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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

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The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

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EQUIVALENT QUASI-NORMS INVOLVING DIFFERENCES AND MODULI OF CONTINUITY IN ANISOTROPIC NIKOL'SKII-BESOV SPACES

B. Halim, A. Senouci

Communicated by V.S. Guliyev

Key words: equivalent quasi-norms, anisotropic Nikol'skii-Besov spaces.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. In this paper we study the equivalence of quasi-norms in the anisotropic Nikol'skii-Besov $B^l_{p,\theta}(\mathbb{R}^n)$ spaces involving differences and moduli of continuity for $0 < p, \theta \leq \infty$.

1 Introduction

Definition 1 Let $l > 0$, $\sigma \in \mathbb{N}$, $\sigma > l$, $0 < p, \theta \leq \infty$. Then $f \in B^l_{p,\theta}(\mathbb{R}^n)$, the Nikol'skii-Besov space, if f is measurable on $\mathbb{R}^n$ and

$$\|f\|_{B^l_{p,\theta}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{b^l_{p,\theta}(\mathbb{R}^n)} < \infty, \tag{1.1}$$

where

$$\|f\|_{b^l_{p,\theta}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\frac{\|\Delta^\sigma_h f\|_{L_p(\mathbb{R}^n)}}{|h|^l} \right)^\theta \frac{dh}{|h|^n} \right)^{\frac{1}{\theta}}, \tag{1.2}$$

if $0 < \theta < \infty$,

$$\|f\|_{b^l_{p,\infty}(\mathbb{R}^n)} = \sup_{h \in \mathbb{R}^n, h \neq 0} \frac{\|\Delta^\sigma_h f\|_{L_p(\mathbb{R}^n)}}{|h|^l}, \tag{1.3}$$

if $\theta = \infty$, and $\Delta^\sigma_h f$ is the difference of the function f of order σ with step $h \in \mathbb{R}^n$.

Definition 2 Let $l > 0$, $\sigma \in \mathbb{N}$, $\sigma > l$, $0 < p, \theta \leq \infty$. Then $f \in \widetilde{B}^l_{p,\theta}(\mathbb{R}^n)$, the Nikol'skii-Besov space, if f is measurable on $\mathbb{R}^n$ and

$$\|f\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{\widetilde{b}^l_{p,\theta}(\mathbb{R}^n)} < \infty, \tag{1.4}$$

where

$$\|f\|_{\widetilde{b}^l_{p,\theta}(\mathbb{R}^n)} = \left(\int_0^\infty \left(\frac{\omega^{(\sigma)}(\delta; f)_p}{\delta^l} \right)^\theta \frac{d\delta}{\delta} \right)^{\frac{1}{\theta}} < \infty, \tag{1.5}$$

if $0 < \theta < \infty$,

$$\|f\|_{\widetilde{b}^l_{p,\infty}(\mathbb{R}^n)} = \sup_{\delta > 0} \frac{\omega^{(\sigma)}(\delta; f)_p}{\delta^l}, \tag{1.6}$$

if $\theta=\infty$, and $\omega^{(\sigma)}(\delta;f)_p=\sup_{|h|\le\delta}\|\Delta_h^\sigma f\|_p$ is the L_p- modulus of continuity of the function f of order σ.

It has been known that if $1\le p,\theta\le\infty$ then the quasi-norms $\|.\|_{B^l_{p.\theta}(\mathbb{R}^n)}$ corresponding to different $\sigma\in\mathbb{N},\sigma>l,l>0$ are equivalent, $\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)=B^l_{p,\theta}(\mathbb{R}^n)$, and the norms $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$ and $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}$ with $\sigma>l$ are equivalent (see, for example [2], [6]). By the result of H. Triebel (see [8]) it also follows that similar statements holds for arbitrary $0<p,\theta\le\infty$ under the assumption that $l>n(\frac{1}{p}-1)_+$, where the a_+ denotes the positive part of a. The aim of article [3] was to get rid of the assumption $l>n(\frac{1}{p}-1)_+$, and the following result was obtained.

Theorem 1 *Let* $l>0$, $\sigma\in\mathbb{N}$, $0<p,\theta\le\infty$.

1. *The quasi-norms* $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$ *corresponding to different* $\sigma>l$ *are equivalent.*
2. *The quasi-norms* $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$ *and the quasi-norms* $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}$ *with the same* $\sigma>l$ *are equivalent and* $\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)=B^l_{p,\theta}(\mathbb{R}^n)$.
3. *The quasi-norms* $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}$ *corresponding to different* $\sigma>l$ *are equivalent.*

The aim of this work is to obtain similar results for the anistropic Nikol'skii-Besov spaces.

2 Main results for the anisotropic Nikol'skii-Besov spaces

We introduce some notation. If $p=(p_1,...,p_n)$, $\theta=(\theta_1,...,\theta_n)$, $l=(l_1,...,l_n)$, $\sigma=(\sigma_1,...,\sigma_n)$, we write $\sigma\ge l$ if and only if $\sigma_j\ge l_j$, (in particular $0=(0,...,0)$, $\infty=(\infty,...,\infty)$, $0<p\le\infty\Leftrightarrow 0<p_j\le\infty$, $j=\overline{1,n}$).

Definition 3 Let $l=(l_1,...,l_n)$, $\sigma=(\sigma_1,...,\sigma_n)$, $\sigma_j\in\mathbb{N}$, $p=(p_1,...,p_n)$, $\theta=(\theta_1,...,\theta_n)$, $\sigma>l>0$, $0<p,\theta\le\infty$. Then $f\in B^l_{p,\theta}(\mathbb{R}^n)$, the anistropic Nikol'skii-Besov space, if f is measurable on $\mathbb{R}^n$ and

$$\|f\|_{B^l_{p,\theta}(\mathbb{R}^n)}=\sum_{j=1}^n\|f\|_{B^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)}<\infty, \tag{2.1}$$

where

$$\|f\|_{B^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)}=\|f\|_{L_{p_j}(\mathbb{R}^n)}+\|f\|_{b^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} \tag{2.2}$$

with

$$\|f\|_{b^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)}=\left(\int_0^\infty\left(h^{-l_j}\|\Delta^{\sigma_j}_{h,j}f\|_{p_j}\right)^{\theta_j}\frac{dh}{h}\right)^{\frac{1}{\theta_j}} \tag{2.3}$$

if $0<\theta_j<\infty$ and

$$\|f\|_{b^{l_j}_{p_j,\infty;j}(\mathbb{R}^n)}=\sup_{h>0}\frac{\|\Delta^{\sigma_j}_{h,j}f\|_{L_{p_j}(\mathbb{R}^n)}}{h^{l_j}}. \tag{2.4}$$

Definition 4 Let $l = (l_1, ..., l_n)$, $\sigma = (\sigma_1, ..., \sigma_n)$, $\sigma_j \in \mathbb{N}$, $p = (p_1, ..., p_n)$, $\theta = (\theta_1, ..., \theta_n)$, $\sigma > l > 0$, $0 < p, \theta \leq \infty$. Then $f \in \widetilde{B}^l_{p,\theta}(\mathbb{R}^n)$, the anistropic Nikolskii-Besov space, if f is measurable on $\mathbb{R}^n$ and

$$\|f\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)} = \sum_{j=1}^{n} \|f\|_{\widetilde{B}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} < \infty, \tag{2.5}$$

where

$$\|f\|_{\widetilde{B}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} = \|f\|_{L_{p_j}(\mathbb{R}^n)} + \|f\|_{\widetilde{b}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)}$$

with

$$\|f\|_{\widetilde{b}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} = \left(\int_0^\infty \left(\frac{\omega_j^{\sigma_j}(f,\delta)_{p_j}}{\delta^{l_j}} \right)^{\theta_j} \frac{d\delta}{\delta} \right)^{\frac{1}{\theta_j}} \tag{2.6}$$

if $0 < \theta_j < \infty$ and

$$\|f\|_{\widetilde{b}^{l_j}_{p_j,\infty;j}(\mathbb{R}^n)} = \sup_{\delta > 0} \frac{\omega_j^{\sigma_j}(f,\delta)_{p_j}}{\delta^{l_j}}, \tag{2.7}$$

where

$$\omega_j^{\sigma_j}(f,\delta)_{p_j} = \sup_{|h| \leq \delta} \|\Delta^{\sigma_j}_{h,j} f\|_{L_p(\mathbb{R}^n)}, j = \overline{1,n},$$

is the partial modulus of continuity of the function f of order σ_j.

Theorem 2 *Let* $l = (l_1, ..., l_n) > 0$, $\sigma = (\sigma_1, ..., \sigma_n)$, $\sigma_j \in \mathbb{N}$, $\sigma > l$, $p = (p_1, ..., p_n)$, $\theta = (\theta_1, ..., \theta_n)$, $0 < p, \theta \leq \infty$.

1. *The quasi-norms* $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$ *corresponding to different* σ *satisfying* $\sigma > l$ *are equivalent.*

2. *The quasi-norms* $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$ *and the quasi-norms* $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}$ *with the same* $\sigma > l$ *are equivalent and* $\widetilde{B}^l_{p,\theta}(\mathbb{R}^n) = B^l_{p,\theta}(\mathbb{R}^n)$.

3. *The quasi-norms* $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}$ *corresponding to different* σ *satisfying* $\sigma > l$ *are equivalent.*

The proof of Statement 1 for $0 < p, \theta \leq \infty$ is based on the scheme used in the proof for the isotropic case given in [3] and the proof of Statement 2 is based on the Hardy-type inequality for $0 < \theta < 1$ (see [6]). The proof of Statement 3 follows by Statement 1 and 2. We shall systematically use of the following inequality: if $0 < p < 1$, then for all $f, g \in L_p(\mathbb{R}^n)$, and for all $\epsilon > 0$,

$$\|f + g\|_p \leq (1 + \epsilon)\|f\|_p + c_1(p,\epsilon)\|g\|_p, \tag{2.8}$$

where (see e.g [5], Exercise, 6, p. 36), $c_1(p,\epsilon) = (1 - (1+\epsilon)^{\frac{p}{p-1}})^{\frac{p-1}{p}}$.

2.1 Equivalence of quasi-norms for different $\sigma > l$

Lemma 1 *Let* $l = (l_1, ..., l_n)$, $l_j > 0$, $\sigma = (\sigma_1, ..., \sigma_n)$, $\sigma_j \in \mathbb{N}$, $p = (p_1, ..., p_n)$, $\theta = (\theta_1, ..., \theta_n)$, $\sigma > l > 0$, $0 < p, \theta \le \infty$. *Then the quasi-norms* $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$ *corresponding to different* σ *satisfying* $\sigma > l$ *are equivalent.*

Idea of proof. Denote temporarily quasi-norms (2.1) corresponding to σ by

$$|||f|||^{(\sigma)} = \sum_{j=1}^{n} \|f\|_j^{(\sigma_j)}. \tag{2.9}$$

Let $\sigma = (\sigma_1, ..., \sigma_n)$, $\widetilde{\sigma} = (\widetilde{\sigma}_1, ..., \widetilde{\sigma}_n)$. We need to prove that $|||.|||^{(\sigma)}$ and $|||.|||^{(\widetilde{\sigma})}$ are equivalent where $\sigma > l$, $\widetilde{\sigma} > l$. Let

$$\|f\|_j^{(\sigma_j)} = \|f\|_{L_{p_j}(\mathbb{R}^n)} + \|f\|^{(\sigma_j)},$$

where

$$\|f\|^{(\sigma_j)} = \left(\int_0^\infty \left(h^{-l_j} \|\Delta^{\sigma_j}_{h,j} f\|_{p_j} \right)^{\theta_j} \frac{dh}{h} \right)^{\frac{1}{\theta_j}}, \tag{2.10}$$

if $0 < \theta_j < \infty$ and

$$\|f\|^{(\sigma_j)} = \sup_{h>0} \left(h^{-l_j} \|\Delta^{\sigma_j}_{h,j} f\|_{p_j} \right), \tag{2.11}$$

if $\theta_j = \infty$.

Since

$$\|\Delta^{\sigma_j+1}_{h,j} f\|_{p_j} \le 2^{\frac{1}{p_j}} \|\Delta^{\sigma_j}_{h,j} f\|_{p_j},$$

it follows that

$$\|f\|_j^{(\sigma_j+1)} \le 2^{\frac{1}{p_j}} \|f\|_j^{(\sigma_j)}. \tag{2.12}$$

The proof of the converse inequality is based on the scheme used in the proof given in [3]. Inequality (2.8) will be applied to the identity

$$\Delta^{\sigma_j}_{h,j} f = 2^{-\sigma_j} \Delta^{\sigma_j}_{2h,j} f + P_{\sigma_j-1}(E_{h,j}) \Delta^{\sigma_j+1}_{h,j} f, \tag{2.13}$$

where

$$P_{\sigma_j-1}(z) = -2^{-\sigma_j}(z-1)^{-1}((z+1)^{\sigma_j} - 2^{\sigma_j}) = -\sum_{s=1}^{\sigma_j} \binom{\sigma_j}{s} (z-1)^{s-1} 2^{-s},$$

consequently

$$P_{\sigma_j-1}(E_{h,j}) = -\sum_{s=1}^{\sigma_j} \binom{\sigma_j}{s} (E_{h,j} - I)^{s-1} 2^{-s}.$$

where $E_{h,j}(f)(x) = f(x + he_j)$ for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}$.

Proof. 1. First, we assume that $0 < p_j < 1$, $0 < \theta_j \le \infty$.

Let $\sigma_j \in \mathbb{N}$, and $\|f\|^{(\sigma_j+1)} < \infty$. By [3] (see p. 6) we have ,

$$\|\Delta_{h,j}^{\sigma_j} f\|_{L_{p_j}(\mathbb{R}^n)} \leq 2^{-\sigma_j}(1+\epsilon_j)\|\Delta_{2h,j}^{\sigma_j} f\|_{L_{p_j}(\mathbb{R}^n)} + c_2(\sigma_j, p_j, \epsilon_j)\|\Delta_{h,j}^{\sigma_j+1} f\|_{L_{p_j}(\mathbb{R}^n)}, \tag{2.14}$$

where

$$c_2(\sigma_j, p_j, \epsilon_j) = c_1(p_j, \epsilon_j)\sigma_j^{\frac{1}{p_j}-1} 2^{-\frac{1}{p_j}}((2^{\frac{1}{p_j}-1}+1)^{\sigma_j} - 1).$$

Case $\theta_j = +\infty$. For all $h \in \mathbb{R}$, $h \neq 0$ consider the functional

$$\phi_j(h) = \frac{\|\Delta_{h,j}^{\sigma_j} f\|_{p_j}}{h^{l_j}}.$$

Clearly $\Phi_j(h) < +\infty$ for all $h \neq 0$.

It has been proved in [3] that if $\epsilon_j > 0$ is such that $2^{(l_j-\sigma_j)}(1+\epsilon_j) < 1$, say if $\epsilon_j = 2^{\sigma_j - l_j - 1} - 2^{-1}$, then

$$\begin{aligned}\|f\|^{(\sigma_j)} = \sup_{h>0} \phi_j(h) &\leq [1 - 2^{l_j-\sigma_j}(1+\epsilon_j)]^{-1} c_2 \|f\|^{(\sigma_j+1)} \\ &= 2(1-2^{l_j-\sigma_j})^{-1} c_2 \|f\|^{(\sigma_j+1)} \\ &= c_3 \|f\|^{(\sigma_j+1)}.\end{aligned}$$

So

$$\|f\|^{(\sigma_j)} \leq c_3(p_j, l_j, \sigma_j)\|f\|^{(\sigma_j+1)}, \tag{2.15}$$

where

$$\begin{aligned}c_3(p_j, l_j, \sigma_j) &= 2\left(1-2^{l_j-\sigma_j}\right)^{-1} c_2 \\ &= 2(1-2^{l_j-\sigma_j})^{-1}\sigma_j^{\frac{1}{p_j}-1} 2^{\frac{-1}{p_j}}[(2^{\frac{1}{p_j}-1}+1)^{\sigma_j} - 1][1-(2^{\sigma_j-l_j-1}+2^{-1})^{\frac{p_j}{p_j-1}}]^{\frac{p_j-1}{p_j}}.\end{aligned}$$

Since $\sigma_j > l_j$, $\widetilde{\sigma}_j > l_j$, $0 < p_j < 1$ it follows that $c_3(p_j, l_j, \sigma_j) > 1$, hence we obtain

$$\|f\|_j^{(\sigma_j)} \leq c_3(p_j, l_j, \sigma_j)\|f\|_j^{(\sigma_j+1)}. \tag{2.16}$$

If $\widetilde{\sigma}_j \geq \sigma_j$, then by inequality (2.12)

$$\|f\|_j^{(\widetilde{\sigma}_j)} \leq 2^{\frac{\widetilde{\sigma}_j - \sigma_j}{p_j}} \|f\|_j^{(\sigma_j)}.$$

If $\widetilde{\sigma}_j < \sigma_j$, then by inequality (2.16)

$$\|f\|_j^{(\widetilde{\sigma}_j)} \leq c_3^{\sigma_j - \widetilde{\sigma}_j}(p_j, l_j, \sigma_j)\|f\|_j^{(\sigma_j)}.$$

Hence

$$\|f\|_j^{(\widetilde{\sigma_j})} \leq A_2 \|f\|_j^{(\sigma_j)}, \tag{2.17}$$

where

$$A_2 = \max\left(\max_{j, \widetilde{\sigma}_j \geq \sigma_j} 2^{\frac{\widetilde{\sigma}_j - \sigma_j}{p_j}}, \max_{j, \widetilde{\sigma}_j < \sigma_j} c_3^{\sigma_j - \widetilde{\sigma}_j}\right).$$

If $\widetilde{\sigma}_j \geq \sigma_j$, then by inequality (2.16)

$$c_3^{\sigma_j-\widetilde{\sigma}_j}(p_j, l_j, \sigma_j)\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma}_j)}.$$

If $\widetilde{\sigma}_j < \sigma_j$, then by inequality (2.12) we have

$$2^{\frac{\widetilde{\sigma}_j-\sigma_j}{p_j}}\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma}_j)}.$$

Hence

$$A_1\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma_j})}, \tag{2.18}$$

where

$$A_1 = \min\left(\min_{j,\widetilde{\sigma}_j<\sigma_j} 2^{\frac{\widetilde{\sigma}_j-\sigma_j}{p_j}}, \min_{j,\widetilde{\sigma}_j\geq\sigma_j} c_3^{\sigma_j-\widetilde{\sigma}_j}\right).$$

Consequently

$$A_1\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma_j})} \leq A_2\|f\|_j^{(\sigma_j)}. \tag{2.19}$$

Case $0 < \theta_j < +\infty$. Now consider, for all $\delta > 0$, the quantity

$$\Psi_j(\delta) = \left[\int_{h\geq\delta}\left(\frac{\|\Delta_{h,j}^{\sigma_j}f\|_{L_{p_j}(\mathbb{R}^n)}}{h^{l_j}}\right)^{\theta_j}\frac{dh}{h}\right]^{\frac{1}{\theta_j}}.$$

Clearly $\Psi_j(\delta) < +\infty$ for all $\delta > 0$.

If $1 \leq \theta_j < \infty$, by inequalities (2.14) and (2.8) we get

$$\Psi_j(\delta) \leq 2^{l_j-\sigma_j}(1+\epsilon_j)\Psi_j(2\delta) + c_2(\sigma_j, p_j, \epsilon_j)\|f\|^{(\sigma_j+1)},$$

with the same value of ϵ_j similar arguments leads to inequality (2.16) and consequently we obtain inequality (2.19).

If $0 < \theta_j < 1$, then by [3] we get

$$\Psi_j(\delta) \leq (1+\epsilon_j)^2 2^{(l_j-\sigma_j)}\Psi_j(2\delta) + c_1(\theta_j, \epsilon_j)c_2(p_j, \sigma_j, \epsilon_j)\|f\|^{(\sigma_j+1)},$$

which leads to

$$\|f\|^{(\sigma_j)} \leq c_2 c_1(\theta_j, \epsilon_j)(1-(1+\epsilon_j)^2 2^{l_j-\sigma_j})^{-1}\|f\|^{(\sigma_j+1)}. \tag{2.20}$$

If $\epsilon_j > 0$ is such that $2^{l_j-\sigma_j}(1+\epsilon_j)^2 < 1$ say if $\epsilon_j = \sqrt{2^{\sigma_j-l_j-1}+2^{-1}} - 1$, then we have

$$c_1(\theta_j, \epsilon_j) = \left[1-(1+\epsilon_j)^{\frac{\theta_j}{\theta_j-1}}\right]^{\frac{\theta_j-1}{\theta_j}} = \left[1-(2^{\sigma_j-l_j-1}+2^{-1})^{\frac{\theta_j}{2(\theta_j-1)}}\right]^{\frac{\theta_j-1}{\theta_j}},$$

$$c_2 = \sigma_j^{-1}(\frac{2}{\sigma_j})^{\frac{-1}{p_j}}\left[(2^{\frac{1}{p_j}-1}+1)^{\sigma_j}-1\right] \times \left[1-(2^{\sigma_j-l_j-1}+2^{-1}+2^{-1})^{\frac{p_j}{2(p_j-1)}}\right]^{\frac{p_j-1}{p_j}} > 1,$$

and

$$[1-(1+\epsilon_j)^2 2^{l_j-\sigma_j}]^{-1} = [1-(2^{\sigma_j-l_j-1}+2^{-1})2^{l_j-\sigma_j}]^{-1} = 2(1-2^{l_j-\sigma_j})^{-1} < 1.$$

Let $c_4(p_j,\theta_j,l_j,\sigma_j) = c_2(p_j,\sigma_j,\epsilon_j)c_1(\theta_j,l_j)2(1-2^{l_j-\sigma_j})^{-1}$. Thus

$$\|f\|_j^{(\sigma_j)} \leq c_4(p_j,\theta_j,l_j,\sigma_j)\|f\|_j^{(\sigma_j+1)}, \tag{2.21}$$

and by (2.12) and (2.21) we obtain

$$A_3\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma_j})} \leq A_4\|f\|_j^{(\sigma_j)}, \tag{2.22}$$

where

$$A_3 = \min\left(\min_{j,\widetilde{\sigma}_j<\sigma_j} 2^{\frac{\widetilde{\sigma}_j-\sigma_j}{p_j}}, \min_{j,\widetilde{\sigma}_j\geq\sigma_j} c_4^{\sigma_j-\widetilde{\sigma}_j}\right),$$

and

$$A_4 = \max\left(\max_{j,\widetilde{\sigma}_j\geq\sigma_j} 2^{\frac{\widetilde{\sigma}_j-\sigma_j}{p_j}}, \max_{j,\widetilde{\sigma}_j<\sigma_j} c_4^{\sigma_j-\widetilde{\sigma}_j}\right).$$

2. If $p_j \geq 1$ we apply the standard Minkovski inequality to identity (2.13) instead of inequality (2.8), (see [4], p. 205) then instead of (2.14) we get

$$\|\Delta_{h,j}^{\sigma_j} f\|_{L_{p_j}(\mathbb{R}^n)} \leq 2^{-\sigma_j}\|\Delta_{2h,j}^{\sigma_j} f\|_{L_{p_j}(\mathbb{R}^n)} + 2^{-1}(2^{\sigma_j}-1)\|\Delta_{h,j}^{\sigma_j+1} f\|_{L_{p_j}(\mathbb{R}^n)}, \tag{2.23}$$

If $1 \leq \theta_j \leq \infty$, we have

$$\|f\|_j^{(\sigma_j)} \leq c_5(p_j,l_j,\sigma_j)\|f\|_j^{(\sigma_j+1)}, \tag{2.24}$$

where

$$c_5(p_j,l_j,\sigma_j) = [2^{-1}(2^{\sigma_j}-1)](1-2^{l_j-\sigma_j})^{-1}.$$

Since

$$\|\Delta_{h,j}^{\sigma_j+1} f\|_{p_j} \leq 2\|\Delta_{h,j}^{\sigma_j} f\|_{p_j},$$

it follows that

$$\|f\|_j^{(\sigma_j+1)} \leq 2\|f\|_j^{(\sigma_j)}, \tag{2.25}$$

If $\widetilde{\sigma}_j \geq \sigma_j$, then by inequality (2.25)

$$\|f\|_j^{(\widetilde{\sigma}_j)} \leq 2^{\widetilde{\sigma}_j-\sigma_j}\|f\|_j^{(\sigma_j)}.$$

If $\widetilde{\sigma}_j < \sigma_j$, then by inequality (2.24)

$$\|f\|_j^{(\widetilde{\sigma}_j)} \leq c_5^{\sigma_j-\widetilde{\sigma}_j}(p_j,l_j,\sigma_j)\|f\|_j^{(\sigma_j)}.$$

Hence

$$\|f\|_j^{(\widetilde{\sigma_j})} \leq A_6\|f\|_j^{(\sigma_j)}, \tag{2.26}$$

where

$$A_6 = \max\left(\max_{j,\widetilde{\sigma}_j\geq\sigma_j} 2^{\widetilde{\sigma}_j-\sigma_j}, \max_{j,\widetilde{\sigma}_j<\sigma_j} c_5^{\sigma_j-\widetilde{\sigma}_j}\right).$$

If $\widetilde{\sigma}_j \geq \sigma_j$, then by inequality (2.24)

$$c_5^{\sigma_j-\widetilde{\sigma}_j}(p_j, l_j, \sigma_j)\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma}_j)}.$$

thus

$$\min_j c_5^{\sigma_j-\widetilde{\sigma}_j}(p_j, l_j, \sigma_j)\|f\|^{(\sigma)} \leq \|f\|^{(\widetilde{\sigma})}.$$

If $\widetilde{\sigma}_j < \sigma_j$, then by inequality (2.25) we have

$$2^{\widetilde{\sigma}_j-\sigma_j}\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma}_j)}.$$

so

$$\min_j 2^{\widetilde{\sigma}_j-\sigma_j}\|f\|^{(\sigma)} \leq \|f\|^{(\widetilde{\sigma})}.$$

Hence

$$A_5\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma_j})}, \tag{2.27}$$

where

$$A_5 = \min\left(\min_{j,\widetilde{\sigma}_j<\sigma_j} 2^{\widetilde{\sigma}_j-\sigma_j}, \min_{j,\widetilde{\sigma}_j\geq\sigma_j} c_5^{\sigma_j-\widetilde{\sigma}_j}\right).$$

Consequently

$$A_5\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma_j})} \leq A_6\|f\|_j^{(\sigma_j)}. \tag{2.28}$$

If $0 < \theta_j < 1$, then by (2.8) it follows from (2.23) that

$$\Psi_j(\delta) \leq 2^{l_j-\sigma_j}(1+\epsilon_j)\Psi_j(2\delta) + 2^{-1}(2^{\sigma_j}-1)c_1(\theta_j,\epsilon_j)\|f\|^{(\sigma_j+1)},$$

where $\epsilon_j > 0$ is such that $2^{l_j-\sigma_j}(1+\epsilon_j) < 1$, say if $\epsilon_j = 2^{-1}(2^{\sigma_j-l_j}-1)$, in which case $1+\epsilon_j = 2^{-1}(2^{\sigma_j-l_j}+1)$, and we get the following analogue of inequality (2.24)

$$\|f\|_j^{(\sigma_j)} \leq c_6(\theta_j, l_j, \sigma_j)\|f\|_j^{(\sigma_j+1)}, \tag{2.29}$$

where

$$\begin{aligned} c_6(\theta_j, l_j, \sigma_j) &= 2^{-1}(2^{\sigma_j}-1)c_1(\theta_j,\epsilon_j)2^{(\sigma_j+1)}(2^{\sigma_j}-2^{l_j})^{-1} \\ &= 2^{\sigma_j}(2^{\sigma_j}-1)(2^{\sigma_j}-2^{l_j})^{-1}\left[1-(2^{-1}+2^{\sigma_j-l_j})^{\frac{\theta_j}{\theta_j-1}}\right]^{\frac{\theta_j-1}{\theta_j}}. \end{aligned}$$

Similar arguments lead to the following inequalities

$$A_7\|f\|_j^{(\sigma_j)} \leq \|f\|_j^{(\widetilde{\sigma}_j)} \leq A_8\|f\|_j^{(\sigma_j)}, \tag{2.30}$$

where

$$A_7 = \min\left(\min_{j,\widetilde{\sigma}_j<\sigma_j} 2^{\widetilde{\sigma}_j-\sigma_j}, \min_{j,\widetilde{\sigma}_j\geq\sigma_j} c_6^{\sigma_j-\widetilde{\sigma}_j}\right),$$

and

$$A_8 = \max\left(\max_{j,\widetilde{\sigma}_j\geq\sigma_j} 2^{\widetilde{\sigma}_j-\sigma_j}, \max_{j,\widetilde{\sigma}_j<\sigma_j} c_6^{\sigma_j-\widetilde{\sigma}_j}\right).$$

So, for all $0 < p_j \le \infty$, if $1 \le \theta_j \le \infty$,

$$\min(A_1, A_5)\|f\|_j^{(\sigma_j)} \le \|f\|_j^{(\widetilde{\sigma}_j)} \le \max(A_2, A_6)\|f\|_j^{(\sigma_j)},$$

if $0 < \theta_j < 1$,

$$\min(A_3, A_7)\|f\|_j^{(\sigma_j)} \le \|f\|_j^{(\widetilde{\sigma}_j)} \le \max(A_4, A_8)\|f\|_j^{(\sigma_j)}.$$

Thus for all $0 < \theta_j \le \infty$, $0 < p_j \le \infty$, we have

$$\min(A_1, A_3, A_5, A_7)\|f\|_j^{(\sigma_j)} \le \|f\|_j^{(\widetilde{\sigma}_j)} \le \max(A_2, A_4, A_6, A_8)\|f\|_j^{(\sigma_j)}.$$

Finally we obtain

$$\min(A_1, A_3, A_5, A_7)\sum_{j=1}^{n}\|f\|_j^{(\sigma_j)} \le \sum_{j=1}^{n}\|f\|_j^{(\widetilde{\sigma}_j)} \le \max(A_2, A_4, A_6, A_8)\sum_{j=1}^{n}\|f\|_j^{(\sigma_j)}.$$

□

2.2 Equivalence of the quasi-norms $\|.\|_{B^l_{p,\theta}(\mathbb{R}^n)}$, $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}$

Lemma 2. *Let* $l = (l_1, ..., l_n)$, $\sigma = (\sigma_1, ..., \sigma_n)$, $\sigma_j \in \mathbb{N}$, $p = (p_1, ..., p_n)$, $\theta = (\theta_1, ..., \theta_n)$, $\sigma > l > 0$, $0 < p, \theta \le \infty$. *Then the quasi-norms* $\|.\|_{B^l_{p,\theta}(\mathbb{R})^n}$ *and* $\|.\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R})^n}$ *are equivalent.*

To prove this lemma we need the following statement.

Lemma 3. *Let* $\sigma_j \in \mathbb{N}$, $0 < p_j \le \infty$. *Then there exist* $c_8 > 0$ *depending only on* p_j, σ_j, θ_j *such that for all* $\delta > 0$ *and* $f \in L_p(\mathbb{R}^n)$

$$\omega_j^{(\sigma_j)}(f, \delta)_{p_j} \le c_8 \int_0^\delta \|\Delta^\delta_{\eta,j} f\|_{p_j} \frac{d\eta}{\eta}. \tag{2.31}$$

where $c_8 = c_7\left(1 - \frac{1}{3\sigma_j}\right)$, and $c_7 = \sigma_j^{(\frac{1}{p_j}-1)} 3^{\frac{1}{\theta_j}} 2^{(\frac{1}{p_j}-1)+\sigma_j+\frac{\sigma_j+2}{\theta_j}}$. For $p_j \ge 1$ (see [4], [2]) and for $0 < p_j < 1$(see [3], p. 9).

Proof of Lemma 2. Let $0 < p_j$, $\theta_j \le \infty$. We have for all $h > 0$

$$\frac{\|\Delta^{\sigma_j}_{h,j} f\|_{p_j}}{h^{l_j}} \le \frac{\omega_j^{(\sigma_j)}(f, h)_{p_j}}{h^{l_j}},$$

hence for all $j = \overline{1, n}$

$$\|f\|_{B^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} \le \|f\|_{\widetilde{B}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)},$$

and

$$\|f\|_{B^l_{p,\theta}(\mathbb{R}^n)} \le \|f\|_{\widetilde{B}^l_{p,\theta}(\mathbb{R}^n)}.$$

By Lemma 3 and the Hardy-type inequality for $0 < \theta_j < 1$ (see [7], p. 114), we get

$$\|f\|_{\tilde{b}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} = \left[\int_0^\infty \left(\frac{\omega_j^{(\sigma_j)}(f,\delta)_{p_j}}{\delta^{l_j}}\right)^{\theta_j} \frac{d\delta}{\delta}\right]^{\frac{1}{\theta_j}}$$
$$\leq c_8 \left\| \delta^{\alpha_j} \frac{1}{\delta}\int_0^\delta \|\Delta^{\sigma_j}_{h,j} f\|_{p_j} \frac{dh}{h}\right\|_{L_{\theta_j}(0,\infty)},$$
$$\leq c_8 c_9 \left\| h^{\alpha_j} \left\|\Delta^{\sigma_j}_{h,j} f\right\|_{p_j}\right\|_{L_{\theta_j}(0,\infty)},$$

where $\alpha_j = 1 - l_j - \frac{1}{\theta_j}$ and $c_9 = (l_j\theta_j)^{\frac{-1}{\theta_j}} \left[c_7\left(1-\frac{1}{3\sigma_j}\right)^{1-\frac{1}{\theta_j}}\right]^{1-\theta_j}$.
Let $c_{10} = c_8 c_9$, then

$$\|f\|_{\tilde{b}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} \leq c_{10}\|f\|_{b^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)}.$$

Consequently

$$\|f\|_{\tilde{B}^l_{p,\theta}(\mathbb{R}^n)} \leq A_9\|f\|_{B^l_{p,\theta}(\mathbb{R}^n)},$$

where

$$A_9 = \max\left(1, \max_{j=1,..,n} c_{10}\right).$$

For the case $\theta_j \geq 1$ we apply Lemma 3 and the standard Hardy inequality (see [4], p. 208) and similar arguments lead to the analogous result

$$\|f\|_{\tilde{b}^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)} \leq c_{11}\|f\|_{b^{l_j}_{p_j,\theta_j;j}(\mathbb{R}^n)},$$

where

$$c_{11} = \sigma_j^{(\frac{1}{p_j}-1)} 3^{\frac{1}{\theta_j}} 2^{(\frac{1}{p_j}-1)+\sigma_j+\frac{\sigma_j+2}{\theta_j}} \left(1-\frac{1}{3\sigma_j}\right)(l_j)^{-1}.$$

Hence

$$\|f\|_{\tilde{B}^l_{p,\theta}(\mathbb{R}^n)} \leq A_{10}\|f\|_{B^l_{p,\theta}(\mathbb{R}^n)},$$

where $A_{10} = \max(1, \max_{j=1,..,n} c_{11})$. □

Remark. If $p_j = \theta_j, j = \overline{1,n}$, then the proof of Theorem 2 can be reduced, by applying the Fubini theorem, to the one-dimensional case (Theorem 1 with $n = 1$). Indeed, denote

$$\|f\|^{(\sigma_j)} = \|f\|_{L_{p_j}(\mathbb{R}^n)} + \|f\|_{b^{l_j}_{p_j,p_j;j}(\mathbb{R}^n)}$$
$$= \left(\int_{\mathbb{R}^n} |f(x)|^{p_j} dx\right)^{\frac{1}{p_j}} + \left(\int_0^\infty \left(h^{-l_j}\|\Delta^{\sigma_j}_{h,j} f\|_{p_j}\right)^{p_j} \frac{dh}{h}\right)^{\frac{1}{p_j}}.$$

Since for all $a, b > 0, 0 < p < \infty$

$$\min(2^{p-1}, 1)(a^p + b^p) \leq (a + b)^p \leq \max(2^{p-1}, 1)(a^p + b^p), \tag{2.32}$$

we have [1]

$$(\|f\|^{(\sigma_j)})^{p_j} \approx \int_{\mathbb{R}^n} |f(x)|^{p_j} dx + \left(\int_0^\infty \left(h^{-p_j l_j} \int_{\mathbb{R}^n} |\Delta_{h,j}^{\sigma_j} f(x)|^{p_j} dx \right) \frac{dh}{h} \right). \tag{2.33}$$

Next

$$\|f\|^{(\sigma_j)} \approx \left(\int_{\mathbb{R}^n} |f(x)|^{p_j} dx + \int_0^\infty \left(h^{-p_j l_j} \int_{\mathbb{R}^n} |\Delta_{h,j}^{\sigma_j} f(x)|^{p_j} dx \right) \frac{dh}{h} \right)^{\frac{1}{p_j}}$$
$$\approx \left[\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |f(\overline{x_j}, x_j)|^{p_j}\, dx_j \right) d\overline{x_j} \right.$$
$$\left. + \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty h^{-p_j l_j} \int_{\mathbb{R}} |\Delta_{h,j}^{\sigma_j} f(\overline{x_j}, x_j)|^{p_j}\, dx_j \frac{dh}{h} \right) d\overline{x_j} \right]^{\frac{1}{p_j}}$$
$$\approx \left(\int_{\mathbb{R}^{n-1}} \left(\|f\|^{(\sigma_j)}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R})} \right)^{p_j} d\overline{x_j} \right)^{\frac{1}{p_j}},$$

where $\overline{x_j} = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$. Since

$$\|f\|^{(\sigma_j)}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R})} \approx \|f\|^{(\sigma_j')}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R})}, \quad \sigma_j \neq \sigma_j', \;\; \sigma_j' \in \mathbb{N}, \;\; \sigma_j, \sigma_j' > l_j,$$

we have

$$\|f\|^{(\sigma_j)} \approx \left(\int_{\mathbb{R}^{n-1}} \left(\|f\|^{(\sigma_j)}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R})} \right)^{p_j} d\overline{x_j} \right)^{\frac{1}{p_j}}$$
$$\approx \left(\int_{\mathbb{R}^{n-1}} \left(\|f\|^{(\sigma_j')}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R})} \right)^{p_j} d\overline{x_j} \right)^{\frac{1}{p_j}} \approx \|f\|^{(\sigma_j')}.$$

Consequently we obtain,

$$\sum_{j=1}^n \|f\|^{(\sigma_j)}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R}^n)} \approx \sum_{j=1}^n \|f\|^{(\sigma_j')}_{B^{l_j}_{p_j,p_j;j}(\mathbb{R}^n)}. \tag{2.34}$$

Thus, Statement 1 of Theorem 2 follows. Statements 2 and 3 can be proved in a similar way.

Note that if $p_j \neq \theta_j$, for at least one j, then this argument does not work.

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[1] The symbol $\approx$ denotes the equivalence of norms (quasi-norms).

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Benali Halim
Departement of Mathematics
University Abdelhamid Ibn Badis,
Mostaganem, Algeria
E-mail: benalihalim19@yahoo.fr

Abdelkader Senouci
Departement of Mathematics
Ibn Khaldoun University
Tiaret, Algeria
E-mail: kamer295@yahoo.fr

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