ISSN 2077–9879

Eurasian Mathematical Journal

2016, Volume 7, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http : //publicationethics.org/files/u2/New$_{C}$ode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

https://scholar.google.com/citations?user=Zay4fxkAAAAJ&hl=ru&authuser=1

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 2 (2016), 38 – 49

ON ALMOST BINARITY IN WEAKLY CIRCULARLY MINIMAL STRUCTURES

B.Sh. Kulpeshov

Communicated by J.A. Tussupov

Key words: weak circular minimality, $\aleph_0$-categoricity, convexity rank, orthogonality, binarity.

AMS Mathematics Subject Classification: 03C64, 03C35.

Abstract. We prove that $\aleph_0$–categorical non-1-transitive weakly circularly minimal theories of convexity rank 1 are almost binary.

1 Introduction

Let $M = \langle M, =, <\rangle$ be a linear ordering. If we connect two endpoints of a linearly ordered set M (possibly, these are $-\infty$ and $+\infty$) then we obtain a circular ordering.

More formally, a *circular ordering* is described by a ternary relation K that satisfies the following conditions:

(co1) $\forall x\forall y\forall z(K(x, y, z) \to K(y, z, x))$;
(co2) $\forall x\forall y\forall z(K(x, y, z) \land K(y, x, z) \Leftrightarrow x = y \lor y = z \lor z = x)$;
(co3) $\forall x\forall y\forall z(K(x, y, z) \to \forall t[K(x, y, t) \lor K(t, y, z)])$;
(co4) $\forall x\forall y\forall z(K(x, y, z) \lor K(y, x, z))$.

The following observation relates linear orders and circular orders.

Proposition 1.1. [4] *(i) If $\langle M, \le\rangle$ is a linear ordering and K is the ternary relation derived from $\le$ by the rule*

$$K(x, y, z) :\Leftrightarrow (x \le y \le z) \lor (z \le x \le y) \lor (y \le z \le x)$$

then K is a circular order relation on M.

(ii) If $\langle N, K\rangle$ is a circular ordering and $a \in N$ then the relation $\le_a$ defined on $M := N \setminus \{a\}$ by the rule

$$y \le_a z :\Leftrightarrow K(a, y, z)$$

is a linear order. Furthermore, if we extend this linear order to one, denoted by $\le'$, on N, specifying that $a \le' b$ for all $b \in M$, then the derived circular order relation is the original circular order K.

The present work is concerned with the notion of *weak circular minimality* introduced and originally deeply studied in [5]. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M; =, K^3, \ldots\rangle$ such that any definable (with parameters) subset of M is a finite union of convex sets in M. Convexity rank of a formula with one free variable was introduced in [6]. In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relation with infinitely many infinite convex classes. Obviously, any o-minimal theory has convexity rank 1. Recall that a complete theory T is *binary* if every formula is equivalent to a boolean combination of formulas with at most two free variables. A. Pillay and C. Steinhorn completely described $\aleph_0$–categorical o-minimal theories [11]. Their description implies binarity of these theories. In the work [7] $\aleph_0$–categorical weakly o-minimal binary theories of convexity rank 1 were described, and in the work [8] the binarity of $\aleph_0$–categorical weakly o-minimal theories of convexity rank 1 has been established. Observe that since a circular ordering is determined by a ternary relation there is no binary weakly circularly minimal structure. We say that a weakly circularly minimal theory T is *almost binary* if any formula is equivalent to a boolean combination of formulas with at most two free variables and the formula $K(x, y, z)$ (expressing the relation of circular ordering). We say that a complete theory T is *1-transitive* if for any $M \models T$ every element of M satisfies the same type over $\emptyset$. In this paper we prove the almost binarity of $\aleph_0$–categorical non-1-transitive weakly circularly minimal theories of convexity rank 1.

First, we recall necessary definitions and some preliminary results.

Proposition 1.2. [5] *Let $M = \langle M, K, \ldots\rangle$ be a weakly circularly minimal structure, and let b be an arbitrary element of M. Define the relation $\leq_b$ on $M \setminus \{b\}$ and $\leq'_b$ on M as in Fact 1.1.*

(i) Let M_b be the structure with domain $M \setminus \{b\}$, the order $\leq_b$, and a relation symbol for each b-definable relation of M on a power of $M \setminus \{b\}$. Then M_b is a weakly o-minimal structure.

(ii) Let $\hat{M}_b$ be the structure with domain M, the order $\leq'_b$, and a relation symbol for each b-definable relation of M on powers of M. Then $\hat{M}_b$ is also weakly o-minimal.

Proposition 1.3. [5] *Let M be a weakly circularly minimal structure, $\phi(x)$ be an arbitrary $\emptyset$-definable formula. Then for any $a \in M$ $\phi(M)$ is a finite union of $\{a\}$-definable convex sets.*

Notation 1. (1) $K_0(x, y, z) := K(x, y, z) \wedge x \neq y \wedge y \neq z \wedge z \neq x$.

(2) $K(u_1, \ldots, u_n)$ denotes the formula saying that all subtriples of the tuple $\langle u_1, \ldots, u_n\rangle$ (in increasing order) satisfy K; likewise with K_0 in place of K.

(3) Let A, B, C be pairwise disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ whenever $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend this notation in the natural way, writing for example $K_0(A, b, C, D)$.

For an arbitrary complete 1-type p we denote by $p(M)$ the set of realisations of the type p in M. By $<_{lex}$ we denote the relation of lexicographic ordering. For an arbitrary tuple $\bar{b} = \langle b_1, b_2, \ldots, b_n\rangle$ of length n by $\bar{b}_{n-i}$ we denote the tuple $\langle b_1, b_2, \ldots, b_i\rangle$ for every $1 \leq i \leq n - 1$.

Definition 1. [5] Let M be a circularly ordered structure.

(i) Let $p \in S_1(\emptyset)$. We say that p is *m-convex* if for any elementary extension N of M $p(N)$ is the disjoint union of m maximal convex sets (which are called the *convex components* of $p(N)$). We say that p is *convex* if p is 1-convex. Otherwise, we say that p is *non-convex.*

(ii) We say that M is *m-convex* if every type $p \in S_1(\emptyset)$ is m-convex, and we say that $Th(M)$ is *m-convex* if this holds for all $N \equiv M$.

Theorem 1.1. [5] *Let M be a weakly circularly minimal structure. Then there is $m < \omega$ such that M is m-convex.*

As a corollary we obtain, in particular, that if M is a weakly circularly minimal structure and $p \in S_1(\emptyset)$ then $p(M)$ is a finite union of convex sets.

Let $p \in S_1(\emptyset)$ and $F(x,y)$ be an $\emptyset$-definable formula such that $F(M,b)$ is convex infinite co-infinite for each $b \in p(M)$, $F(M,b) \subset p(M)$. Let $F^l(y)$ be the formula saying that y is a left endpoint of $F(M,y)$:

$$\exists z_1 \exists z_2 [K_0(z_1,y,z_2) \land \forall t_1 (K(z_1,t_1,y) \land t_1 \neq y \to \neg F(t_1,y)) \land$$

$$\land \forall t_2 (K(y,t_2,z_2) \land t_2 \neq y \to F(t_2,y))]$$

We say that $F(x,y)$ is *p-stable convex-to-right* if for every $b \in p(M)$

$$M \models \forall x [F(x,b) \to F^l(b) \land \forall z (K(b,z,x) \to F(z,b))]$$

and there is $a \in p(M)$ such that $\neg F(a,b)$ and $K_0(b,M,a) \subseteq p(M)$.

Let $F_1(x,y), F_2(x,y)$ be arbitrary p-stable convex-to-right formulas. We say that $F_2(x,y)$ is *bigger than* $F_1(x,y)$ if there is $a \in p(M)$ with $F_1(M,a) \subset F_2(M,a)$. This gives a total ordering on the (finite) set of all p-stable convex-to-right formulas $F(x,y)$ (viewed up to equivalence modulo $Th(M)$). We write $f(y) := \text{rend}\, F(M,y)$, meaning that $f(y)$ ia a right endpoint of $F(M,y)$ which lies in the definable completion $\overline{M}$ of M. Then f is a function which maps $p(M)$ into $\overline{M}$. Analogously we can consider p-stable convex-to-left formulas and write $f(y) := \text{lend}\, F(M,y)$, meaning that $f(y)$ is a left endpoint of $F(M,y)$ which lies also in general in the definable completion $\overline{M}$ of M.

Let $F(x,y)$ be a p-stable convex-to-right formula. Slightly adapting the definition in [2], we say that $F(x,y)$ is *equivalence-generating* if for any $\alpha, \beta \in p(M)$ such that $M \models F(\beta, \alpha)$ the following holds:

$$M \models \forall x (K(\beta, x, \alpha) \land x \neq \alpha \to [F(x,\alpha) \leftrightarrow F(x,\beta)])$$

Lemma 1.1. [1] *Let M be an $\aleph_0$–categorical weakly circularly minimal structure, $p \in S_1(\emptyset)$ be non-algebraic, $F(x,y)$ be a p-stable convex-to-right formula which is equivalence-generating. Then there is an $\emptyset$-definable equivalence relation partitioning $p(M)$ into infinitely many infinite convex classes.*

Theorem 1.2. [1] *Let M be an $\aleph_0$–categorical m-convex weakly circularly minimal structure, $m > 1$, $p \in S_1(\emptyset)$ be non-algebraic. Then any p-stable convex-to-right formula is equivalence-generating.*

Let f be a unary function to $\overline{M}$ with $\text{Dom}(f) = I \subseteq M$, where I is an open convex set. We say f is *monotonic-to-right (left) on I* if it preserves (reverses) the relation K_0, i.e. for any $a,b,c \in I$ such that $K_0(a,b,c)$ we have $K_0(f(a),f(b),f(c))$ $(K_0(f(c),f(b),f(a)))$.

Proposition 1.4. *Let M be an $\aleph_0$–categorical non-1-transitive weakly circularly minimal structure of convexity rank 1, $p \in S_1(\emptyset)$ be non-algebraic. Then any function f the domain of which contains $p(M)$ is strictly monotonic or constant on $p(M)$.*

Definition 2. [3] Let M be a weakly o-minimal structure, $A \subseteq M$, M be $|A|^+$–saturated, $p_1, p_2 \in S_1(A)$ be non-algebraic. We say p_1 is not *weakly orthogonal* to p_2 ($p_1 \not\perp^w p_2$) if there are an A-definable formula $H(x, y)$, $\alpha \in p_1(M)$ and $\beta_1, \beta_2 \in p_2(M)$ such that $\beta_1 \in H(\alpha, M)$ and $\beta_2 \notin H(\alpha, M)$.

We extend the last definition on weakly circularly minimal structures.

Lemma 1.2. ([3], Corollary 34) *The relation $\not\perp^w$ is an equivalence relation on $S_1(A)$.*

It is not difficult to see that the last assertion is also true in the case of weakly circularly minimal structures.

2 The main theorem

Theorem 2.1. *Any $\aleph_0$–categorical non-1-transitive weakly circularly minimal theory of convexity rank 1 is almost binary.*

First we introduce necessary definitions and prove a series of assertions that will be culminated in the proof of Theorem 2.1.

Let M be a non-1-transitive weakly circularly minimal structure. Then M is m-convex for some $m \geq 1$. In case $m = 1$ it is not difficult to see that there is an $\emptyset$-definable linear order regarding of which M is a weakly o-minimal structure. Therefore further we consider the case $m > 1$. Also observe that if M is m-convex for some $m > 1$, M is non-1-transitive.

Let $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic, $U_1, U_2, \ldots, U_s$ be convex components of the types $p_1, p_2, \ldots, p_s$ respectively. We say that a family of convex components $\{U_1, \ldots, U_s\}$ is *weakly orthogonal over* $\emptyset$ if every s-tuple $\langle a_1, \ldots, a_s\rangle \in U_1 \times \ldots \times U_s$ satisfies the same type over $\emptyset$. We say that a family of convex components $\{U_1, \ldots, U_s\}$ is *orthogonal over* $\emptyset$ if for any sequence $(n_1, n_2, \ldots, n_s) \in \omega^s$ every $(n_1 + n_2 + \ldots + n_s)$–tuple $\langle a_1^1, a_1^2, \ldots, a_1^{n_1}; \ldots; a_2^1, a_2^2, \ldots, a_2^{n_2}; \ldots; a_s^1, a_s^2, \ldots, a_s^{n_s}\rangle \in (U_1)^{n_1} \times (U_2)^{n_2} \times \ldots \times (U_s)^{n_s}$ with $K_0(a_1^1, a_1^2, \ldots, a_1^{n_1}; \ldots; a_2^1, a_2^2, \ldots, a_2^{n_2}; \ldots; a_s^1, a_s^2, \ldots, a_s^{n_s})$ satisfies the same type over $\emptyset$.

Proposition 2.1. [9] *Let T be an $\aleph_0$-categorical m-convex weakly circularly minimal theory of convexity rank 1, $m > 1$, $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic, $U_1, U_2, \ldots, U_s$ be convex components of the types $p_1, p_2, \ldots, p_s$ respectively. Suppose that $\{U_1, U_2, \ldots, U_s\}$ is weakly orthogonal over $\emptyset$. Then it is orthogonal over $\emptyset$.*

Proposition 2.2. [9] *Let T be an $\aleph_0$-categorical m-convex weakly circularly minimal theory of convexity rank 1, $m > 1$, $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic, $U_1, U_2, \ldots, U_s$ be convex components of the types $p_1, p_2, \ldots, p_s$ respectively. Suppose that $\{U_1, U_2, \ldots, U_s\}$ is pairwise weakly orthogonal over $\emptyset$. Then $\{U_1, U_2, \ldots, U_s\}$ is weakly orthogonal over $\emptyset$.*

Let $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, $\{U_1, U_2\}$ be convex components of types p_1 and p_2 respectively. We say that an $\emptyset$–definable formula $\phi(x, y)$ is a (U_1, U_2)*–splitting formula* if there is $a \in U_1$ such that $\phi(a, M) \subset U_2$, $\phi(a, M)$ convex and $\operatorname{lend} \phi(a, M) = \operatorname{lend} U_2$.

Lemma 2.1. *Let M be an $\aleph_0$-categorical m-convex weakly circularly minimal structure, $m > 1$, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, U_1, U_2 be convex components of the types p_1 and p_2 respectively. Suppose that $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$. Then there is at least one (U_1, U_2)–splitting formula.*

Proof. Since $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$, there are $a_1, a'_1 \in U_1, a_2, a'_2 \in U_2$ such that $tp(\langle a_1, a_2\rangle/\emptyset) \neq tp(\langle a'_1, a'_2\rangle/\emptyset)$. Consequently, there is an $\emptyset$-definable formula $R(x, y)$ such that

$$M \models R(a_1, a_2) \wedge \neg R(a'_1, a'_2).$$

Note that there exists $a''_2 \in U_2$ such that $M \models \neg R(a_1, a''_2)$. If this is not true then let $A(x, y)$ be an $\emptyset$-definable formula isolating $tp(\langle a_1, a_2\rangle/\emptyset)$. Then $M \models \theta(a_1)$, where

$$\theta(a_1) := \forall y[A(a_1, y) \rightarrow R(a_1, y)]$$

Since $tp(a_1/\emptyset) = tp(a'_1/\emptyset)$, we have: $M \models \theta(a'_1)$, contradicting our assumption. Thus, we have:

$$M \models R(a_1, a_2) \wedge \neg R(a_1, a''_2).$$

Without loss of generality, suppose that $K_0(a_1, a_2, a''_2)$. By the weak circular minimality $R(a_1, M)$ is a finite union of $\{a_1\}$-definable convex sets, and let $R_0(a_1, x)$ define a convex set containing $\{a_2\}$. Then the formula

$$\phi(x, y) := U_{p_1}(x) \wedge U_{p_2}(y) \wedge \exists t[R_0(x, t) \wedge K(x, y, t)]$$

is a (U_1, U_2)–splitting formula. □

Let $\phi_1(x, y)$, $\phi_2(x, y)$ be (U_1, U_2)–splitting formulas. We say that $\phi_1(x, y)$ *less than* $\phi_2(x, y)$ if there is $a \in U_1$ such that $\phi_1(a, M) \subset \phi_2(a, M)$. Obviously the set of all (U_1, U_2)–splitting formulas is linearly ordered.

Lemma 2.2. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $M \models T$, $A \subseteq M$, A be finite, $A \neq \emptyset$, $p_1, p_2 \in S_1(A)$ be non-algebraic, $p_1 \not\perp^w p_2$. Then*

1. If there exists $a \in p_1(M)$ such that $dcl(A \cup \{a\}) \cap p_2(M) = \emptyset$ then there is a unique (p_1, p_2)–splitting formula.

2. If there exists $a \in p_1(M)$ such that $dcl(A \cup \{a\}) \cap p_2(M) \neq \emptyset$ then there are exactly two (p_1, p_2)–splitting formulas $\phi_1(x, y), \phi_2(x, y)$ so that $\phi_1(x, y)$ is less than $\phi_2(x, y)$ and $|\phi_2(a, M) \setminus \phi_1(a, M)| = 1$ for any $a \in p_1(M)$.

Proof. Take an arbitrary element $a \in A$ and consider the structure $\hat{M}_a$. By Proposition 1.2 $\hat{M}_a$ is a weakly o-minimal structure. The further proof is analogous to the proof of Lemma 6 in [8]. □

Lemma 2.3. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, U_1, U_2 be convex components of the types p_1 and p_2 respectively. Suppose that $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$. Then*

1. If there is $a \in U_1$ such that $dcl(\{a\}) \cap U_2 = \emptyset$ then there exists a unique (U_1, U_2)–splitting formula.

2. If there is $a \in U_1$ such that $dcl(\{a\}) \cap U_2 \neq \emptyset$ then there are exactly two (U_1, U_2)–splitting formulas $\phi_1(x, y), \phi_2(x, y)$ so that $\phi_1(x, y)$ is less than $\phi_2(x, y)$ and $|\phi_2(a, M) \setminus \phi_1(a, M)| = 1$ for any $a \in U_1$.

Proof. 1. Suppose that $dcl(\{a\}) \cap U_2 = \emptyset$ for some $a \in U_1$. Assume the contrary: there are at least two (U_1, U_2)-splitting formulas $\phi_1(x, y)$ and $\phi_2(x, y)$, and let for definiteness $\phi_1(x, y)$ is less than $\phi_2(x, y)$. Obviously, $|\phi_2(a, M) \setminus \phi_1(a, M)| > 1$. Indeed if $|\phi_2(a, M) \setminus \phi_1(a, M)| = 1$ then there is $b \in U_2$ such that $b \in \phi_2(a, M) \setminus \phi_1(a, M)$ and $b \in dcl(\{a\})$, contradicting our assumption. Since $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$, there are $a \in U_1$, $b_1, b_2 \in U_2$ such that

$$M \models \phi_1(a, b_1) \wedge \neg\phi_1(a, b_2) \wedge \phi_2(a, b_2).$$

Without loss of generality, we assume that $K_0(a, b_1, b_2)$ and $K(b_1, M, b_2) \subset U_2$. Then consider the following formula:

$$F(x, b_2) := \exists t[U_{p_1}(t) \wedge \neg\phi_1(t, b_2) \wedge \phi_2(t, b_2) \wedge \phi_2(t, x) \wedge K(b_2, x, t) \wedge U_{p_2}(x)]$$

It is not difficult to see that $F(x, y)$ is a p_2-stable convex-to-right formula, contradicting that T has the convexity rank 1.

2. Let $b \in dcl(\{a\}) \cap U_2$ for some $a \in U_1$. Then there is an $\emptyset$–definable formula $\theta(x, y)$ such that

$$M \models \theta(a, b) \wedge \exists! y\theta(a, y).$$

Consider the following formulas:

$$\phi_1(a, y) := \exists t[\theta(a, t) \wedge K(a, y, t) \wedge y \neq t \wedge U_{p_2}(y)]$$

$$\phi_2(a, y) := \exists t[\theta(a, t) \wedge K(a, y, t) \wedge U_{p_2}(y)]$$

Obviously, $\phi_1(x, y)$, $\phi_2(x, y)$ are (U_1, U_2)–splitting formulas and $\phi_2(a, M) \setminus \phi_1(a, M) = \{b\}$. By analogy with Point 1 one can prove that there is no other (U_1, U_2)–splitting formulas. □

Lemma 2.4. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $p_1, p_2, p_3 \in S_1(\emptyset)$ be non-algebraic, U_1, U_2, U_3 be convex components of the types p_1, p_2 and p_3 respectively. Suppose that $\{U_1, U_2\}$ and $\{U_2, U_3\}$ are not weakly orthogonal over $\emptyset$. Then for any $a, a' \in U_1$, $b, b' \in U_2$, $c, c' \in U_3$ such that $tp(\langle a, b\rangle/\emptyset) = tp(\langle a', b'\rangle/\emptyset)$, $tp(\langle a, c\rangle/\emptyset) = tp(\langle a', c'\rangle/\emptyset)$, $tp(\langle b, c\rangle/\emptyset) = tp(\langle b', c'\rangle/\emptyset)$ we have $tp(\langle a, b, c\rangle/\emptyset) = tp(\langle a', b', c'\rangle/\emptyset)$.*

Proof. Assume the contrary: there exist $a, a' \in U_1$, $b, b' \in U_2$, $c, c' \in U_3$ satisfying the hypotheses of the lemma and there is an $\emptyset$–definable formula $R(x, y, z)$ so that

$$M \models R(a, b, c) \wedge \neg R(a', b', c').$$

Without loss of generality, we assume that $K_0(U_1, U_2, U_3)$. Note that the elements a, b and c are pairwise algebraically independent over $\emptyset$. Indeed, if for example $b \in dcl(\{a\})$ then there is an $\emptyset$–definable formula $\theta(x, y)$ so that

$$M \models \theta(a, b) \wedge \exists! y\theta(a, y).$$

Consider the following formula: $R'(x, z) := \forall y[\theta(x, y) \rightarrow R(x, y, z)]$. Then

$$M \models R'(a, c) \wedge \neg R'(a', c'),$$

contradicting the hypothesis $tp(\langle a, c\rangle/\emptyset) = tp(\langle a', c'\rangle/\emptyset)$.

Let $A(x, y)$ and $B(x, y)$ be $\emptyset$–definable formulas isolating $tp(\langle a, c\rangle/\emptyset)$ and $tp(\langle b, c\rangle/\emptyset)$ respectively. Then there is $c'' \in U_3$ such that

$$M \models A(a, c'') \wedge B(b, c'') \wedge \neg R(a, b, c'')$$

If this is not true then $M \models D(a, b)$, where

$$D(a, b) := \forall y[U_3(y, a) \wedge A(a, y) \wedge B(b, y) \rightarrow R(a, b, y)]$$

Here $U_3(y, a)$ is an $\{a\}$-definable formula determining the set U_3 (a convex component of any non-algebraic 1-type which is not $\emptyset$-definable is $\{a\}$-definable for any $a \in M$). But then $M \models D(a', b')$, contradicting our assumption. Thus, we have:

$$M \models R(a, b, c) \wedge \neg R(a, b, c'').$$

Transforming if necessary we can assume that $R(a, b, M)$ is convex, $R(a, b, M) \subset U_3$ and $\mathrm{lend}\, R(a, b, M) = \mathrm{lend}\, U_3$.

Let $\phi_{13}(x, y)$ be a (U_1, U_3)–splitting formula, $\phi_{23}(x, y)$ be a (U_2, U_3)–splitting formula. Then either $\phi_{13}(a, M) \subseteq \phi_{23}(b, M)$ or $\phi_{23}(b, M) \subset \phi_{13}(a, M)$. If $\phi_{13}(a, M) = \phi_{23}(b, M)$ then by strict monotonicity of $f(x) := \mathrm{rend}\, \phi_{23}(x, M)$ on $p_2(M)$ we have that $b \in dcl(\{a\})$, contradicting our supposition. If $|\phi_{23}(b, M) \setminus \phi_{13}(a, M)| = 1$ or $|\phi_{13}(a, M) \setminus \phi_{23}(b, M)| = 1$ then we also can see that $b \in dcl(\{a\})$.

Suppose that $\phi_{13}(a, M) \subset \phi_{23}(b, M)$. Then $|\phi_{23}(b, M) \setminus \phi_{13}(a, M)| > 1$. Since $tp(\langle a, c\rangle/\emptyset) = tp(\langle a, c''\rangle/\emptyset)$, we have either $c, c'' \in \phi_{13}(a, M)$ or $c, c'' \in \neg\phi_{13}(a, M)$. Without loss of generality, suppose the first. Let $p'_1 := tp(a/\{b\})$, $p'_3 := tp(c/\{b\})$. Obviously, $p'_1 \not\perp^w p'_3$ and $R(x, b, z)$, $\phi_{13}(x, z)$ are (p'_1, p'_3)–splitting formulas and $|\phi_{13}(a, M) \setminus R(a, b, M)| > 1$, contradicting Lemma 2.2. The case $\phi_{23}(b, M) \subset \phi_{13}(a, M)$ is considered analogously. □

Lemma 2.5. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $M \models T$, $A \subseteq M$, A be finite, $A \neq \emptyset$, $p_1, p_2, p_3 \in S_1(A)$ be non-algebraic. Then for any $a, a' \in p_1(M)$, $b, b' \in p_2(M)$, $c, c' \in p_3(M)$ such that $tp(\langle a, b\rangle/A) = tp(\langle a', b'\rangle/A)$, $tp(\langle a, c\rangle/A) = tp(\langle a', c'\rangle/A)$, $tp(\langle b, c\rangle/A) = tp(\langle b', c'\rangle/A)$ we have $tp(\langle a, b, c\rangle/A) = tp(\langle a', b', c'\rangle/A)$.*

Proof. Take an arbitrary element $a \in A$ and consider the structure $\hat{M}_a$. By Proposition 1.2 $\hat{M}_a$ is a weakly o-minimal structure. The further proof is analogous to the proof of Lemma 8 in [8]. □

Lemma 2.6. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $p_1, p_2, p_3 \in S_1(\emptyset)$ be non-algebraic, U_1, U_2, U_3 be convex components of the types p_1, p_2 and p_3 respectively. Then for any $a, a' \in U_1$, $b, b' \in U_2$, $c, c' \in U_3$ such that $tp(\langle a, b\rangle/\emptyset) = tp(\langle a', b'\rangle/\emptyset)$, $tp(\langle a, c\rangle/\emptyset) = tp(\langle a', c'\rangle/\emptyset)$, $tp(\langle b, c\rangle/\emptyset) = tp(\langle b', c'\rangle/\emptyset)$ we have $tp(\langle a, b, c\rangle/\emptyset) = tp(\langle a', b', c'\rangle/\emptyset)$.*

Proof. Without loss of generality, we assume that $K_0(U_1, U_2, U_3)$. If U_1, U_2 and U_3 are pairwise weakly orthogonal over $\emptyset$ then the conclusion of the lemma follows from Proposition 2.2. If the convex components U_1, U_2 and U_3 are not pairwise weakly orthogonal over $\emptyset$ then it follows by Lemma 2.4. Therefore suppose that $\{U_1, U_2\}$ is weakly orthogonal over $\emptyset$, but $\{U_2, U_3\}$ is not weakly orthogonal over $\emptyset$. Then $\{U_1, U_3\}$ is weakly orthogonal over $\emptyset$, since if not we obtain that $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$.

Assume that the conclusion of the lemma does not hold and consequently there are $a, a' \in U_1$, $b, b' \in U_2$, $c, c' \in U_3$ satisfying the hypotheses of the lemma and there is an $\emptyset$–definable formula $R(x, y, z)$ such that

$$M \models R(a, b, c) \wedge \neg R(a', b', c').$$

By analogy with Lemma 2.4 we can see that a, b and c are pairwise algebraically independent over $\emptyset$. Let $A(x, y)$ and $B(y, z)$ isolate $tp(\langle a, c\rangle/\emptyset)$ and $tp(\langle b, c\rangle/\emptyset)$ respectively. Note that there is $c'' \in U_3$ such that

$$M \models A(a, c'') \wedge B(b, c'') \wedge \neg R(a, b, c'').$$

If this is not true then $M \models \theta(a, b)$, where

$$\theta(a, b) := \forall y[U_3(y, a) \wedge A(a, y) \wedge B(b, y) \rightarrow R(a, b, y)]$$

Here the formula $U_3(y, a)$ determines the set U_3. Since $tp(\langle a, b\rangle/\emptyset) = tp(\langle a', b'\rangle/\emptyset)$, $M \models \theta(a', b')$, contradicting our assumption. Thus, we have:

$$M \models R(a, b, c) \wedge \neg R(a, b, c'').$$

Without loss of generality, suppose that $K_0(a, b, c, c'')$. Transforming if necessary, we can assume that $R(a, b, M)$ is convex, $R(a, b, M) \subset U_3$ and $\operatorname{lend} R(a, b, M) = \operatorname{lend} U_3$. Let $p_2' := tp(b/\{a\})$, $p_3' := tp(c/\{a\})$. Then $p_2'(M) = U_2$, $p_3'(M) = U_3$ and consequently $R(a, y, z)$ is a (p_2', p_3')–splitting formula.

Since $\{U_2, U_3\}$ is not weakly orthogonal over $\emptyset$, there is a (U_2, U_3)–splitting formula $\phi_{23}(x, y)$. Obviously, it is also a (p_2', p_3')–splitting formula.

Since $tp(\langle b, c\rangle/\emptyset) = tp(\langle b, c''\rangle/\emptyset)$ we have that either $c, c'' \in \phi_{23}(b, M)$ or $c, c'' \in \neg\phi_{23}(b, M)$. Without loss of generality, suppose the first. Then $R(a, b, M) \subset \phi_{23}(b, M)$ and $|\phi_{23}(b, M) \setminus R(a, b, M)| > 1$, contradicting Lemma 2.2.

The case when $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$ and $\{U_2, U_3\}$ is weakly orthogonal over $\emptyset$ is considered analogously. □

Lemma 2.7. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, U_1, U_2 be convex components of the types p_1 and p_2 respectively. Suppose that $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$. Then for any $a, a' \in U_1$, $\langle b_1, b_2\rangle, \langle b_1', b_2'\rangle \in [U_2]^2$ such that $K_0(a, b_1, b_2)$, $K_0(a', b_1', b_2')$, $tp(\langle a, b_1\rangle/\emptyset) = tp(\langle a', b_1'\rangle/\emptyset)$, $tp(\langle a, b_2\rangle/\emptyset) = tp(\langle a', b_2'\rangle/\emptyset)$ we have $tp(\langle a, b_1, b_2\rangle/\emptyset) = tp(\langle a', b_1', b_2'\rangle/\emptyset)$.*

Proof. Assume the contrary: there is an $\emptyset$–definable formula $R(x, y, z)$ such that $M \models R(a, b_1, b_2) \wedge \neg R(a', b_1', b_2')$ for some $a, a' \in U_1$, $\langle b_1, b_2\rangle, \langle b_1', b_2'\rangle \in [U_2]^2$ with $K_0(a, b_1, b_2)$, $K_0(a', b_1', b_2')$, $tp(\langle a, b_1\rangle/\emptyset) = tp(\langle a', b_1'\rangle/\emptyset)$ and $tp(\langle a, b_2\rangle/\emptyset) = tp(\langle a', b_2'\rangle/\emptyset)$.

First note that if $b_1 \in dcl(\{a\})$ then $b_2 \notin dcl(\{a\})$. Indeed, if $b_1, b_2 \in dcl(\{a\})$ then we can show that $b_2 \in dcl(\{b_1\})$, and consequently we can prove that $dcl(\{b_1\})$ is infinite, contradicting the $\aleph_0$–categoricity of T. Suppose that $b_1 \in dcl(\{a\})$. Then there is an $\emptyset$–definable formula $\theta(x,y)$ such that

$$M \models \theta(a,b_1) \wedge \exists! y \theta(a,y).$$

Consider the following formula:

$$R'(x,z) := \forall y[\theta(x,y) \to R(x,y,z)]$$

According to our assumption we obtain that

$$M \models R'(a,b_2) \wedge \neg R'(a',b_2'),$$

contradicting the hypothesis that $tp(\langle a,b_2\rangle/\emptyset) = tp(\langle a',b_2'\rangle/\emptyset)$. Thus, we can assume that $b_1, b_2 \notin dcl(\{a\})$.

Let $A_2(x,y)$ be an $\emptyset$–definable formula isolating $tp(\langle a,b_2\rangle/\emptyset)$. Then there is $b_2'' \in U_2$ such that

$$M \models K_0(a,b_1,b_2'') \wedge A_2(a,b_2'') \wedge \neg R(a,b_1,b_2'')$$

If it is not true then $M \models \theta(a,b_1)$, where

$$\theta(a,b_1) := \forall y[K_0(a,b_1,y) \wedge A_2(a,y) \to R(a,b_1,y)]$$

Since $tp(\langle a,b_1\rangle/\emptyset) = tp(\langle a',b_1'\rangle/\emptyset)$, we have $M \models \theta(a',b_1')$, contradicting our assumption. Thus, we have:

$$M \models R(a,b_1,b_2) \wedge \neg R(a,b_1,b_2'').$$

Case 1. $tp(\langle a,b_1\rangle/\emptyset) = tp(\langle a,b_2\rangle/\emptyset)$.

Let $p_2'(y) := \{A_2(a,y)\}$. Then p_2' determines a type over $\{a\}$, since $A_2(a,M)$ is 1-indiscernible over $\{a\}$. Without loss of generality, suppose that $K_0(a,b_1,b_2,b_2'')$. By weak circular minimality $R(a,b_1,M)$ is a finite union of $\{a,b_1\}$–definable convex sets, and let $R_0(a,b_1,z)$ determines the convex set containing $\{b_2\}$. Then consider the following formula:

$$H(a,b_1,z) := K(a,b_1,z) \wedge U_2(a,b_1) \wedge U_2(a,z) \wedge \exists t[R_0(a,b_1,t) \wedge K(a,b_1,z,t)]$$

It is not difficult to see that $H(a,y,z)$ is a p_2'–stable formula, contradicting that T has the convexity rank 1.

Case 2. $tp(\langle a,b_1\rangle/\emptyset) \neq tp(\langle a,b_2\rangle/\emptyset)$.

Since $\{U_1,U_2\}$ is not weakly orthogonal over $\emptyset$ and $b_1 \notin dcl(\{a\})$, there is a (U_1,U_2)–splitting formula $\phi(x,y)$ such that $\phi(a,M)$ is 1-indiscernible over $\{a\}$ and

$$M \models \phi(a,b_1) \wedge \neg\phi(a,b_2).$$

Since $b_2 \notin dcl(\{a\})$ and $\mu(x) := \text{rend}\,\phi(x,M)$ is strictly monotonic on $p_1(M)$, there is $a_1 \in U_1$ such that

$$M \models K_0(a,a_1,b_1) \wedge \phi(a_1,b_1) \wedge \neg\phi(a_1,b_2)$$

Consider the following functions: $f_a(y) := \text{rend}\,R(a,y,M)$, $g_{b_1}(x) := \text{rend}\,R(x,\,b_1,\,M)$.

Further we can conduct reasons in the weakly o-minimal structure $\hat{M}_a$, and therefore the further proof repeats the proof of Lemma 9 in [8]. □

Lemma 2.8. *Let T be an $\aleph_0$–categorical m-convex weakly circularly minimal theory of convexity rank 1, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, U_1, U_2 be convex components of the types p_1 and p_2 respectively. Suppose that $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$. Then for any $n_1, n_2 < \omega$ and any tuples $\bar{a} = \langle a_1, a_2, \ldots, a_{n_1}\rangle$, $\bar{a}' = \langle a'_1, a'_2, \ldots, a'_{n_1}\rangle \in [U_1]^{n_1}$, $\bar{b} = \langle b_1, b_2, \ldots, b_{n_2}\rangle$, $\bar{b}' = \langle b'_1, b'_2, \ldots, b'_{n_2}\rangle \in [U_2]^{n_2}$ such that $K_0(a_1, a_2, \ldots, a_{n_1})$, $K_0(a'_1, a'_2, \ldots, a'_{n_1})$, $K_0(b_1, b_2, \ldots, b_{n_2})$, $K_0(b'_1, b'_2, \ldots, b'_{n_2})$ and for any $i, j : 1 \le i \le n_1$, $1 \le j \le n_2$ $tp(\langle a_i, b_j\rangle/\emptyset) = tp(\langle a'_i, b'_j\rangle/\emptyset)$ we have $tp(\langle \bar{a}, \bar{b}\rangle/\emptyset) = tp(\langle \bar{a}', \bar{b}'\rangle/\emptyset)$.*

Proof. We prove by induction on $(n_1, n_2) \ge_{lex} (1, 1)$. Step $(1, 1)$ is trivial. Step $(1, 2)$ has been established in Lemma 2.7. Suppose that the lemma has been established for all (k_1, k_2) with $(1, 2) \le_{lex} (k_1, k_2) <_{lex} (n_1, n_2)$. Prove the lemma for case (n_1, n_2). Assume the contrary: there are an $\emptyset$–definable formula $R(\bar{x}, \bar{y})$ and tuples $\bar{a}, \bar{a}' \in [p_1(M)]^{n_1}$, $\bar{b}, \bar{b}' \in [p_2(M)]^{n_2}$ satisfying the hypotheses of the lemma so that

$$M \models R(\bar{a}, \bar{b}) \wedge \neg R(\bar{a}', \bar{b}').$$

By analogy with the proof of Lemma 2.7 we can see that there is $b''_{n_2} \in U_2$ such that $K_0(b_1, \ldots, b_{n_2-1}, b''_{n_2})$, $tp(\langle a_i, b_{n_2}\rangle/\emptyset) = tp(\langle a_i, b''_{n_2}\rangle/\emptyset)$, $1 \le i \le n_1$ и

$$M \models R(\bar{a}, \bar{b}_{n_2-1}, b_{n_2}) \wedge \neg R(\bar{a}, \bar{b}_{n_2-1}, b''_{n_2})$$

Let $B := \{\bar{a}_{n_1-1}, \bar{b}_{n_2-2}\}$. Then obviously $B \ne \emptyset$, and we can further conduct reasons in the weakly o-minimal structure $\hat{M}_{b_1}$, whereas the further proof repeats the proof of Lemma 10 in [8]. □

Proof of Theorem 2.1. By the $\aleph_0$–categoricity there are only finitely many non-algebraic 1-types over $\emptyset$. Let $\{U_1, U_2, \ldots, U_s\}$ be a complete list of convex components of non-algebraic 1-types of $S_1(\emptyset)$. Prove by induction on $s \ge 2$ that for any $n_1, n_2, \ldots, n_s < \omega$, for any $\bar{a}_{n_1}$, $\bar{a}'_{n_1} \in [U_1]^{n_1}$, $\bar{a}_{n_2}$, $\bar{a}'_{n_2} \in [U_2]^{n_2}$, $\ldots$, $\bar{a}_{n_s}$, $\bar{a}'_{n_s} \in [U_s]^{n_s}$, such that for any i_1, i_2, j, k : $1 \le i_1 < i_2 \le s$, $1 \le j \le n_{i_1}$, $1 \le k \le n_{i_2}$ $tp(\langle a^j_{n_{i_1}}, a^k_{n_{i_2}}\rangle/\emptyset) = tp(\langle (a^j_{n_{i_1}})', (a^k_{n_{i_2}})'\rangle/\emptyset)$, we have

$$tp(\langle \bar{a}_{n_1}, \bar{a}_{n_2}, \ldots, \bar{a}_{n_s}\rangle/\emptyset) = tp(\langle \bar{a}'_{n_1}, \bar{a}'_{n_2}, \ldots, \bar{a}'_{n_s}\rangle/\emptyset) \qquad (*)$$

Step $s = 2$. If $\{U_1, U_2\}$ is weakly orthogonal over $\emptyset$ then by Proposition 2.1 $\{U_1, U_2\}$ is orthogonal over $\emptyset$, i.e. $(*)$ holds. If $\{U_1, U_2\}$ is not weakly orthogonal over $\emptyset$ then $(*)$ follows by Lemma 2.8.

Step $s = 3$. Case $n_1 = 1, n_2 = 1, n_3 = 1$ has been established in Lemma 2.6. Suppose that $(*)$ has been established for all (k_1, k_2, k_3) with $(1, 1, 1) \le_{lex} (k_1, k_2, k_3) <_{lex} (n_1, n_2, n_3)$ and prove it for (n_1, n_2, n_3). Assume the contrary: there are $\bar{a}_{n_1}$, $\bar{a}'_{n_1} \in [U_1]^{n_1}$, $\bar{a}_{n_2}$, $\bar{a}'_{n_2} \in [U_2]^{n_2}$, $\bar{a}_{n_3}$, $\bar{a}'_{n_3} \in [U_3]^{n_3}$ such that for any $i_1, i_2, j, k : 1 \le i_1 < i_2 \le 3$, $1 \le j \le n_{i_1}$, $1 \le k \le n_{i_2}$ $tp(\langle a^j_{n_{i_1}}, a^k_{n_{i_2}}\rangle/\emptyset) = tp(\langle (a^j_{n_{i_1}})', (a^k_{n_{i_2}})'\rangle/\emptyset)$ and there is an $\emptyset$–definable formula $R(\bar{x}_{n_1}, \bar{x}_{n_2}, \bar{x}_{n_3})$ so that

$$M \models R(\bar{a}_{n_1}, \bar{a}_{n_2}, \bar{a}_{n_3}) \wedge \neg R(\bar{a}'_{n_1}, \bar{a}'_{n_2}, \bar{a}'_{n_s})$$

Then we can see that there exists $(a^{n_3}_{n_3})'' \in U_3$ such that $K_0(\bar{a}_{n_1}, \bar{a}_{n_2}, \bar{a}_{n_3-1}, (a^{n_3}_{n_3})'')$, $tp(\langle a^j_{n_i}, a^{n_3}_{n_3}\rangle/\emptyset) = tp(\langle a^j_{n_i}, (a^{n_3}_{n_3})''\rangle/\emptyset)$ for any $1 \le i \le 2$ and $1 \le j \le n_i$ and

$$M \models \neg R(\bar{a}_{n_1}, \bar{a}_{n_2}, \bar{a}_{n_3-1}, (a^{n_3}_{n_3})'').$$

Obviously, $n_i > 1$ for some $i \leq 3$. Without loss of generality, suppose that $n_3 > 1$. Let $A := \{\bar{a}_{n_1-1}, \bar{a}_{n_2-1}, \bar{a}_{n_3-1}\}$. Then $A \neq \emptyset$, and considering the types $p'_1 := tp(a^{n_1}_{n_1}/A), p'_2 := tp(a^{n_2}_{n_2}/A), p'_3 := tp(a^{n_3}_{n_3}/A)$, we have a contradiction with Lemma 2.5.

Suppose that $(*)$ has been established for all k with $3 \leq k < s$. Prove it for s. We further conduct reasons in the weakly o-minimal structure $\hat{M}_a$ for an arbitrary $a \in \{a^1_{n_1}, \ldots, a^{n_1}_{n_1}\} \subseteq U_1$, whereas the further proof repeats the proof of Theorem 2 in [8]. □

Acknowledgments

This work was supported by Grant of Science Committee of Education and Science Ministry of the Republic of Kazakhstan No. 0830/GF4.

References

[1] A.B. Altayeva, B.Sh. Kulpeshov, *Equivalence-generating formulas in weakly circularly minimal structures,* Reports of the National Academy of Sciences of the Republic of Kazakhstan, 2014, no. 2, 5–10.

[2] B.S. Baizhanov, B.Sh. Kulpeshov, *On behaviour of 2-formulas in weakly o-minimal theories,* Mathematical Logic in Asia: Proceedings of the 9th Asian Logic Conference, Singapore: World Scientific, 2006, 31–40.

[3] B.S. Baizhanov, *Expansion of a model of a weakly o-minimal theory by a family of unary predicates,* The Journal of Symbolic Logic 66 (2001), 1382–1414.

[4] M. Bhattacharjee, H.D. Macpherson, R.G. Möller, P.M. Neumann, *Notes on infinite permutation groups,* Lecture Notes in Mathematics 1698, Springer, 1998, 202 pp.

[5] B.Sh. Kulpeshov, H.D. Macpherson, *Minimality conditions on circularly ordered structures,* Mathematical Logic Quarterly 51 (2005), 377–399.

[6] B.Sh. Kulpeshov, *Weakly o-minimal structures and some of their properties,* The Journal of Symbolic Logic 63 (1998), 1511–1528.

[7] B.Sh. Kulpeshov, *The property of being binary for $\aleph_0$-categorical weakly o-minimal theories,* Algebra and Logic 44 (2005), no. 4, 256–263.

[8] B.Sh. Kulpeshov, *Binarity of $\aleph_0$-categorical weakly o-minimal theories of convexity rank 1,* Siberian Electronic Mathematical Reports 3 (2006), 185–196.

[9] B.Sh. Kulpeshov, *On indiscernibility of a set in circularly ordered structures,* Siberian Electronic Mathematical Reports 12 (2015), 255–266.

[10] H.D. Macpherson, C. Steinhorn, *On variants of o-minimality,* Annals of Pure and Applied Logic 79 (1996), 165–209.

[11] A. Pillay, C. Steinhorn, *Definable sets in ordered structures I,* Transactions of the American Mathematical Society 295 (1986), 565–592.

Beibut Shaiykovich Kulpeshov
Department of Mathematical and Computer Modeling
International Information Technology University
34A Manas St / 8A Zhandosov St
050040 Almaty, Kazakhstan
E-mail: b.kulpeshov@iitu.kz

Received: 12.12.2015