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The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 70th birthday)

On May 5, 2016 was the 70th birthday of Tynysbek Sharipovich Kal'menov, member of the Editorial Board of the Eurasian Mathematical Journal, general director of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, laureate of the Lenin Komsomol Prize of the Kazakh SSR (1978), doctor of physical and mathematical sciences (1983), professor (1986), honoured worker of science and technology of the Republic of Kazakhstan (1996), academician of the National Academy of Sciences (2003), laureate of the State Prize in the field of science and technology (2013).

T.Sh. Kal'menov was born in the South-Kazakhstan region of the Kazakh SSR. He graduated from the Novosibirsk State University (1969) and completed his postgraduate studies there in 1972.

He obtained seminal scientific results in the theory of partial differential equations and in the spectral theory of differential operators.

For the Lavrentiev-Bitsadze equation T.Sh. Kal'menov proved the criterion of strong solvability of the Tricomi problem in the L_p-spaces. He described all well-posed boundary value problems for the wave equation and equations of mixed type within the framework of the general theory of boundary value problems.

He solved the problem of existence of an eigenvalue of the Tricomi problem for the Lavrentiev-Bitsadze equation and the general Gellerstedt equation on the basis of the new extremum principle formulated by him.

T.Sh. Kal'menov proved the completeness of root vectors of main types of Bitsadze-Samarskii problems for a general elliptic operator. Green's function of the Dirichlet problem for the polyharmonic equation was constructed. He established that the spectrum of general differential operators, generated by regular boundary conditions, is either an empty or an infinite set. The boundary conditions characterizing the volume Newton potential were found. A new criterion of well-posedness of the mixed Cauchy problem for the Poisson equation was found.

On the whole, the results obtained by T.Sh. Kal'menov have laid the groundwork for new perspective scientific directions in the theory of boundary value problems for hyperbolic equations, equations of the mixed type, as well as in the spectral theory.

More than 50 candidate of sciences and 9 doctor of sciences dissertations have been defended under his supervision. He has published more than 120 scientific papers. The list of his basic publications can be viewed on the web-page

https://scholar.google.com/citations?user=Zay4fxkAAAAJ&hl=ru&authuser=1

The Editorial Board of the Eurasian Mathematical Journal congratulates Tynysbek Sharipovich Kal'menov on the occasion of his 70th birthday and wishes him good health and new creative achievements!

Short communications

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YOUNG'S INEQUALITY FOR CONVOLUTIONS IN MORREY-TYPE SPACES

V.I. Burenkov, T.V. Tararykova

Communicated by V.S. Guliyev

Key words: convolutions of functions, local and global Morrey-type spaces.

AMS Mathematics Subject Classification: 47H30, 46E35.

Abstract. An analogue of the classical Young's inequality for convolutions of functions is proved in the case of the general global Morrey-type spaces. The form of this analogue is different from Young's inequality for convolutions in the case of the Lebesgue spaces.

1 Introduction

Over the last three decades, the general local and global Morrey-type spaces have been in the focus of many studies. In particular, for a certain range of the numerical parameters $0 < p_1, p_2, \theta_1, \theta_2 \leq \infty$ of the general local Morrey-type spaces $LM_{p_1\theta_1,w_1(\cdot)}$ and $LM_{p_2\theta_2,w_2(\cdot)}$, necessary and sufficient conditions on the functional parameters w_1 and w_2 have been obtained under which the maximal operator, the fractional maximal operator, the Riesz potential, genuine singular integral operators, and the Hardy operator are bounded as operators acting from the space $LM_{p_1\theta_1,w_1(\cdot)}$ to the space $LM_{p_2\theta_2,w_2(\cdot)}$.

The recent survey papers [1, 2, 6, 7, 10, 11, 9, 5, 8] describe in detail the history of studying general Morrey-type spaces, the present state of the operator theory in the general Morrey-type spaces and various applications of this theory.

One of the most common definitions of the general Morrey-type spaces is as follows. Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ of radius $r > 0$ with center at a point $x \in \mathbb{R}^n$.

Definition 1. Let $0 < p, \theta \leq \infty$, and let w be a nonnegative Lebesgue measurable function on the half-axis $(0,\infty)$ that is not equivalent to zero. The *local Morrey-type space* $LM_{p\theta,w(\cdot)} \equiv LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ with finite quasinorm

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)}.$$

The *global Morrey-type space* $GM_{p\theta,w(\cdot)} \equiv GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ with finite quasinorm

$$\|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)}.$$

Remark 1. If the function w is equivalent to zero (in short, $w \sim 0$) on (t,∞) for some $t>0$, then we set

$$b = \inf\{t>0:\ w \sim 0 \,\text{on}\, (t,\infty)\}.$$

If $w(r) = 0$ and $\|f\|_{L_p(B(x,r))} = \infty$, then we assume that $w(r)\|f\|_{L_p(B(x,r))} = 0$. Under this agreement,

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,b)}, \qquad \|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n}\big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,b)}.$$

In the case of the local Morrey-type spaces (in contrast to the global Morrey-type spaces), the finiteness of $\|f\|_{LM_{p\theta,w(\cdot)}}$ does not impose any constraints on the behavior of the function f for $|x|\geq b$. For definiteness, we assume that $f(x)=0$ for $|x|\geq b$.

If $\theta=\infty$ and $w(\cdot)\equiv 1$, then $LM_{p\infty,1} = GM_{p\infty,1} = L_p(\mathbb{R}^n)$, while if $\theta=\infty$ and $w(r)=r^{-\lambda}$, $0\leq\lambda\leq n/p$, then

$$GM_{p\infty,r^{-\lambda}} \equiv M_p^\lambda$$

is the classical Morrey space and

$$LM_{p\infty,r^{-\lambda}} \equiv LM_p^\lambda$$

is a local version of the Morrey space.

The spaces M_p^λ are nontrivial (i.e., they consist not only of functions equivalent to zero on $\mathbb{R}^n$) if and only if $0\leq\lambda\leq n/p$. The spaces LM_p^λ are nontrivial if and only if $\lambda\geq 0$. For $\lambda=0$, we have $LM_p^0 = M_p^0 = L_p$. For $\lambda = n/p$, we have $M_p^{n/p} = L_\infty$.

The first natural question concerning the general Morrey-type spaces is to find out for what functions w the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$ are nontrivial. To answer this question, we need the following definition.

Definition 2. Let $0<p,\theta\leq\infty$. Then Ω_θ is the set of all functions w that are nonnegative, Lebesgue measurable on $(0,\infty)$, not equivalent to zero, and are such that

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty$$

for some $t>0$. Further, $\Omega_{p\theta}$ is the set of all functions w that are nonnegative, Lebesgue measurable on $(0,\infty)$, not equivalent to zero, and are such that for some $t>0$

$$\|w(r)r^{n/p}\|_{L_\theta(0,t)} < \infty, \qquad \|w(r)\|_{L_\theta(t,\infty)} < \infty\,.$$

Let

$$a = \inf\big\{t>0:\ \|w\|_{L_\theta(t,\infty)} < \infty\big\}.$$

Note that if $w\in\Omega_{p\theta}$, then $a=0$.

Lemma 1.1 ([3], [4]). *Let $0<p,\theta\leq\infty$, and let w be a nonnegative Lebesgue measurable function on $(0,\infty)$ that is not equivalent to zero.*

The space $LM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w\in\Omega_\theta$, and the space $GM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w\in\Omega_{p\theta}$.

Moreover, if $w\in\Omega_\theta$, then the space $LM_{p\theta,w(\cdot)}$ contains all functions $f\in L_p(\mathbb{R}^n)$ that vanish on $B(0,t)$ for some $t>a$.

If $w\in\Omega_{p\theta}$, then $L_p(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)\subset GM_{p\theta,w(\cdot)}$.

In this note we formulate analogues of the classical Young's inequality for convolutions (Section 2) and truncated convolutions (Section 3) in the case of the general global Morrey-type spaces. The form of these analogues is different from Young's inequality for convolutions in the case of the Lebesgue spaces.

2 An analogue of Young's inequality for convolutions

Let f_1 and f_2 be measurable functions and

$$(f_1 * f_2)(x) = \int_{\mathbb{R}^n} f_1(x-y) f_2(y)\, dy, \qquad x \in \mathbb{R}^n,$$

be the convolution of these functions.

In this section, we formulate an analogue of Young's inequality for convolutions in the Lebesgue spaces:

$$\|f_1 * f_2\|_{L_p} \leq \|f_1\|_{L_{p_1}} \|f_2\|_{L_{p_2}} \tag{2.1}$$

for all $f_k \in L_{p_k}$, $k = 1, 2$, where

$$1 \leq p_1, p_2 \leq p \leq \infty, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1. \tag{2.2}$$

If $1 \leq p_2 = p \leq \infty$, then the inequality takes the form

$$\|f_1 * f_2\|_{L_p} \leq \|f_1\|_{L_1} \|f_2\|_{L_p}. \tag{2.3}$$

Applying twice the generalized Minkowski inequality for integrals, one can easily prove that if $1 \leq p, \theta \leq \infty$ and $w \in \Omega_{p\theta}$, then

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \leq \|f_1\|_{L_1} \|f_2\|_{GM_{p\theta,w(\cdot)}} \tag{2.4}$$

for all $f_1 \in L_1$ and $f_2 \in GM_{p\theta,w(\cdot)}$; in particular, for any $0 \leq \lambda \leq n/p$ and all $f_1 \in L_1$ and $f_2 \in M_p^\lambda$,

$$\|f_1 * f_2\|_{M_p^\lambda} \leq \|f_1\|_{L_1} \|f_2\|_{M_p^\lambda}. \tag{2.5}$$

These are direct analogues of Young's inequality (2.3) (L_p is replaced by $GM_{p\theta,w(\cdot)}$, M_p^λ respectively).

Remark 2. In inequality (2.4), one cannot replace the global space $GM_{p\theta,w(\cdot)}$ by the local space $LM_{p\theta,w(\cdot)}$ even if one adds a constant factor independent of f_1 and f_2 to the right-hand side. In particular, for any $0 < p \leq \infty$, $\lambda > 0$, and any $A > 0$, the inequality

$$\|f_1 * f_2\|_{LM_p^\lambda} \leq A \|f_1\|_{L_1} \|f_2\|_{LM_p^\lambda} \tag{2.6}$$

with arbitrary $f_1 \in L_p$ and $f_2 \in LM_p^\lambda$ fails, as is shown in the following example.[2]

Let $n = 1$ and

$$f_{1k} = \chi_{[-k-1,-k]}, \quad f_{2k} = \chi_{[k,k+1]}, \quad k \in \mathbb{N}.$$

Replacing f_1 with f_{1k} and f_2 with f_{2k} in (2.6), one can verify that this inequality is impossible.

[2] This example was proposed by E.D. Nursultanov.

Remark 3. In the case of the global spaces $GM_{p\theta,w(\cdot)}$, for any p_1, p_2, and p satisfying condition (2.2), the direct analogue of Young's inequality

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{L_{p_1}} \|f_2\|_{GM_{p_2\theta,w(\cdot)}}$$

fails for $p_1 > 1$ even if one adds a constant factor independent of f_1 and f_2 to the right-hand side. In particular, for any $A > 0$, the inequality

$$\|f_1 * f_2\|_{M_p^\lambda} \le A \|f_1\|_{L_{p_1}} \|f_2\|_{M_{p_2}^\lambda} \tag{2.7}$$

with arbitrary $f_1 \in L_{p_1}$ and $f_2 \in M_{p_2}^\lambda$ fails to hold.

This is obvious if $n/p < \lambda \le n/p_2$. Indeed, it follows from (2.7) that $f_1 * f_2 \in M_p^\lambda$ with $\lambda > n/p$, which implies that the convolution $f_1 * f_2$ is equivalent to zero on $\mathbb{R}^n$ for all $f_1 \in L_{p_1}$ and $f_2 \in M_{p_2}^\lambda$, but this is impossible.

For $n = 1$ and any $0 < \lambda \le 1/p$, this is confirmed by the following example.[3] Let $\alpha = 1/(\lambda p_2)$ and

$$f_1 = \sum_{k=2}^{\infty} k^{-1/p_1} (\ln k)^{-2/p_1} \chi_{[-k^\alpha - 1, -k^\alpha + 1]}, \quad f_2 = \sum_{k=2}^{\infty} \chi_{[k^\alpha, k^\alpha + 1]}.$$

It is obvious that $f_1 \in L_{p_1}$ and $f_2 \notin L_{p_2}$.One can verify that $f_2 \in M_{p_2}^\lambda$ and that $\|f_1 * f_2\|_{M_p^\lambda} = \infty$ and, more generally, $\|f_1 * f_2\|_{M_q^\nu} = \infty$ for all $0 < q \le \infty$ and $0 \le \nu \le 1/q$.

For this reason, in the following statement we additionally assume that $f_1 \in L_{p_1}$ and $f_2 \in L_{p_2}$.

Theorem 2.1. *Let*

$$1 \le p_1, p_2 \le p \le \infty, \quad \frac{p_1 p_2}{p} \le \theta_1, \theta_2 \le \infty, \quad 0 \le \alpha_1, \alpha_2 \le 1,$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \quad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = \frac{1}{p}, \quad \frac{\alpha_1}{\theta_1} + \frac{\alpha_2}{\theta_2} = \frac{1}{\theta}.$$

Let, next, $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, *and*

$$w(r) = w_1^{\alpha_1}(r) w_2^{\alpha_2}(r), \quad r > 0.$$

Then $w \in \Omega_{p\theta}$, *for all* $f_k \in GM_{p_k\theta_k,w_k(\cdot)} \cap L_{p_k}$, $k = 1, 2$, *the convolution* $f_1 * f_2$ *exists almost everywhere on* $\mathbb{R}^n$, *and*

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{GM_{p_1\theta_1,w_1(\cdot)}}^{\alpha_1} \|f_1\|_{L_{p_1}}^{1-\alpha_1} \|f_2\|_{GM_{p_2\theta_2,w_2(\cdot)}}^{\alpha_2} \|f_2\|_{L_{p_2}}^{1-\alpha_2}. \tag{2.8}$$

Let us distinguish the following particular cases of inequality (2.8).

1. If $\alpha_1 = 0$ and $p_1 = 1$, then $p_2 = p$, $\alpha_2 = 1$, $\theta_2 = \theta$, and $w_2(\cdot) = w(\cdot)$, and this is inequality (2.4). In this case, it suffices to assume that $f_2 \in GM_{p\theta,w(\cdot)}$.

[3] This example was also proposed by E.D. Nursultanov.

2. If $\alpha_1 = 0$ and $p_1 > 1$, then $p_2 < p$, $\alpha_2 = p_2/p$, and $\theta_2 = (p_2/p)\theta$, where $p_1 \le \theta \le \infty$, $w_2(\cdot) = w^{p/p_2}(\cdot)$, and

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{L_{p_1}} \|f_2\|^{p_2/p}_{GM_{p_2,(p_2/p)\theta,w^{p/p_2}(\cdot)}} \|f_2\|^{1-p_2/p}_{L_{p_2}} \tag{2.9}$$

for $w \in \Omega_{p\theta}$.

3. If $\theta_1 = \theta_2 = \theta = \infty$, $0 \le \lambda_1 \le n/p_1$, $0 \le \lambda_2 \le n/p_2$, $w_1(r) = r^{-\lambda_1}$, and $w_2(r) = r^{-\lambda_2}$, then $w(r) = r^{-(\alpha_1\lambda_1+\alpha_2\lambda_2)}$ and

$$\|f_1 * f_2\|_{M_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \le \|f_1\|^{\alpha_1}_{M_{p_1}^{\lambda_1}} \|f_1\|^{1-\alpha_1}_{L_{p_1}} \|f_2\|^{\alpha_2}_{M_{p_2}^{\lambda_2}} \|f_2\|^{1-\alpha_2}_{L_{p_2}}. \tag{2.10}$$

4. If $\theta_1 = \theta_2 = \infty$, $\alpha_1 = 0$, $\alpha_2 = p_2/p$, and $w_2(r) = r^{-\lambda_2}$, then $\theta = \infty$, $w(r) = r^{-(p_2/p)\lambda_2}$, and

$$\|f_1 * f_2\|_{M_p^{(p_2/p)\lambda_2}} \le \|f_1\|_{L_{p_1}} \|f_2\|^{p_2/p}_{M_{p_2}^{\lambda_2}} \|f_2\|^{1-p_2/p}_{L_{p_2}} \tag{2.11}$$

for $0 \le \lambda_2 \le n/p_2$.

If $f \in GM_{p\theta,w(\cdot)}$ (in particular, if $f \in M_p^\lambda$), then this does not generally imply that $f \in L_p$. For example, if $0 < \lambda < n/p$, then $|x|^{\lambda-n/p} \in M_p^\lambda$, but $|x|^{\lambda-n/p} \notin L_p$.

In this connection, consider the modified global Morrey-type spaces

$$\widehat{GM}_{p\theta,w(\cdot)} = GM_{p\theta,w(\cdot)} \cap L_p$$

with the quasinorm

$$\|f\|_{\widehat{GM}_{p\theta,w(\cdot)}} = \max\{\|f\|_{GM_{p\theta,w(\cdot)}}, \|f\|_{L_p}\},$$

including the spaces

$$\widehat{M}_p^\lambda = M_p^\lambda \cap L_p$$

with the quasinorm

$$\|f\|_{\widehat{M}_p^\lambda} = \max\{\|f\|_{M_p^\lambda}, \|f\|_{L_p}\}.$$

Corollary 2.1. *Under the hypotheses of Theorem 2.1,*

$$\|f_1 * f_2\|_{\widehat{GM}_{p\theta,w(\cdot)}} \le \|f_1\|_{\widehat{GM}_{p_1\theta_1,w_1(\cdot)}} \|f_2\|_{\widehat{GM}_{p_2\theta_2,w_2(\cdot)}}. \tag{2.12}$$

If $\theta_1 = \theta_2 = \theta = \infty$, $0 \le \lambda_1 \le n/p_1$, $0 \le \lambda_2 \le n/p_2$, $w_1(r) = r^{-\lambda_1}$, and $w_2(r) = r^{-\lambda_2}$, then inequality (2.12) takes the form

$$\|f_1 * f_2\|_{\widehat{M}_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \le \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{\widehat{M}_{p_2}^{\lambda_2}}. \tag{2.13}$$

Note that the spaces $\widehat{M}_p^\lambda$ possess the monotonicity property with respect to the parameter λ: if $0 \le \lambda \le \mu \le n/p$, then

$$\widehat{M}_p^\mu \subset \widehat{M}_p^\lambda \quad \text{and} \quad \|f\|_{\widehat{M}_p^\lambda} \le \|f\|_{\widehat{M}_p^\mu}.$$

In inequality (2.13), for fixed p_1, p_2, p, λ_1, and λ_2, the maximal value of the parameter ν for which $f_1 * f_2 \in M_p^\nu$ is equal to $\max\{p_1\lambda_1/p, p_2\lambda_2/p\}$ (the maximum is attained either for $\alpha_1 = 0$ or for $\alpha_2 = 0$). Hence, the "best" inequality among those of form (2.13) is the inequality [4]

$$\|f_1 * f_2\|_{\widehat{M}_p^{\lambda}} \le \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{\widehat{M}_{p_2}^{\lambda_2}} \quad \text{with} \quad \lambda = \max\left\{\frac{p_1\lambda_1}{p}, \frac{p_2\lambda_2}{p}\right\}.$$

3 An analogue of Young's inequality for truncated convolutions

Let $\Omega \subset \mathbb{R}^n$ be an open set and $0 < p, \theta \le \infty$. For a function f defined on Ω, we will denote by f° its extension by zero to $\mathbb{R}^n$. For $w \in \Omega_\theta$, by definition, $f \in LM_{p\theta,w(\cdot)}(\Omega)$ if $f^\circ \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and, accordingly, for $w \in \Omega_{p\theta}$, $f \in GM_{p\theta,w(\cdot)}(\Omega)$ if $f^\circ \in GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. In this case,

$$\|f\|_{LM_{p\theta,w(\cdot)}(\Omega)} \equiv \|f^\circ\|_{LM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p(\Omega\cap B(0,r))}\big\|_{L_\theta(0,\infty)}$$

and

$$\|f\|_{GM_{p\theta,w(\cdot)}(\Omega)} \equiv \|f^\circ\|_{GM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \big\|w(r)\|f\|_{L_p(\Omega\cap B(x,r))}\big\|_{L_\theta(0,\infty)}.$$

In the case of the local spaces $LM_{p\theta,w(\cdot)}(\Omega)$, it is assumed that $0 \in \Omega$.

Consider a "truncated" convolution

$$(f_1 * f_2)_{\Omega_2}(x) = \int_{\Omega_2} f_1(x-y) f_2(y)\, dy$$

for $x \in \Omega_1$, where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are measurable sets, f_2 is a measurable function on Ω_2, and f_1 is a measurable function on $\Omega_1 - \Omega_2 = \{x - y : x \in \Omega_1,\ y \in \Omega_2\}$.

Let f_2° be the zero extension of f_2 to $\mathbb{R}^n \setminus \Omega_2$ and f_1° be the zero extension of f_1 to $\mathbb{R}^n \setminus (\Omega_1 - \Omega_2)$. Then, for $x \in \Omega_1$,

$$(f_1 * f_2)_{\Omega_2}(x) = \int_{\Omega_2} f_1^\circ(x-y) f_2(y)\, dy = \int_{\mathbb{R}^n} f_1^\circ(x-y) f_2^\circ(y)\, dy = (f_1^\circ * f_2^\circ)(x).$$

Therefore under the hypotheses of Theorem 2.1, for all $f_1 \in GM_{p_1\theta_1,w_1(\cdot)}(\Omega_1 - \Omega_2) \cap L_{p_1}(\Omega_1 - \Omega_2)$ and $f_2 \in GM_{p_2\theta_2,w_2(\cdot)}(\Omega_2) \cap L_{p_2}(\Omega_2)$, the convolution $(f_1 * f_2)_{\Omega_2}$ exists almost everywhere on Ω_1 and

$$\|(f_1 * f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} \le \|f_1\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}(\Omega_1-\Omega_2)} \|f_1\|^{1-\alpha_1}_{L_{p_1}(\Omega_1-\Omega_2)} \|f_2\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}(\Omega_2)} \|f_2\|^{1-\alpha_2}_{L_{p_2}(\Omega_2)}.$$

If $\alpha_1 = p_1/p$, $\alpha_2 = 0$, $\theta_1 = (p_1/p)\theta$, and $w_1(\cdot) = w^{p/p_1}(\cdot)$, then this inequality takes the form

$$\|(f_1 * f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} \le \|f_1\|^{p_1/p}_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-\Omega_2)} \|f_1\|^{1-p_1/p}_{L_{p_1}(\Omega_1-\Omega_2)} \|f_2\|_{L_{p_2}(\Omega_2)}.$$

Tracing the proof of Theorem 2.1, we can sharpen this estimate.

[4] It would be of interest to construct, if possible, functions $f_1 \in \widehat{M}_{p_1}^{\lambda_1}$, $f_2 \in \widehat{M}_{p_2}^{\lambda_2}$ such that $f_1 * f_2 \notin \widehat{M}_p^{\nu}$ for any $\nu > \lambda$.

Theorem 3.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be measurable sets,*

$$1 \le p_1, p_2 \le p \le \infty, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \quad p_2 \le \theta \le \infty,$$

and $w \in \Omega_{p\theta}$. Then the convolution $(f_1 * f_2)_{\Omega_2}$ exists almost everywhere on Ω_1 and

$$\|(f_1 * f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} \le \Big(\sup_{y\in\Omega_2} \|f_1\|_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-y)}\Big)^{p_1/p} \Big(\sup_{x\in\Omega_1} \|f_1\|_{L_{p_1}(x-\Omega_2)}\Big)^{1-p_1/p} \|f_2\|_{L_{p_2}(\Omega_2)} \quad (3.1)$$

for all measurable functions f_1 on $\Omega_1 - \Omega_2$ and f_2 on Ω_2 for which the right-hand side of this inequality is finite.

Remark 4. Since

$$(f_1 * f_2)_{\Omega_2}(x) = \int_{x-\Omega_2} f_2(x-y) f_1(y)\, dy,$$

this does not allow us to obtain a variant of inequality (3.1) in which Ω_1, f_1 and Ω_2, f_2 are interchanged.

Remark 5. Theorem 3.1 remains valid if we define the global Morrey-type spaces $GM_{p\theta,w(\cdot)}$ as the spaces of all measurable functions f on Ω such that

$$\|f\|^{(1)}_{GM_{p\theta,w(\cdot)}(\Omega)} = \sup_{x\in\Omega} \big\| w(r) \|f\|_{L_p(\Omega\cap B(x,r))} \big\|_{L_\theta(0,\infty)} < \infty.$$

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References

[1] V.I. Burenkov, *Recent progress in the problem of boundedness of the classical operators of real analysis in general Morrey-type spaces. I.* Eurasian Math. J. 3 (2012), no. 3, 8-27.

[2] V.I. Burenkov, *Recent progress in the problem of boundedness of the classical operators of real analysis in general Morrey-type spaces. II.* Eurasian Math. J. 4 (2013), no. 1, 21-45.

[3] V.I. Burenkov, H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces.* Studia Mathematica 163 (2004), no. 2, 157-176.

[4] V.I. Burenkov, P. Jain, T.V. Tararykova, *On boundedness of the Hardy operator in Morrey-type spaces.* Eurasian Math. J. 2 (2011), no. 1, 52–80.

[5] V.I. Burenkov, Y. Sawano, *Necessary and sufficient conditions for the boundedness of classical operators of real analysis in general Morrey-type spaces.* Proceedings of the Symposium on real analysis at the Ibaraki University, 2013, 81-90.

[6] V.S. Guliyev, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators.* Eurasian Math. J. 3 (2012), no. 3, 33–61.

[7] P.G. Lemarié-Rieusset, *The role of Morrey spaces in the study of Navier-Stokes and Euler equations.* Eurasian Math. J. 3 (2012), no. 3, 62-93.

[8] H. Rafeiro, N. Samko, S. Samko, *Morrey-Campanato spaces: an overview.* Oper. Theory Adv. Appl. Birkhäuser/Springer, Basel AG 228 (2013), 293-323.

[9] M.A. Ragusa, *Operators in Morrey-type spaces and applications.* Eurasian Math. J. 3 (2012), no. 3, 94-109.

[10] W. Sickel, *Smoothness spaces related to Morrey spaces – a survey. I.* Eurasian Math. J. 3 (2012), no. 3, 110-149.

[11] W. Sickel, *Smoothness spaces related to Morrey spaces – a survey. II.* Eurasian Math. J. 4 (2013), no. 1, 82-124.

Victor Ivanovich Burenkov
Department of Mathematical Analysis
and Theory of Functions
RUDN University
6 Miklukho Maklay St
117198 Moscow, Russia

and

School of Mathematics
Cardiff University
Cardiff CF24 4AG, UK
E-mail: burenkov@cf.ac.uk

Tamara Vasil'evna Tararykova
School of Mathematics
Cardiff University
Cardiff CF24 4AG, UK
E-mail: tararykovat@cf.a.c.uk

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