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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

https://scholar.google.com/citations?user=Zay4fxkAAAAJ&hl=ru&authuser=1

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

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THE COMPOSITION OPERATOR IN SOBOLEV MORREY SPACES

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Abstract. In this paper we prove sufficent conditions on a map f from the real line to itself in order that the composite map $f \circ g$ belongs to a Sobolev Morrey space of real valued functions on a domain of the n-dimensional space for all functions g in such a space. Then we prove sufficient conditions on f in order that the composition operator T_f defined by $T_f[g] \equiv f \circ g$ for all functions g in the Sobolev Morrey space is continuous, Lipschitz continuous and differentiable in the Sobolev Morrey space. We confine the attention to Sobolev Morrey spaces of order up to one.

1 Introduction

In this paper we consider the composition operator in Sobolev Morrey spaces of the first order. Let Ω be a bounded open subset of $\mathbb{R}^n$ with the cone property. Let $W_p^{1,\lambda}(\Omega)$ be the Sobolev space of all functions with derivatives up to order 1 in the Morrey space $M_p^{\lambda}(\Omega)$ with exponents $\lambda \in [0, n/p]$, $p \in [1, +\infty]$.

Let Ω_1 be a bounded open subset of $\mathbb{R}$. Let $W_p^{1,\lambda}(\Omega, \Omega_1)$ denote the set of functions of $W_p^{1,\lambda}(\Omega)$ which map Ω to Ω_1.

Let $C^{0,1}(\bar{\Omega}_1)$ denote the space of all Lipschitz continuous functions from $\bar{\Omega}_1$ to $\mathbb{R}$. Let r be a natural number. Let $C^r(\bar{\Omega}_1)$ denote the space of all r times continuously differentiable functions from $\bar{\Omega}_1$ to $\mathbb{R}$.

Then we prove the following results.

(j) We prove that if $f \in C^{0,1}(\bar{\Omega}_1)$ and if $g \in W_p^{1,\lambda}(\Omega)$ has values in Ω_1, then the composite function $f \circ g$ belongs to $W_p^{1,\lambda}(\Omega)$ and the norm of $f \circ g$ can be estimated in terms of the norms of f and of g. We note that in the case $\lambda = 0$, which corresponds to the classical Sobolev space such a result is well known (see Marcus and Mizel [13]).

(jj) We exploit an abstract scheme of [11] and prove that if $1 + \lambda > n/p$, then the composition map T from $C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ which takes a pair (f, g) to the composite function $f \circ g$ is r-times continuously Fréchet differentiable. We note that in the case $\lambda = 0$ the result of the present paper improves a corresponding result of Valent [16] for the case $r = 1$.

(jjj) We prove that if $f \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and if $1+\lambda > n/p$, then the map which takes g to $f \circ g$ is Lipschitz continuous on bounded subsets of $W^{1,\lambda}_p(\Omega)$. For a related result in the Besov space setting, we refer to the paper [2] of Bourdaud and the second named author.

We believe that our sufficient conditions on f of (j), (jj), (jjj) are optimal, just as they have been shown to be optimal in the frame of Sobolev spaces, which corresponds to case $\lambda = 0$ (see Appell and Zabreiko [1, Ch. 9], Runst and Sickel [15, Ch. 5], Bourdaud and the second named author [2].)

The composition operator has been considered by several authors. For extensive references, we refer to the monographs of Appell and Zabreiko [1, Ch. 9], of Runst and Sickel [15], of Dudley and Norvaisa [7], and to the recent survey paper Bourdaud and Sickel [3]. In particular, the continuity, the Lipschitz continuity and the higher order differentiability of $f \circ g$ as a function of both f and g has long been investigated.

2 Composition operator in Morrey spaces

Throughout the paper, n is a nonzero natural number. Let $B_n(x,r)$ be the open ball in $\mathbb{R}^n$ of radius $r > 0$ and with center at the point $x \in \mathbb{R}^n$.

Definition 1. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Let $p \in]0,+\infty]$, $\lambda \in [0, \frac{n}{p}]$. Let

$$w_\lambda(\rho) = \begin{cases} \rho^{-\lambda}, & \rho \in]0,1], \\ 1, & \rho \geq 1. \end{cases}$$

We denote by $M^\lambda_p(\Omega)$ the space of all (equivalence classes of) real-valued measurable functions g on Ω for which

$$\|g\|_{M^\lambda_p(\Omega)} = \sup_{(x,r)\in\Omega\times]0,\infty[} w_\lambda(r)\|g\|_{L_p(B(x,r)\cap\Omega)} < \infty.$$

In the above definition and in the sequel, we set $\frac{n}{p} \equiv 0$ if $p = +\infty$. We note that $M^\lambda_p(\Omega) \subset L_p(\Omega)$, where we retain the standard notation of $L_p(\Omega)$ for the space of all real-valued p-summable measurable functions on Ω.

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^n$ of finite measure. Let Ω_1 be a Borel subset of $\mathbb{R}$. Let $p \in [1,+\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$. Let $g \in M^\lambda_p(\Omega)$ be such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$. Let f be a Borel measurable function from Ω_1 to $\mathbb{R}$. Assume that there exist $a, b \in]0,+\infty[$ such that*

$$|f(\xi)| \leq a|\xi| + b, \quad \forall\, \xi \in \Omega_1. \tag{2.1}$$

Then $f \circ g \in M^\lambda_p(\Omega)$ and

$$\|f \circ g\|_{M^\lambda_p(\Omega)} \leq a\|g\|_{M^\lambda_p(\Omega)} + b\|1\|_{M^\lambda_p(\Omega)}\,. \tag{2.2}$$

Proof. Since $1 \in M^\lambda_p(\Omega)$, we have

$$\|f \circ g\|_{M^\lambda_p(\Omega)} \leq \|a|g| + b\|_{M^\lambda_p(\Omega)} \leq a\|g\|_{M^\lambda_p(\Omega)} + b\|1\|_{M^\lambda_p(\Omega)}.$$

□

Remark 1. Let Ω, Ω_1 be as in Lemma 2.1. First we note that for a Lipschitz continuous map f from Ω_1 to $\mathbb{R}$ with the Lipschitz constant $\mathrm{Lip}(f)$ and for a measurable function g such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$ the following inequality

$$|f(g(x))| \leq \mathrm{Lip}(f)|g(x)| + \mathrm{Lip}(f)|y| + |f(y)| \tag{2.3}$$

holds for almost all $x \in \Omega$ and for all $y \in \Omega_1$.

Moreover, if f is a Lipschitz continuous function on Ω_1 and if $y \in \Omega_1$, condition (2.1) is satisfied with $a = \mathrm{Lip}(f)$, $b = \mathrm{Lip}(f)|y| + |f(y)|$. Hence, Lemma 2.1 implies that

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}(\mathrm{Lip}(f)|y| + |f(y)|), \quad \forall\, y \in \Omega_1. \tag{2.4}$$

Corollary 2.1. *Let the assumptions of Lemma 2.1 be satisfied. If also $0 \in \Omega_1$ and f is a real-valued Lipschitz continuous function on Ω_1, then*

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)} + |f(0)| \cdot \|1\|_{M_p^\lambda(\Omega)}.$$

Corollary 2.2. *Let the assumptions of Lemma 2.1 be satisfied. If also $0 \in \Omega_1$, $f(0) = 0$ and f is a real-valued Lipschitz continuous function on Ω_1, then*

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)}.$$

For each Borel measurable function f from $\mathbb{R}$ to $\mathbb{R}$ and measurable function g from Ω to $\mathbb{R}$ we set

$$T_f[g] \equiv f \circ g\,.$$

Then we have the following.

Corollary 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$. Let f be a locally Lipschitz continuous function from $\mathbb{R}$ to itself. Then*

$$T_f[M_p^\lambda(\Omega) \cap L_\infty(\Omega)] \subseteq M_p^\lambda(\Omega) \cap L_\infty(\Omega).$$

(Note that in general $M_p^\lambda(\Omega) \nsubseteq L_\infty(\Omega)$).

Proof. Let $g \in M_p^\lambda(\Omega) \cap L_\infty(\Omega)$. We set $\Omega_1 = \left[-\|g\|_{L_\infty(\Omega)}, \|g\|_{L_\infty(\Omega)}\right]$. Since Ω_1 is a bounded interval and f is Lipschitz continuous on Ω_1, Corollary 2.1 implies that

$$\|f \circ g\|_{M_p^\lambda(\Omega)} < +\infty.$$

We also have

$$\|f \circ g\|_{L_\infty(\Omega)} \leq \|f\|_{L_\infty(\Omega_1)} < +\infty.$$

Hence, $f \circ g \in M_p^\lambda(\Omega) \cap L_\infty(\Omega)$. □

3 Composition operator in Sobolev Morrey spaces

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $p \in [1, +\infty]$ and $\lambda \in \left[0, \frac{n}{p}\right]$. Then we define the Sobolev space of order 1 built on the Morrey space $M_p^\lambda(\Omega)$, as the set

$$W_p^{1,\lambda}(\Omega) \equiv \left\{g \in M_p^\lambda(\Omega) : D_j g \in M_p^\lambda(\Omega),\ \forall\, j \in \{1, \ldots, n\}\right\},$$

where $D_j g$ is the distributional derivative of g with respect to the j-th variable. Then we set

$$\|g\|_{W_p^{1,\lambda}(\Omega)} = \|g\|_{M_p^\lambda(\Omega)} + \sum_{j=1}^{n} \|D_j g\|_{M_p^\lambda(\Omega)}, \quad \forall\, g \in W_p^{1,\lambda}(\Omega).$$

In particular, $W_p^{0,\lambda}(\Omega) = M_p^\lambda(\Omega)$ and $W_p^{1,0}(\Omega) = W_p^1(\Omega)$, where $W_p^1(\Omega)$ denotes the classical Sobolev space with the exponents $1, p$ in Ω. Obviously, $W_p^{1,\lambda}(\Omega) \subset W_p^1(\Omega)$.

Next we try to understand whether the Lipschitz continuity of a function f of a real variable is enough to ensure that $T_f[W_p^{1,\lambda}(\Omega)] \subseteq W_p^{1,\lambda}(\Omega)$ under suitable conditions on the exponents. To do so, we face the problem of taking the distributional derivatives of the composite function $f \circ g$, and we expect that

$$D_j(f \circ g) = (f' \circ g) D_j g, \quad \forall\, j \in \{1, \ldots, n\}.$$

However, it is not clear what $f' \circ g$ should mean. Indeed, f' is defined only up to the set of measure zero N_f of points where f is not differentiable and $g^{\leftarrow}(N_f)$ may have a positive measure, and even fill the whole of Ω, and $f'(g(x))$ makes no sense when $x \in g^{\leftarrow}(N_f)$. Classically, one circumvents such a difficulty by introducing a result of de la Vallée-Poussin which states that both $D_j(f \circ g)$ and $D_j g$ vanish almost everywhere on $g^{\leftarrow}(N_f)$. Accordingly, it suffices to define $(f' \circ g)(x)$ when $x \in \Omega \setminus g^{\leftarrow}(N_f)$, and to replace $(f' \circ g)(x)$ by 0 in $g^{\leftarrow}(N_f)$. We find convenient to introduce a symbol for the function which equals $(f' \circ g)(x)$ when $x \in \Omega \setminus g^{\leftarrow}(N_f)$ and 0 elsewhere. Then we introduce the following.

Definition 3. Let Ω be an open subset of $\mathbb{R}^n$. Let Ω_1 be a Borel subset of $\mathbb{R}$. Let g be a measurable function from Ω to $\mathbb{R}$. Let the set $N_g \equiv \{x \in \Omega\colon\ g(x) \notin \Omega_1\}$ have measure zero.

Let H be a Borel subset of Ω_1. Let h be a Borel measurable function from $\Omega_1 \setminus H$ to $\mathbb{R}$. Let $h \tilde{\circ} g$ be the function from Ω to $\mathbb{R}$ defined by

$$h \tilde{\circ} g \equiv \begin{cases} 0, & \text{if } x \in g^{\leftarrow}(H) \cup N_g, \\ h(g(x)), & \text{if } x \in \Omega \setminus (g^{\leftarrow}(H) \cup N_g). \end{cases} \tag{3.1}$$

By definition, the function $h \tilde{\circ} g$ is measurable. Next we note that the following holds.

Lemma 3.1. *Let Ω, Ω_1, h, H be as in Definition 3. Let g, g_1 be measurable functions from Ω to $\mathbb{R}$ such that $g(x)$, $g_1(x) \in \Omega_1$ for almost all $x \in \Omega$. If $g(x) = g_1(x)$ for almost all $x \in \Omega$, then $(h \tilde{\circ} g)(x) = (h \tilde{\circ} g_1)(x)$ for almost all $x \in \Omega$.*

Proof. Let N be a measurable subset of measure zero of Ω such that $g(x) = g_1(x)$ and $g(x)$, $g_1(x) \in \Omega_1$ for all $x \in \Omega \setminus N$. Since N has measure zero, it suffices to show that $(h \tilde{\circ} g)(x) = (h \tilde{\circ} g_1)(x)$ for all $x \in \Omega \setminus N$.

If $x \in (\Omega \setminus N) \cap g^{\leftarrow}(H)$, then $g_1(x) = g(x) \in H$ and $x \in (\Omega \setminus N) \cap g_1^{\leftarrow}(H)$, and accordingly $(h\tilde{\circ}g_1)(x) = 0 = (h\tilde{\circ}g)(x)$. If instead $x \in (\Omega \setminus N) \cap (\Omega \setminus g^{\leftarrow}(H))$, then $g_1(x) = g(x) \notin H$ and accordingly $x \in (\Omega \setminus N) \cap (\Omega \setminus g_1^{\leftarrow}(H))$ and $(h\tilde{\circ}g_1)(x) = h(g_1(x)) = h(g(x)) = (h\tilde{\circ}g)(x)$. Hence, $(h\tilde{\circ}g_1)(x) = (h\tilde{\circ}g)(x)$ for all $x \in \Omega \setminus N$. □ By the previous Lemma, it makes sense to introduce the following.

Definition 4. Let Ω, Ω_1, h, H be as in Definition 3. If G is an equivalence class of measurable functions g from Ω to $\mathbb{R}$ such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$, then we define $h\tilde{\circ}G$ to be the equivalence class of measurable functions from Ω to $\mathbb{R}$ which are equal to $h\tilde{\circ}g$ almost everywhere for at least a $g \in G$.

If Ω be an open subset of $\mathbb{R}^n$. We say that (an equivalence class of functions) g of $L_1^{\text{loc}}(\Omega)$ vanishes on a subset A of Ω provided that $\tilde{g}(x) = 0$ for almost all $x \in A$, for at least a representative $\tilde{g}$ of g (and thus for all representatives of g).

Remark 2. Let g_1, g_2 be measurable locally summable functions from Ω to $\mathbb{R}$. Let $g_1 = g_2$ almost everywhere in Ω. If A is a subset of $\mathbb{R}$, then the symmetric difference $g_1^{\leftarrow}(A) \triangle g_2^{\leftarrow}(A)$ has measure zero. Indeed, $g_1^{\leftarrow}(A) \triangle g_2^{\leftarrow}(A) \subseteq \{x \in \Omega\colon\ g_1(x) \neq g_2(x)\}$.

Then we have the following n-dimensional form of the result of de la Vallée-Poussin [6]. For a proof, we refer to Marcus and Mizel [12, p. 298].

Theorem 3.1 ([6]). *Let Ω be an open subset of $\mathbb{R}^n$. Let $g \in W_1^{1,\text{loc}}(\Omega)$. If N is a subset of $\mathbb{R}$ of measure zero, then $(D_1 g, \dots, D_n g) = 0$ on $\tilde{g}^{\leftarrow}(N)$ for any representative $\tilde{g}$ of g.*

Here we note that the equalities $D_1 g = 0, \dots, D_n g = 0$ on $\tilde{g}^{\leftarrow}(N)$ have to be understood in the sense of Definition 4.

Next we introduce the following form of the chain rule (see Marcus and Mizel [12, p. 300]).

Lemma 3.2 ([12]). *Let Ω be an open subset of $\mathbb{R}^n$. Let Ω_1 be an interval of $\mathbb{R}$. Let f be Lipschitz continuous function from Ω_1 to $\mathbb{R}$. Let*

$$W_1^{1,\text{loc}}(\Omega, \Omega_1) \equiv \{g \in W_1^{1,\text{loc}}(\Omega)\colon\ \tilde{g}(x) \in \Omega_1 \text{ for almost all } x \in \Omega \text{ for all representatives } \tilde{g} \text{ of } g\}.$$

Let N_f be the subset of Ω_1 such that $\Omega_1 \setminus N_f$ is the set of points of Ω_1 where f is differentiable. Let $g \in W_1^{1,\text{loc}}(\Omega, \Omega_1)$. Let $f'\tilde{\circ}g$ be defined as in Definition 4 (with $h = f'$, $H = N_f$). Then the chain rule

$$D_j(f \circ g) = (f'\tilde{\circ}g) D_j g, \tag{3.2}$$

holds in the sense of distributions in Ω for all $j \in \{1, \dots, n\}$.

We note that the set N_f of Lemma 3.2 is a Borel subset of Ω_1 (of measure zero) and that f' is a Borel measurable function on $\Omega_1 \setminus N_f$, and that accordingly it makes sense to apply Definition 4 to $f'\tilde{\circ}g$ (cf. Federer [8, p. 211].)

Then we have the following statement, which can be verified by exploiting the definition of the norm in a Morrey space.

Theorem 3.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1, +\infty]$. Let $\lambda \in [0, \frac{n}{p}]$. Then the pointwise multiplication is bilinear and continuous from $M_p^\lambda(\Omega) \times L_\infty(\Omega)$ to $M_p^\lambda(\Omega)$ and*

$$\|uv\|_{M_p^\lambda(\Omega)} \leq \|u\|_{M_p^\lambda(\Omega)} \|v\|_{L_\infty(\Omega)} \qquad \forall (u, v) \in M_p^\lambda(\Omega) \times L_\infty(\Omega)\,.$$

Next we recall the following Sobolev-Morrey Embedding Theorem, which is a combination of results obtained by S.L. Sobolev and C. Morrey. See also Campanato [5, Theorem II.2, p. 75].

Theorem 3.3. *Let $p \in [1, +\infty[$, $\lambda \in \left[0, \frac{n}{p}\right]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let $1 + \lambda > \frac{n}{p}$. Then $W_p^{1,\lambda}(\Omega)$ is continuously imbedded into $L_\infty(\Omega)$.*

Next we introduce an elementary multiplication lemma which extends to Sobolev Morrey spaces on domains a well known result for classical Sobolev spaces. Multiplication theorems in classical Sobolev spaces are well known and we mention Zolesio [18], Valent [16], Runst and Sickel [15]. For Sobolev Morrey spaces in $\mathbb{R}^n$, we cite the contribution of Yuan, Sickel and Yang [17, Theorem 6.3, p. 156].

Lemma 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let $p \in [1, +\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$,*

$$1 + \lambda > \frac{n}{p}. \tag{3.3}$$

Then the pointwise product from $W_p^{1,\lambda}(\Omega) \times W_p^{1,\lambda}(\Omega)$ to $W_p^{1,\lambda}(\Omega)$ is bilinear and continuous.

Proof. We want to prove that if $u, v \in W_p^{1,\lambda}(\Omega)$, then $uv \in W_p^{1,\lambda}(\Omega)$. To do so, we observe that

$$(uv)_{x_j} = u_{x_j} v + u v_{x_j}\,,$$
$$u_{x_j} \in M_p^\lambda(\Omega), \;\; v_{x_j} \in M_p^\lambda(\Omega)\,,$$

for all $j \in \{1, \ldots, n\}$. Since $1+\lambda > \frac{n}{p}$, Theorem 3.3 implies that $W_p^{1,\lambda}(\Omega)$ is continuously embedded into $L_\infty(\Omega)$. Then by Theorem 3.2 the pointwise product is bilinear and continuous from $M_p^\lambda(\Omega) \times L_\infty(\Omega)$ to $M_p^\lambda(\Omega)$ and from $L_\infty(\Omega) \times M_p^\lambda(\Omega)$ to $M_p^\lambda(\Omega)$. Thus, $uv \in M_p^\lambda(\Omega)$ and

$$u_{x_j} v \in M_p^\lambda(\Omega), \;\; u v_{x_j} \in M_p^\lambda(\Omega),$$

and accordingly $(uv)_{x_j} \in M_p^\lambda(\Omega)$ for all $j \in \{1, \ldots, n\}$. Hence, the statement holds true. □

Next we introduce the following sufficient condition for Sobolev Morrey spaces of order one.

Lemma 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let Ω_1 be an interval of $\mathbb{R}$. Let $p \in [1, +\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$. Let f be Lipschitz continuous function from Ω_1 to $\mathbb{R}$. Let*

$$W_p^{1,\lambda}(\Omega, \Omega_1) \equiv \{g \in W_p^{1,\lambda}(\Omega) \colon \; \tilde{g}(x) \in \Omega_1 \text{ for almost all } x \in \Omega \text{ for all representatives } \tilde{g} \text{ of } g\}.$$

Then

$$T_f[W_p^{1,\lambda}(\Omega,\Omega_1)] \subseteq W_p^{1,\lambda}(\Omega).$$

Let N_f be the subset of Ω_1 such that $\Omega_1 \backslash N_f$ is the set of points of Ω_1 where f is differentiable. Let $g \in W_p^{1,\lambda}(\Omega,\Omega_1)$. Let $f'\tilde{\circ}g$ be defined as in Definition 4 (with $h = f'$, $H = N_f$). Then $f'\tilde{\circ}g \in L_\infty(\Omega)$ and the chain rule formula (3.2) holds in the sense of distributions in Ω for all $j \in \{1,\dots,n\}$. Moreover,

$$\|f \circ g\|_{W_p^{1,\lambda}(\Omega)} \leq \{(\mathrm{Lip}(f)|y| + |f(y)|) + \mathrm{Lip}(f)\}(\|g\|_{W_p^{1,\lambda}(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}), \tag{3.4}$$

for all $g \in W_p^{1,\lambda}(\Omega,\Omega_1)$ and for all $y \in \Omega_1$.

Proof. By Remark 1, we know that inequality (2.4) holds for all $g \in W_p^{1,\lambda}(\Omega,\Omega_1) \subseteq M_p^\lambda(\Omega)$ and for all $y \in \Omega_1$.

Now let $g \in W_p^{1,\lambda}(\Omega,\Omega_1)$. Let $\tilde{g}$ be a representative of g. Let $N_{\tilde{g}}$ be a subset of measure 0 of Ω such that

$$\tilde{g}(x) \in \Omega_1, \quad \forall\, x \in \Omega \setminus N_{\tilde{g}}.$$

If $x \in \Omega \setminus (N_{\tilde{g}} \cup \tilde{g}^{\leftarrow}(N_f))$, then

$$|f'(\tilde{g}(x))| = \left| \lim_{\eta \to \tilde{g}(x)} \frac{f(\tilde{g}(x)) - f(\eta)}{\tilde{g}(x) - \eta} \right| \leq \mathrm{Lip}(f).$$

Since $f'\tilde{\circ}\tilde{g} = 0$ for all $x \in N_{\tilde{g}} \cup \tilde{g}^{\leftarrow}(N_f)$, we conclude that

$$|(f'\tilde{\circ}\tilde{g})(x)| \leq \mathrm{Lip}(f) \quad \text{ a.e. in } \Omega.$$

and accordingly that $f'\tilde{\circ}g \in L_\infty(\Omega)$ and that $\|f'\tilde{\circ}g\|_{L_\infty(\Omega)} \leq \mathrm{Lip}(f) < +\infty$. Then by the multiplication Theorem 3.2 and by the membership of $D_j g$ in $M_p^\lambda(\Omega)$, we have $(f'\tilde{\circ}g)D_j g \in M_p^\lambda(\Omega)$ and

$$\|(f'\tilde{\circ}g)D_j g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|D_j g\|_{M_p^\lambda(\Omega)} \tag{3.5}$$

for all $j \in \{1,\dots,n\}$. Thus the right hand side of equality (3.2) belongs to $M_p^\lambda(\Omega)$ for all $g \in W_p^{1,\lambda}(\Omega)$.

By formula (3.2) for the chain rule, inequalities (2.4), (3.5) imply that

$$\begin{aligned} &\|f \circ g\|_{W_p^{1,\lambda}(\Omega)} \\ &\quad = \|f \circ g\|_{M_p^\lambda(\Omega)} + \sum_{j=1}^n \|(f'\tilde{\circ}g)D_j g\|_{M_p^\lambda(\Omega)} \\ &\quad \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}(\mathrm{Lip}(f)|y| + |f(y)|) + \sum_{j=1}^n \mathrm{Lip}(f)\|D_j g\|_{M_p^\lambda(\Omega)} \\ &\quad \leq \{(\mathrm{Lip}(f)|y| + |f(y)|) + \mathrm{Lip}(f)\}(\|g\|_{W_p^{1,\lambda}(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}), \end{aligned} \tag{3.6}$$

for all $g \in W_p^{1,\lambda}(\Omega)$ and $y \in \Omega_1$, and thus inequality (3.4) holds true. □

Corollary 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let $p \in [1,+\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$. Let f be a function from $\mathbb{R}$ to itself. Then the following statements hold.*

(i) If $1+\lambda > \frac{n}{p}$ *and if* f *is locally Lipschitz continuous, then* $T_f[W^{1,\lambda}_p(\Omega)] \subseteq W^{1,\lambda}_p(\Omega)$.

(ii) If $1+\lambda \leq \frac{n}{p}$ *and if* f *is Lipschitz continuous, then* $T_f[W^{1,\lambda}_p(\Omega)] \subseteq W^{1,\lambda}_p(\Omega)$.

Proof. We first consider statement (i). The Sobolev Embedding Theorem 3.3 implies that $W^{1,\lambda}_p(\Omega)$ is continuously embedded into $L_\infty(\Omega)$. Thus if $g \in W^{1,\lambda}_p(\Omega)$, there exists a bounded interval Ω_1 of $\mathbb{R}$ such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$. Since $f\big|_{\Omega_1}$ is Lipschitz continuous, Lemma 3.4 implies that $f \circ g \in W^{1,\lambda}_p(\Omega)$.

Statement (ii) is an immediate consequence of Lemma 3.4 with $\Omega_1 = \mathbb{R}$. □

If Ω_1 is a bounded interval of $\mathbb{R}$, then $C^{0,1}(\bar{\Omega}_1)$ denotes the Banach space of all real valued Lipschitz continuous functions from $\bar{\Omega}_1$ to $\mathbb{R}$ with the norm

$$\|h\|_{C^{0,1}(\bar{\Omega}_1)} \equiv \sup_{\bar{\Omega}_1} |h| + \mathrm{Lip}\,(h), \qquad \forall\, h \in C^{0,1}(\bar{\Omega}_1)\,.$$

Then we summarize in the following statement some facts we need in the sequel in the case $1+\lambda > \frac{n}{p}$ and which are immediate consequences of Lemma 3.3 and Lemma 3.4.

Corollary 3.2. *Let* $p \in [1,+\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $1+\lambda > \frac{n}{p}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Let* Ω_1 *be a bounded open interval of* $\mathbb{R}$. *Then the following statements hold.*

(i) $W^{1,\lambda}_p(\Omega)$ *is a Banach algebra.*

(ii) If $(f,g) \in C^{0,1}(\bar{\Omega}_1) \times W^{1,\lambda}_p(\Omega,\Omega_1)$, *then* $f \circ g \in W^{1,\lambda}_p(\Omega)$. *Moreover, there exists an increasing function* ψ *from* $[0,+\infty[$ *to itself such that*

$$\|f \circ g\|_{W^{1,\lambda}_p(\Omega)} \leq \|f\|_{C^{0,1}(\bar{\Omega}_1)} \psi(\|g\|_{W^{1,\lambda}_p(\Omega)}) \tag{3.7}$$

for all $(f,g) \in C^{0,1}(\bar{\Omega}_1) \times W^{1,\lambda}_p(\Omega,\Omega_1)$.

4 Continuity of the composition operator in Sobolev Morrey spaces

Corollary 3.2 shows that if $1+\lambda > \frac{n}{p}$, then the composition T maps $C^{0,1}(\bar{\Omega}_1) \times W^{1,\lambda}_p(\Omega,\Omega_1)$ to $W^{1,\lambda}_p(\Omega)$. Now we want to understand for which f's the composition operator T_f is continuous in $W^{1,\lambda}_p(\Omega,\Omega_1)$. By following [10], [11], the idea is that if f is a polynomial, then T_f is continuous in $W^{1,\lambda}_p(\Omega)$. Indeed, for $1+\lambda > \frac{n}{p}$, the space $W^{1,\lambda}_p(\Omega)$ is a Banach algebra. Then we exploit inequality (3.7) to show that if f is a limit of polynomials, then T_f is continuous. Actually, such a scheme can be applied in a somewhat abstract general setting, which we now introduce. Let $\mathcal{X}$ be a commutative Banach algebra with unity. Let $\mathbb{N}$ denote the set natural numbers including 0. Let $m \in \mathbb{N}\backslash\{0\}$. In the applications of the present paper, we are interested in the specific case $m = 1$, but here we present a more general case, which can be applied to analyze vector valued functions with components in Sobolev Morrey spaces.

We first note that if p belongs to the space $\mathcal{P}(\mathbb{R}^m)$ of polynomials in m real variables with real coefficients, then it makes perfectly sense to compose p with some $x \equiv (x_1, \ldots, x_m) \in \mathcal{X}^m$. Namely, if p is defined by the equality

$$p(\xi_1, \ldots, \xi_m) \equiv \sum_{|\eta| \leq deg\, p,\, \eta \in \mathbb{N}^m} a_\eta \xi_1^{\eta_1} \ldots \xi_m^{\eta_m}, \quad \text{with } a_\eta \in \mathbb{R}, \quad (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, \tag{4.1}$$

then we set

$$\tau_p[x] \equiv \sum_{|\eta| \leq deg\, p,\, \eta \in \mathbb{N}^m} a_\eta x_1^{\eta_1} \ldots x_m^{\eta_m}, \quad \forall\, x \equiv (x_1, \ldots, x_m) \in \mathcal{X}^m, \tag{4.2}$$

where the product of the x_j's is that of $\mathcal{X}$, and where we understand that x^0 is the unit element of $\mathcal{X}$, for all $x \in \mathcal{X}$. Next we state the following result of [11, Theorem 3.1].

Theorem 4.1 ([11]). *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\| \cdot \|_{\mathcal{Y}}$ be a norm on $\mathcal{P}(\mathbb{R}^m)$. Let $\mathcal{Y}$ be the completion of $\mathcal{P}(\mathbb{R}^m)$ with respect to the norm $\| \cdot \|_{\mathcal{Y}}$. Let $\mathcal{X}$ be a real commutative Banach algebra with unity. Let $\tilde{\mathcal{X}}$ be a real Banach space. Assume that there exists a linear continuous and injective map $\mathcal{J}$ of $\mathcal{X}$ into $\tilde{\mathcal{X}}$. Let $\mathcal{A}$ be a subset of $\mathcal{X}^m$. Assume that there exists an increasing function ψ of $[0, +\infty)$ to itself such that*

$$\|\mathcal{J}[p(x_1, \ldots, x_m)]\|_{\tilde{\mathcal{X}}} \leq \|p\|_{\mathcal{Y}}\, \psi\left(\|(x_1, \ldots, x_m)\|_{\mathcal{X}^m}\right), \tag{4.3}$$

for all $(p, (x_1, \ldots, x_m)) \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{A}$. Then there exists a unique map $\tilde{A}$ from $\mathcal{Y} \times \mathcal{A}$ to $\tilde{\mathcal{X}}$ such that the following two conditions hold.

(i) $\tilde{A}[p, x] = \mathcal{J}[p(x)]$, for all $(p, x) \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{A}$.

(ii) For all fixed $x \equiv (x_1, \ldots, x_m) \in \mathcal{A}$, the map $y \longmapsto \tilde{A}[y, x]$ is continuous from $\mathcal{Y}$ to $\tilde{\mathcal{X}}$.

Furthermore, the map $\tilde{A}[\cdot, x]$ of (ii) is linear, and $\tilde{A}$ is continuous from $\mathcal{Y} \times \mathcal{A}$ to $\tilde{\mathcal{X}}$, and if $y \in \mathcal{Y}$, $y = \lim_{j \to \infty} p_j$ in $\mathcal{Y}$, $p_j \in \mathcal{P}(\mathbb{R}^m)$, $x \equiv (x_1, \ldots, x_m) \in \mathcal{A}$, then

(iii) $\tilde{A}[y, x] = \lim_{j \to \infty} \mathcal{J}[p_j(x)]$ in $\tilde{\mathcal{X}}$;

(iv) $\|\tilde{A}[y, x]\|_{\tilde{\mathcal{X}}} \leq \|y\|_{\mathcal{Y}}\, \psi\left(\|x\|_{\mathcal{X}^m}\right)$.

We shall call $\tilde{A}[y, x]$ the 'abstract' composition of y and x.

We now turn to apply the above theorem to the case of Sobolev Morrey spaces. To do so, we need the following.

Lemma 4.1. *Let Ω_1 be a nonempty bounded open interval of $\mathbb{R}$. Then the Banach space $C^1(\bar{\Omega}_1)$ with the norm*

$$\|f\|_{C^1(\bar{\Omega}_1)} = \sup_{\bar{\Omega}_1} |f| + \sup_{\bar{\Omega}_1} |f'|, \qquad \forall f \in C^1(\bar{\Omega}_1)\,,$$

is a completion of the space $(\mathcal{P}(\mathbb{R}), \| \cdot \|_{C^{0,1}(\bar{\Omega}_1)})$.

Proof. We first note that

$$\sup_{\bar{\Omega}_1} |f'| = \mathrm{Lip}(f), \quad \forall\, f \in C^1(\bar{\Omega}_1).$$

Then we have

$$\|f\|_{C^1(\bar{\Omega}_1)} = \sup |f|_{\bar{\Omega}_1} + \sup_{\bar{\Omega}_1} |f'| = \sup |f|_{\bar{\Omega}_1} + \mathrm{Lip}(f) = \|f\|_{C^{0,1}(\bar{\Omega}_1)} \tag{4.4}$$

for all $f \in C^1(\bar{\Omega}_1)$. Hence,

$$\|f\|_{C^{0,1}(\bar{\Omega}_1)} = \|f\|_{C^1(\bar{\Omega}_1)}, \quad \forall\, p \in \mathcal{P}(\mathbb{R}).$$

Since $C^1(\bar{\Omega}_1)$ is a Banach space and the restriction map from $\mathcal{P}(\mathbb{R})$ to $C^1(\bar{\Omega}_1)$ which takes $p \in \mathcal{P}(\mathbb{R})$ to $p\big|_{\bar{\Omega}_1}$ is a linear isometry from $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^1(\bar{\Omega}_1)})$ to $\left(\{p\big|_{\bar{\Omega}_1} :\ p \in \mathcal{P}(\mathbb{R})\}, \|\cdot\|_{C^1(\bar{\Omega}_1)}\right)$, it suffices to show that $\{p\big|_{\bar{\Omega}_1} :\ p \in \mathcal{P}(\mathbb{R})\}$ is dense in $C^1(\bar{\Omega}_1)$. However, such a density is a well known consequence of the Weierstrass Approximation Theorem (cf. *e.g.* Rohlin and Fuchs [14, p. 185].) □

Then by applying Theorem 4.1, we obtain the following.

Theorem 4.2. *Let $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $1 + \lambda > \frac{n}{p}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let Ω_1 be a bounded open interval of $\mathbb{R}$. Then the composition operator T is continuous from $C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$.*

Proof. We set $\|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)}$, $\mathcal{X} = \tilde{\mathcal{X}} = W_p^{1,\lambda}(\Omega)$, $\mathcal{A} = W_p^{1,\lambda}(\Omega, \Omega_1)$, $m = 1$. Then we take $\mathcal{J}$ to be the identity map. As we have shown, $C^1(\bar{\Omega}_1)$ is a completion of $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)})$. By Corollary 3.2, $W_p^{1,\lambda}(\Omega)$ is a Banach algebra and there exists a function ψ as in (3.7). Then by Theorem 4.1, there exists a unique map $\tilde{A}$ from $C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$ such that the following two conditions hold

(i) $\tilde{A}[p, g] = \tau_p[g]$ for all $(p, g) \in \mathcal{P}(\mathbb{R}) \times \mathcal{A}$.

(ii) For each fixed $g \in W_p^{1,\lambda}(\Omega, \Omega_1)$, the map from $C^1(\bar{\Omega}_1)$ to $W_p^{1,\lambda}(\Omega)$ which takes f to $f \circ g$ is continuous.

Moreover, $\tilde{A}$ is continuous. Clearly, T satisfies (i), and inequality (3.7) implies that T satisfies (ii). Hence, we must necessarily have

$$\tilde{A}[f, g] = T[f, g], \quad \forall (f, g) \in C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1).$$

As a consequence, T is continuous on $C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$. □

5 Lipschitz continuity of the composition operator in Sobolev Morrey spaces

Next we prove a Lipschitz continuity statement for the composition operator by exploiting an argument of Bourdaud and the second named author [2].

Theorem 5.1. *Let $p \in [1,+\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $1+\lambda > \frac{n}{p}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. If $f \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$, then T_f maps $W^{1,\lambda}_p(\Omega)$ to itself and is Lipschitz continuous on the bounded subsets of $W^{1,\lambda}_p(\Omega)$.*

Proof. Let $\mathcal{B}$ be a bounded subset of $W^{1,\lambda}_p(\Omega)$. Since $W^{1,\lambda}_p(\Omega)$ is continuously embedded into $L_\infty(\Omega)$, the set $\mathcal{B}$ is a bounded subset of $L_\infty(\Omega)$ and there exists a closed interval B of $\mathbb{R}$ such that

$$\left[-\|g\|_{L_\infty(\Omega)}, \|g\|_{L_\infty(\Omega)}\right] \subseteq B, \quad \forall\, g \in \mathcal{B}.$$

Now let g_1, $g_2 \in \mathcal{B}$. Since f is continuously differentiable, we can write

$$\begin{aligned}&(f\circ g_2)(x) - (f\circ g_1)(x)\\ &\qquad = \int_0^1 f'[g_1(x)+t(g_2(x)-g_1(x))](g_2(x)-g_1(x))dt, \quad \forall\, x\in\Omega.\end{aligned}$$

Next we fix $x \in \Omega$, $r \in]0,+\infty[$. By the Minkowski inequality for integrals, we have

$$\begin{aligned}&w_\lambda(r)\|f\circ g_2 - f\circ g_1\|_{L_p(\Omega\cap\mathbb{B}_n(x,r))}\\ &\qquad \le \int_0^1 w_\lambda(r)\|f'[g_1(\cdot)+t(g_2(\cdot)-g_1(\cdot))](g_2(\cdot)-g_1(\cdot))\|_{L_p(\Omega\cap\mathbb{B}_n(x,r))}\,dt\\ &\qquad \le \int_0^1 w_\lambda(r)\sup_B|f'|\,\|g_2-g_1\|_{L_p(\Omega\cap\mathbb{B}_n(x,r))}\,dt\\ &\qquad \le \sup_B|f'|\,\|g_2-g_1\|_{M^\lambda_p(\Omega)}.\end{aligned}$$

Hence,

$$\|f\circ g_2 - f\circ g_1\|_{M^\lambda_p(\Omega)} \le \sup_B|f'|\,\|g_2-g_1\|_{M^\lambda_p(\Omega)}. \tag{5.1}$$

Next we fix $j \in \{1,\dots,n\}$ and we try to estimate

$$\begin{aligned}&\|D_j\{f\circ g_2 - f\circ g_1\}\|_{M^\lambda_p(\Omega)} \qquad (5.2)\\ &\qquad = \|f'(g_2)D_j\,g_2 - f'(g_1)D_j\,g_1\}\|_{M^\lambda_p(\Omega)}\\ &\qquad \le \|f'\circ g_2 - f'\circ g_1\|_{L_\infty(\Omega)}\,\|D_j\,g_2\|_{M^\lambda_p(\Omega)}\\ &\qquad +\|f'\circ g_1\|_{L_\infty(\Omega)}\,\|D_j\,g_2 - D_j\,g_1\|_{M^\lambda_p(\Omega)}\\ &\qquad \le \mathrm{Lip}(f'|_B)\|g_2-g_1\|_{L_\infty(\Omega)}\sup_{g\in\mathcal{B}}\|g\|_{W^{1,\lambda}_p(\Omega)} + \sup_B|f'|\,\|g_2-g_1\|_{W^{1,\lambda}_p(\Omega)}\\ &\qquad \le \left\{\mathrm{Lip}(f'|_B)\|I\|_{\mathcal{L}(W^{1,\lambda}_p(\Omega),L_\infty(\Omega))}\sup_{g\in\mathcal{B}}\|g\|_{W^{1,\lambda}_p(\Omega)} + \sup_B|f'|\right\}\|g_2-g_1\|_{W^{1,\lambda}_p(\Omega)},\end{aligned}$$

where $\mathcal{L}(W^{1,\lambda}_p(\Omega), L_\infty(\Omega))$ denotes the Banach space of all linear and continuous maps from $W^{1,\lambda}_p(\Omega)$ to $L_\infty(\Omega)$ and I denotes the inclusion map of $W^{1,\lambda}_p(\Omega)$ into $L_\infty(\Omega)$. Indeed, $N_{f'} = \varnothing$ and $f'\tilde{\circ}g_1 = f'\circ g_1$, $f'\tilde{\circ}g_2 = f'\circ g_2$.

By inequalities (5.1) and (5.2), we conclude that

$$\|f\circ g_2 - f\circ g_1\|_{W_p^{1,\lambda}(\Omega)} \leq \left\{(1+n)\sup_B |f'| + n\mathrm{Lip}(f'\big|_B)\|I\|_{\mathcal{L}(W_p^{1,\lambda}(\Omega),L_\infty(\Omega))}\sup_{g\in\mathcal{B}}\|g\|_{W_p^{1,\lambda}(\Omega)}\right\}\|g_2-g_1\|_{W_p^{1,\lambda}(\Omega)}.$$

□

6 Differentiability properties of the composition operator in Sobolev Morrey spaces

Next we turn to the question of differentiability, and by following [11], we note that the following holds.

Lemma 6.1 ([11]). *Let $m\in\mathbb{N}\setminus\{0\}$. Let $\mathcal{X}$ be a commutative real Banach algebra with the unity $1_{\mathcal{X}}$. Let $\mathcal{P}(\mathbb{R}^m)$ be the set of real polynomials in m real variables. Let $p\in\mathcal{P}(\mathbb{R}^m)$ be defined by*

$$p(\eta)\equiv\sum_{|\gamma|\leq deg\, p} a_\gamma\eta_1^{\gamma_1}\dots\eta_m^{\gamma_m},\quad \forall\,\eta\equiv(\eta_1,\dots,\eta_m)\in\mathbb{R}^m.$$

The map τ_p of $\mathcal{X}^m$ to $\mathcal{X}$ defined by setting

$$\tau_p[x_1,\dots,x_m]\equiv\sum_{|\gamma|\leq deg\, p} a_\gamma x_1^{\gamma_1}\dots x_m^{\gamma_m},\quad \forall\,(x_1,\dots,x_m)\in\mathcal{X}^m,$$

with the understanding that $x^0\equiv 1_{\mathcal{X}}$, for all $x\in\mathcal{X}$, is of class $C^r(\mathcal{X}^m,\mathcal{X})$, for all $r\in\mathbb{N}\cup\{\infty\}$. Furthermore, the differential of $\tau_p[\cdot]$ at $x^\sharp\equiv(x_1^\sharp,\dots,x_m^\sharp)$ is delivered by the map

$$\mathcal{X}^m\ni(h_1,\dots,h_m)\mapsto\sum_{i=1}^m\tau_{\frac{\partial p}{\partial x_i}}[x^\sharp]\,h_i\in\mathcal{X}.$$

Once more, we plan to proceed by approximation and show that T_f is of class C^r if f is a limit of polynomials with an appropriate norm. As we shall see, it turns out that a right choice for the norm is the following

$$\|p\|_{\mathcal{Y}_r}=\sum_{|\gamma|\leq r,\,\gamma\in\mathbb{N}^m}\|D^\gamma p\|_{\mathcal{Y}},\quad\forall p\in\mathcal{P}(\mathbb{R}^m). \tag{6.1}$$

Then we define $\mathcal{Y}_r$ to be the completion of the space $(\mathcal{P}(\mathbb{R}^m),\|\cdot\|_{\mathcal{Y}_r})$. As is well known, $\mathcal{Y}_r$ is unique up to a linear isometry, and we always choose $\mathcal{Y}_r\subseteq\mathcal{Y}$. Then we have the following obvious

Remark 3. If $r,s\in\mathbb{N}$, $s\leq r$, then

$$\mathcal{Y}_r\subseteq\mathcal{Y}_s,\quad \|y\|_{\mathcal{Y}_s}\leq\|y\|_{\mathcal{Y}_r},\quad\forall\,y\in\mathcal{Y}_r.$$

Now we have the following (cf. [11, Theorem 2.4]).

Theorem 6.1 ([11]). *Let $m \in \mathbb{N} \setminus \{0\}$, $r, s \in \mathbb{N}$, $\gamma \in \mathbb{N}^m$, $r - |\gamma| = s$. Let $\|\cdot\|_{\mathcal{Y}}$ be a norm on $\mathcal{P}(\mathbb{R}^m)$, and let $\|\cdot\|_{\mathcal{Y}_r}$ be the norm defined in (6.1), and let $\mathcal{Y}_r$ be the completion of $(\mathcal{P}(\mathbb{R}^m), \|\cdot\|_{\mathcal{Y}_r})$. Then there exists one and only one linear and continuous operator of $\mathcal{Y}_r$ to $\mathcal{Y}_s$ which coincides with the ordinary partial derivation of multi index γ on the elements of $\mathcal{P}(\mathbb{R}^m)$. By abuse of notation, we shall denote such operator by D^γ, just as the usual partial derivative of multi index γ. We have*

$$D^\gamma y = \lim_{j\to\infty} D^\gamma p_j \quad \text{in } \mathcal{Y}_s, \text{ whenever } \lim_{j\to\infty} p_j = y \text{ in } \mathcal{Y}_r, \tag{6.2}$$

and

$$\|y\|_{\mathcal{Y}_r} = \sum_{|\gamma|\le r,\, \gamma\in\mathbb{N}^m} \|D^\gamma y\|_{\mathcal{Y}}, \quad \forall\, y \in \mathcal{Y}_r\,.$$

Then we state the following result of [11, Theorem 4.1].

Theorem 6.2 ([11]). *Let $r, m \in \mathbb{N}\backslash\{0\}$. Let $\|\cdot\|_{\mathcal{Y}}$ be a norm on $\mathcal{P}(\mathbb{R}^m)$. Let $\mathcal{Y}_r$ be the completion of $\mathcal{P}(\mathbb{R}^m)$ with respect to the norm $\|\cdot\|_{\mathcal{Y}_r}$ defined in (6.1). Let $\mathcal{X}$ be a real commutative Banach algebra with unity. Let $\tilde{\mathcal{X}}$ be a real Banach space. Assume that there exists a linear continuous and injective map $\mathcal{J}$ of $\mathcal{X}$ into $\tilde{\mathcal{X}}$. Let $(\cdot) * (\cdot)$ be a continuous and bilinear map of $\tilde{\mathcal{X}} \times \mathcal{X}$ to $\tilde{\mathcal{X}}$. Let $*$ satisfy the following condition:*

$$\mathcal{J}[x_1] * x_2 = \mathcal{J}[x_1, x_2], \quad \forall x_1, x_2 \in \mathcal{X}. \tag{6.3}$$

Let $\mathcal{A}$ be an open subset of $\mathcal{X}^m$. Assume that there exists an increasing function ψ of $[0, +\infty)$ to itself satisfying condition (4.3), for all $(p, x) \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{A}$. Then the restriction of the map $\tilde{A}$ of Theorem 4.1 to $\mathcal{Y}_r \times \mathcal{A}$ is of class C^r from $\mathcal{Y}_r \times \mathcal{A}$ to $\tilde{\mathcal{X}}$. (Note that $\mathcal{Y}_r \subseteq \mathcal{Y}_0$, and that $\mathcal{Y}_0$ equals the space $\mathcal{Y}$ defined in Theorem 4.1.) Furthermore, the differential of $\tilde{A}$ at each $(y^\#, x^\#) \in \mathcal{Y}_r \times \mathcal{A}$ is given by

$$(v, w) \longmapsto \tilde{A}[v, x^\#] + \sum_{l=1}^{m} \tilde{A}[D_l y^\#, x^\#] * w_l,$$

for all $(v, w) \equiv (v, (w_1, \dots, w_m)) \in \mathcal{Y}_r \times \mathcal{X}^m$. (For the definition of $D_l y^\#$, see Theorem 6.1)

Now that we have introduced the above result on the r times differentiability of $\tilde{A}$, we introduce a formula for the differentials $d^s\tilde{A}$ of order $s = 1, \dots, r$ of $\tilde{A}$ of [11, p. 932]. In order to write the formulas in a coincise way, we put a hat $\hat{}$ over a term which we wish to suppress. So for example, $\xi_1 \cdots \hat{\xi}_j \cdots \xi_s$ denotes $\prod_{\substack{l=1,\dots,s\\ l\neq j}} \xi_l$.

Proposition 6.1 ([11]). *Let all the assumptions of Theorem 6.2 hold. Let $r, s \in \mathbb{N}$, $1 \le s \le r$. The differential of order s of $\tilde{A}$ at $(y^\#, x^\#) \in \mathcal{Y}_r \times \mathcal{A}$, which can be identified with an element of $\mathcal{L}^{(s)}(\mathcal{Y}_r \times \mathcal{X}^m, \tilde{\mathcal{X}})$, is given by the formula*

$$\begin{aligned} &d^s\tilde{A}[y^\#, x^\#]((v_{[1]}, w_{[1]}), \dots (v_{[s]}, w_{[s]})) \\ &\quad = \sum_{j=1}^{s} \sum_{l_1,\dots,\hat{l}_j,\dots,l_s=1}^{m} \left\{\tilde{A}[D_{l_s} \cdots \widehat{D_{l_j}} \cdots D_{l_1} v_{[j]}, x^\#]\right\} * \left(w_{s,l_s} \cdots \widehat{w_{j,l_j}} \cdots w_{1,l_1}\right) \\ &\quad + \sum_{l_1,\dots,l_s=1}^{m} \left\{\tilde{A}[D_{l_s} \cdots D_{l_1} y^\#, x^\#]\right\} * \left(w_{s,l_s} \cdots w_{1,l_1}\right), \end{aligned} \tag{6.4}$$

for all $(v_{[j]}, w_{[j]} \equiv (w_{j,1}, \dots, w_{j,m})) \in \mathcal{Y}_r \times \mathcal{X}^m$, $j = 1, \dots, s$. *In (6.4), the symbols* $l_1, \dots, l_s$ *denote summation indices ranging from 1 to* m.

Next we return to applications to Sobolev Morrey spaces. For an open nonempty interval Ω_1 of $\mathbb{R}$ and $r \in \mathbb{N}$, we denote by $C^r(\overline{\Omega}_1)$ the space of real valued r-times continuously differentiable functions in $\overline{\Omega}_1$ with the norm

$$\|f\|_{C^r(\overline{\Omega}_1)} = \sum_{j=0}^{r} \sup_{\overline{\Omega}_1} \left| \frac{d^j}{dt^j} f \right|, \qquad \forall f \in C^r(\overline{\Omega}_1).$$

Then we prove the following.

Lemma 6.2. *Let* $r \in \mathbb{N} \setminus \{0\}$. *Let* Ω_1 *be a nonempty bounded interval of* $\mathbb{R}$. *Then* $C^{r+1}(\overline{\Omega}_1)$ *is a completion of the space* $(\mathcal{P}(\mathbb{R}, \|\cdot\|_{\mathcal{Y}_r})$, *where*

$$\|p\|_{\mathcal{Y}_r} \equiv \sum_{l=0}^{r} \left\| \frac{d^l}{dt^l} p \right\|_{C^{0,1}(\overline{\Omega}_1)}, \quad \forall p \in \mathcal{P}(\mathbb{R}).$$

If $f \in C^{r+1}(\overline{\Omega}_1)$ *and if* $\{p_j\}_{j \in \mathbb{N}}$ *is a sequence of* $\mathcal{P}(\mathbb{R})$ *which converges to* f *in the* $\|\cdot\|_{\mathcal{Y}_r}$*-norm and if* $l \in \{0, \dots, r\}$, *then*

$$\frac{d^l}{dt^l} f = \lim_{j \to \infty} \frac{d^l}{dt^l} p_j, \tag{6.5}$$

in $C^{r-l+1}(\overline{\Omega}_1)$.

Proof. As we have already pointed out, we have

$$\|f\|_{C^{0,1}(\overline{\Omega}_1)} = \|f\|_{C^1(\overline{\Omega}_1)}, \quad \forall f \in C^1(\overline{\Omega}_1),$$

(see (4.4).) Hence,

$$\|p\|_{\mathcal{Y}_r} = \sum_{l=0}^{r} \left(\left\| \frac{d^l}{dt^l} p \right\|_{C^0(\overline{\Omega}_1)} + \left\| \frac{d^{l+1}}{dt^{l+1}} p \right\|_{C^0(\overline{\Omega}_1)} \right), \quad \forall p \in \mathcal{P}(\mathbb{R}),$$

and

$$\|p\|_{C^{r+1}(\overline{\Omega}_1)} \le \|p\|_{\mathcal{Y}_r} \le 2\|p\|_{C^{r+1}(\overline{\Omega}_1)}, \quad \forall p \in \mathcal{P}(\mathbb{R}).$$

Hence, the norm $\|\cdot\|_{\mathcal{Y}_r}$ is equivalent to the norm $\|\cdot\|_{C^{r+1}(\overline{\Omega}_1)}$ on $\mathcal{P}(\mathbb{R})$. Since $C^{r+1}(\overline{\Omega}_1)$ is a Banach space and the restriction map which takes p in $\mathcal{P}(\mathbb{R})$ to $p\big|_{\overline{\Omega}_1}$ in $C^{r+1}(\overline{\Omega}_1)$ is linear isometry of $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^{r+1}(\overline{\Omega}_1)})$ onto $\left(\{p\big|_{\overline{\Omega}_1} : \ p \in \mathcal{P}(\mathbb{R})\}, \|\cdot\|_{C^{r+1}(\overline{\Omega}_1)}\right)$, it suffices to show that for each $f \in C^{r+1}(\overline{\Omega}_1)$, there exists a sequence of polynomials $\{p_j\}_{j \in \mathbb{N}}$ in $\mathcal{P}(\mathbb{R})$ such that

$$f = \lim_{j \to \infty} p_j\big|_{\overline{\Omega}_1} \quad \text{in } C^{r+1}(\overline{\Omega}_1),$$

i.e., $\{p\big|_{\overline{\Omega}_1} : \ p \in \mathcal{P}(\mathbb{R})\}$ is dense in $C^{r+1}(\overline{\Omega}_1)$. However, such a density is a well known consequence of the Weierstrass Approximation Theorem (cf. *e.g.* Rohlin and Fuchs [14, p. 185].) □

Remark 4. Under the assumptions of the previous lemma, the operator D^γ defined by (6.2) coincides with the ordinary D^γ-differentiation in $C^{r+1}(\bar{\Omega}_1)$.

Lemma 6.3. *Let $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $1 + \lambda > \frac{n}{p}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ satisfying the cone property. Let Ω_1 be a bounded open interval of $\mathbb{R}$. Then the set*

$$\tilde{W}_p^{1,\lambda}(\Omega, \Omega_1) \equiv \left\{ g \in W_p^{1,\lambda}(\Omega) \colon \ \overline{\tilde{g}(\Omega)} \subseteq \Omega_1, \ \text{where } \tilde{g} \text{ is the only continuous representatives of } g \right\}$$

is open in $W_p^{1,\lambda}(\Omega)$.

Proof. Under our assumptions on p and λ, the space $W_p^{1,\lambda}(\Omega)$ is continuously imbedded into $C^0(\overline{\Omega})$. Let $g_0 \in \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$. By definition of $\tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$ the distance d of $\tilde{g}_0(\Omega)$ to $\mathbb{R} \setminus \Omega_1$ is nonzero. Then if we choose $g \in W_p^{1,\lambda}(\Omega)$ such that

$$\|g - g_0\|_{W_p^{1,\lambda}(\Omega)} < \frac{d}{\|I\|_{\mathcal{L}(W_p^{1,\lambda}(\Omega), L_\infty(\Omega))} + 1}$$

we have $g \in \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$. Hence, $\tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$ is open in $W_p^{1,\lambda}(\Omega)$. □

Theorem 6.3. *Let $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $1 + \lambda > \frac{n}{p}$, $r \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let Ω_1 be a bounded open interval of $\mathbb{R}$. Then the composition operator T from $C^{r+1}(\bar{\Omega}_1) \times \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$ defined by*

$$T[f, g] \equiv f \circ g, \quad \forall\, (f, g) \in C^{r+1}(\bar{\Omega}_1) \times \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$$

is of class C^r. If $(f_0, g_0) \in C^{r+1}(\bar{\Omega}_1) \times \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$, then the first order differential of T at (f_0, g_0) is given by the formula

$$dT[f_0, g_0](v, w) = v \circ g_0 + f'(g_0) w$$

for all $(v, w) \in C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega)$.

If $s \in \{1, \dots, r\}$, then the s-th order differential of T at (f_0, g_0) is given by the formula

$$d^s T[f_0, g_0][(v_{[1]}, w_{[1]}), \dots, (v_{[s]}, w_{[s]})] = \sum_{j=1}^{s} \frac{d^{s-1} v_{[j]}}{dt^{s-1}} \circ g_0\, w_{[1]} \dots \widehat{w_{[j]}} \dots w_{[s]} + \frac{d^s f_0}{dt^s} \circ g_0\, w_{[1]} \dots w_{[s]},$$

for all $(v_{[1]}, w_{[1]}), \dots, (v_{[s]}, w_{[s]}) \in C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega)$.

Proof. We set $\|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)}$, $\mathcal{X} = \tilde{\mathcal{X}} = W_p^{1,\lambda}(\Omega)$, $\mathcal{A} = \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$, $m = 1$. Then we take $\mathcal{J}$ to be the identity map. As we have shown, $C^{r+1}(\bar{\Omega}_1)$ is a completion of $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{\mathcal{Y}_r})$ and thus we can take $\mathcal{Y}_r = C^{r+1}(\bar{\Omega}_1)$. By Corollary 3.2, $W_p^{1,\lambda}(\Omega)$ is a Banach algebra and there exists a function ψ as in (3.7). By Lemma 6.3, the set $\mathcal{A} = \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$ is open in $W_p^{1,\lambda}(\Omega)$. As we have already proved in the proof of Theorem 4.2, the abstract composition $\tilde{A}$ of Theorem 4.1 coincides with T. Then we can invoke Proposition 6.1 and Theorem 6.2 and conclude that T is of class C^r from $\mathcal{Y}_r \times \mathcal{A} = C^{r+1}(\bar{\Omega}_1) \times \tilde{W}_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$ and that the formulas for the differentials hold. □

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References

[1] J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Mathematics 95, Cambridge University Press, 1990.

[2] G. Bourdaud, M. Lanza de Cristoforis, *Regularity of the symbolic calculus in Besov algebras*, Studia Mathematica 184 (2008), 271–298.

[3] G. Bourdaud and W. Sickel, *Composition operators on function spaces with fractional order of smoothness*, RIMS Kokyuroku Bessatsu B26 (2011), 93–132.

[4] V.I. Burenkov, *Sobolev spaces on domains*, Stuttgart-Leipzig: B. G. Teubner Verlagsgesellschaft mbH, 1998.

[5] S. Campanato, *Proprietà di inclusione per spazi di Morrey*, Ricerche Mat. 12 (1963), 67–86.

[6] Ch. J. de la Vallée-Poussin, *Sur l'intégrale de Lebesgue*, Trans. Amer. Math. Soc. 16 (1915), 435–501.

[7] R. M. Dudley, R. Norvaisa, *Concrete functional calculus*, Springer, 2011.

[8] H. Federer, *Geometric measure theory*, Springer-Verlag New York Inc., New York, 1969.

[9] G.B. Folland, *Real analysis: modern techniques and their applications*, United States of America: A Wiley-Interscience publication, 1999.

[10] M. Lanza de Cristoforis, *Differentiability properties of a composition operator*, Rendiconti del Circolo Matematico di Palermo, Serie II 56 (1998), 157–165.

[11] M. Lanza de Cristoforis, *Differentiability properties of an abstract autonomous composition operator*, J. London Math. Soc. (2) 61 (2000), no. 3, 923–936.

[12] M. Marcus, J. Mizel, *Absolute continuity on tracks and mappings of Sobolev spaces*, Arch. Rat. Mech. Anal. 45 (1972), 294–320.

[13] M. Marcus, V.J. Mizel, *Complete characterization of functions which act, via superposition, on Sobolev spaces*, Trans. Amer. Math. Soc. 251 (1979), 187–218.

[14] V. Rohlin, D. Fuchs, *Premier cours de topologie - chapitres géometriques*, Éditions Mir, Moscou.

[15] T. Runst, W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations*, De Gruyter, Berlin, 1996.

[16] T. Valent, *A property of multiplication in Sobolev spaces. Some applications*, Rend. Sem. Mat. Univ. Padova 74 (1985), 63–73.

[17] W. Yuan, W. Sickel, D. Yang, *Morrey and Campanato meet Besov, Lizorkin and Triebel*, Lecture Notes in Math. Vol. 2005, Springer, Berlin, 2010.

[18] J.L. Zolesio, *Multiplication dans les espaces de Besov*, Proceedings of the Royal Society of Edinburgh, Sect. A78 1-2 (1977/78), 113–117.

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