname{Re} \mathfrak{a}_p[u_p - u_1, u_p - u_1] \\ &\leqslant |\mathfrak{a}_1[u_1, u_p - u_1] - \mathfrak{a}_p[u_1, u_p - u_1]| \leqslant h_p(u_1) \|u_p - u_1\|_{\mathring{H}^1(\Omega)} \end{aligned}$$

with the help of (3.1) and (3.2), whence it follows that

$$\|u_p - u_1\|_{\mathring{H}^1(\Omega)} \leqslant \frac{h_p(u_1)}{\gamma} \to 0,\ p \to 1.$$

□

It should be noted that the equivalent condition for strong ellipticity of the limit equation (put $p = 1$)

$$-\sum_{i,j=1}^{n} \left((a_{ij} + b_{ij})u_{x_i}(x) + c_{ij}u_{x_i}(x/q)\right)_{x_j} = f(x) \quad (x \in \Omega)$$

looks as follows (see [10]):

$$\operatorname{Re} \sum_{i,j=1}^{n} (a_{ij} + b_{ij} + c_{ij}\lambda)\, \xi_i \xi_j > 0 \quad (|\lambda| = q^{n/2}, \xi \neq 0),$$

which reduces to

$$q^{n/2}|c(\xi)| < \operatorname{Re} a(\xi) + \operatorname{Re} b(\xi) \quad (\xi \neq 0).$$

The last condition is of course weaker than (2.6). For example, an equation with the operator

$$-\sum_{i=1}^{n} \left((au_{x_i}(x) + bu_{x_i}(x/p) + cu_{x_i}(x/q)\right)_{x_i} \equiv -\Delta \left(au(x) + pbu(x/p) + cqu(x/q)\right),$$

where a and b are positive numbers, and $c \in \mathbb{C}$ is such that

$$a - b \leqslant q^{n/2}|c| < a + b,$$

is strongly elliptic only for $p = 1$ but not for any $p > 1$. In other words, strong ellipticity of the limit equation does not mean strong ellipticity of any prelimit equation.

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