ISSN 2077–9879

Eurasian Mathematical Journal

2016, Volume 7, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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The Eurasian Mathematical Journal (EMJ)
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The L.N. Gumilyov Eurasian National University
Building no. 3
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Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
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NURZHAN BOKAYEV

(to the 60th birthday)

On January 5, 2016 was the 60th birthday of Doctor of Physical-Mathematical Sciences (1996), Professor Nurzhan Adilkhanovich Bokayev. Professor Bokayev is the head of the department "Higher Mathematics" of the L.N. Gumilyov Eurasian National University (since 2009), the Vice-President of Mathematical Society of the Turkic World (since 2014), and a member of the Editorial Board of our journal.

N.A. Bokayevwas born in the Urnek village, Karabalyk district, Kostanay region. He graduated from the E.A. Buketov Karaganda State University in 1977 and the M.V. Lomonosov Moscow State University's full-time postgraduate study in 1984.

Scientific works of Professor Bokayev are devoted to studying problems of the theory of functions, in particular of the theory of orthogonal series.

He proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems; investigated Besov-type function spaces with respect to the Price bases; studied the Price - and Haar-type p-adic operators; introduced new classes of Faber-Schauder-type systems of functions and spaces of multivariable functions of bounded p-variation and of bounded p-fluctuation, obtained estimates for the best approximation of functions in these spaces by polynomials with respect to the Walsh and Haar systems, established weighted integrability conditions of the sum of multiple trigonometric series and series with respect to multiplicative systems, found the embedding criterion for the Nikol'skii-Besov spaces with respect to multiplicative bases and the coefficient criterion for belonging of functions to such spaces.

His scientific results have made essential contribution to the theory of series with respect to the Walsh and Haar systems and multiplicative systems.

N.A. Bokayev has published more than 150 scientific papers. Under his supervision 8 dissertations have been defended: 4 candidate of sciences dissertations and 4 PhD dissertations.

The Editorial Board of the Eurasian Mathematical Journal congratulates Nurzhan Adilkhanovich Bokayev on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

The EMJ has been included in the Emerging Sources Citation Index

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On behalf of the Editorial Board of the EMJ
V.I. Burenkov, T.V. Tararykova, A.M. Temirkhanova

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 1 (2016), 84 – 99

HARDY-TYPE INEQUALITIES FOR THE FRACTIONAL INTEGRAL OPERATOR IN q-ANALYSIS

S. Shaimardan

Communicated by R. Oinarov

Key words: Hardy-type inequalities, integral operator, q-analysis, q-integral.

AMS Mathematics Subject Classification: 26D10, 26D15, 33D05, 39A13.

Abstract. We obtain necessary and sufficient conditions for the validity of a certian Hardy-type inequality involving q-integrals.

1 Introduction

The q-derivative or Jackson's derivative, is a q-analogue of the ordinary derivative. q-differentiation is the inverse of Jackson's q-integration. It was introduced by F. H. Jackson [11] (see also [7]). He was the first to develop q-analysis. After that many q-analogue of classical results and concepts were studied and their applications are investigated.

Concerning recent results on q-analysis and its applications we also refer to the resent book by T. Ernst [8]. Some integral inequalities were obtained by H. Gauchman [10]. A Hardy-type inequality in q-analysis was recently obtained by L. Maligranda, R. Oinarov and L-E. Persson [13].

In this paper we prove a new Hardy-type inequality in which the Hardy operator is replaced by the q-analogue of the infinitesimal fractional operator (see [1] and (1.3) below).

In classical analysis, the hypergeometric function (Gaussian function) is defined for $|z| < 1$ by the power series [9]:

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \quad \forall \alpha, \beta, \gamma \in \mathbb{C}, $$

where $(\alpha)_n$ is the Pochhammer symbol, which is defined by:

$$ (\alpha)_0 = 1, \ (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \ n > 0. $$

If B denotes the Beta function, then

$$ {}_2F_1(\alpha-1, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{2-\alpha} dx, $$

where $Re(\gamma) > Re(\beta) > 0$. When $\beta = \gamma$ we have that

$$ {}_2F_1\left(\alpha-1,\beta;\beta;z\right) = (1-z)^{\alpha-1}. $$

Let $\alpha+\beta<\gamma, \gamma \neq 0,-1,-2,\cdots$. Then the following generalized fractional integral operator was introduced in [14]:

$$ I_\alpha^{\gamma,\beta} f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int_0^x {}_2F_1\left(\alpha-1,\beta;\gamma;\frac{s}{x}\right) ds, \tag{1.1} $$

where $\Gamma(\cdot)$ denotes the Gamma function. If $\beta=\gamma$ then the operator

$$ I_\alpha f(x) := \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int_0^x {}_2F_1\left(\alpha-1,\beta;\beta;\frac{s}{x}\right) ds, $$

is called the Riemann-Liouville fractional integral operator. When $\gamma=1$, $\beta=2$, we have that

$$ \hat{I} f(x) := \lim_{\alpha\longrightarrow 0} \Gamma(\alpha) I_\alpha^{1,2} f(x) = \int_0^x \ln\frac{x}{x-s}\frac{f(s)}{s} ds, \tag{1.2} $$

which is called the infinitesimal fractional integral operator [1].

The purpose of this paper is to find a q-analogue of operator (1.2) and to prove a q-analogue of the following Hardy-type integral inequality [1]:

$$ \left(\int_0^\infty u^r(x)\left(\int_0^x t^{\gamma-1}\ln\frac{x}{x-t} f(t)dt\right)^r dx\right)^{\frac{1}{r}} \leq C\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}, \quad \forall f(\cdot)\geq 0, \tag{1.3} $$

where $C>0$ is independent of f and u is a positive real valued function on $(0,\infty)$ briefly a weight function. We derive necessary and sufficient conditions for the validity of a q-analogue of inequality (1.3) in q-analysis for the case $1<p<\infty$, $0<r<\infty$ and $\gamma>\frac{1}{p}$ (see Theorem 3.1 and Theorem 3.2). We also consider the problem of finding the best constant in a q-analogue of inequality (1.3).

The paper is organized as follows: We present some preliminaries in Section 2. The main results and detailed proofs are presented in Section 3.

2 Preliminaries

First we recall definitions and notions of the theory of q-analysis, our main references are the books [7], [8] and [9].

Let $0<q<1$ be fixed.

For a real number $\alpha \in \mathbb{R}$, the q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, \quad \alpha \in R.$$

It is clear that $\lim\limits_{q\longrightarrow 1} \frac{1-q^\alpha}{1-q} = \alpha$.

The q-analogue of the power $(a-b)^k$ is defined by

$$(a-b)_q^0 = 1, \quad k \in \mathbb{N}, \quad (a-b)_q^k = \prod_{i=0}^{k-1}(a-q^i b), \quad \forall\ a,b \in \mathbb{R},$$

and

$$(1-b)_q^\alpha := \frac{(1-b)_q^\infty}{(1-q^\alpha b)_q^\infty}, \quad \forall\ b,\alpha \in \mathbb{R}. \tag{2.1}$$

and by using well-known relations this can also be written as

$$(1-b)_q^\alpha = \frac{1}{(1-q^\alpha b)_q^{-\alpha}}, \quad \forall b,\alpha \in \mathbb{R}. \tag{2.2}$$

The q-hypergeometric function ${}_2\Phi_1$ is defined by ([9]):

$${}_2\Phi_1\left[\begin{matrix} q^\alpha & q^\beta \\ & q^\gamma \end{matrix} ;q\ ;x\right] := \sum_{n=0}^{\infty} \frac{(q^\alpha;q)_q^n (q^\beta;q)_q^n}{(q^\gamma;q)_q^n (q;q)_q^n} x^n, \quad |x|<1,$$

where $(q^\alpha;q)_q^n = \prod\limits_{i=0}^{n-1}(1-q^{i+\alpha})$ and $\gamma \neq 0,-1,-2,\cdots$. Moreover, this series converges absolutely and $\lim\limits_{q\to 1} \frac{(q^\alpha;q)_q^n}{(1-q)^n} = (a)_n$, so

$$\lim_{q\to 1} {}_2\Phi_1\left[\begin{matrix} q^\alpha & q^\beta \\ & q^\gamma \end{matrix} ;q\ ;x\right] = {}_2F_1(\alpha,\beta;\gamma;x).$$

For $f:[0,b) \longrightarrow R, \quad 0 \le b < \infty$, the q-derivative is defined by:

$$D_q f(x) := \frac{f(x)-f(qx)}{(1-q)x}, \quad x \in (0,b), \tag{2.3}$$

and $D_q f(0) = f'(0)$ provided $f'(0)$ exists. It is clear that if $f(x)$ is differentiable, then $\lim\limits_{q\longrightarrow 1} D_q f(x) = f'(x)$.

Definition 1. The q-Taylor series of $f(x)$ at $x=c$ is defined by

$$f(x) := \sum_{j=0}^{\infty} \left(D_q^j\right)(c) \frac{(x-c)_q^j}{[j]_q!},$$

where

$$[j]_q! = \begin{cases} 1, & \text{if j} = 0, \\ [1]_q \times [2]_q \times \cdots \times [j]_q, & \text{if j} \in \mathbb{N}. \end{cases}$$

The definite q-integral or the q-Jackson integral of a function f is defined by the formula

$$\int_0^x f(t)d_qt := (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x), \quad x \in (0,b), \tag{2.4}$$

and the improper q-integral of a function $f(x) : [0,\infty) \to \mathbb{R}$, is defined by the formula

$$\int_0^\infty f(t)d_qt := (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k). \tag{2.5}$$

Note that the series in the right hand sides of (2.4) and (2.5) converge absolutely.

Definition 2. The function

$$\Gamma_q(\alpha) := \int_0^\infty x^{\alpha-1}E_q^{-qx}d_qx, \quad \alpha > 0,$$

is called the q-Gamma function, where $E_q^{-qx} = (1-(1-q)x)_q^\infty$.

We have that

$$\Gamma_q(\alpha+1) = [\alpha]_q\Gamma_q(\alpha),$$

for any $\alpha > 0$.

Definition 3. The function

$$B_q(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-qt)_q^{\beta-1}d_qt, \quad \alpha,\beta > 0,$$

is called the q-Beta function. Note that

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

for α, $\beta > 0$.

Let Ω be a subset of $(0,\infty)$ and $\mathcal{X}_\Omega(t)$ denote the characteristic function of Ω. For all $z > 0$, we have that(see [5]):

$$\int_0^\infty \mathcal{X}_{(0,z]}(t)f(t)d_qt = (1-q)\sum_{q^i \le z} q^i f(q^i), \tag{2.6}$$

$$\int_0^\infty \mathcal{X}_{[z,\infty)}(t)f(t)d_qt = (1-q)\sum_{q^i \ge z} q^i f(q^i). \tag{2.7}$$

R.P. Agarwal and W.A. Al-Salam (see [2], [3] and [4]) introduced several types of fractional q-integral operators and fractional q-derivatives. In particular, they defined the q-analogue of the fractional integral operator of the Riemann-Liouville type by

$$I_{q,\alpha}f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (1-\frac{qs}{x})_q^{\alpha-1} f(s)d_qs, \quad \alpha\in\mathbb{R}^+.$$

Using formula (2.2), we can rewrite $I_{q,\alpha}$ as follows:

$$I_{q,\alpha}f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \frac{f(s)}{(1-q^\alpha\frac{s}{x})_q^{1-\alpha}}d_qs, \quad \alpha\in\mathbb{R}^+. \tag{2.8}$$

Out next goal is to define a q-analogue of $\ln\frac{x}{x-s}$, but for this we need the following result of independent interest.

Proposition 2.1. *Let $0 < s \le x < \infty$. Then*

$${}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} & q^\beta \\ & & ;q\ ;q^\alpha\frac{s}{x} \\ q^\gamma & \end{matrix}\right] = \frac{1}{B_q(\beta,\gamma)}\int_0^1 \frac{t^{\beta-1}(1-qt)_q^{\gamma-\beta-1}}{(1-q^\alpha t\frac{s}{x})_q^{1-\alpha}}d_qt, \tag{2.9}$$

for $\beta,\gamma > 0$, and

$${}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} & q^\beta \\ & & ;q\ ;q^\alpha\frac{s}{x} \\ q^\beta & \end{matrix}\right] = \frac{1}{(1-q^\alpha\frac{s}{x})_q^{1-\alpha}}, \tag{2.10}$$

for $\beta = \gamma$.

Proof. First we consider equality (2.10). From (2.1) and (2.3), we get that

$$\begin{aligned} D_{q.s}^1\left(\frac{1}{(1-q^\alpha\frac{s}{x})_q^{1-\alpha}}\right) &= D_{q,s}^1\left(\frac{(1-q\frac{s}{x})_q^\infty}{(1-q^\alpha\frac{s}{x})_q^\infty}\right) \\ &= \left[\frac{(1-q^2\frac{s}{x})_q^\infty}{(1-q^{\alpha+1}\frac{s}{x})_q^\infty} - \frac{(1-q\frac{s}{x})_q^\infty}{(1-q^\alpha\frac{s}{x})_q^\infty}\right]\frac{1}{(q-1)s} \\ &= \frac{(1-q^2\frac{s}{x})_q^\infty}{(1-q^\alpha\frac{s}{x})_q^\infty}\left[\frac{(1-q^\alpha\frac{s}{x})-(1-q\frac{s}{x})}{s(q-1)}\right] \\ &= \frac{(1-q^2\frac{s}{x})_q^\infty}{(1-q^\alpha\frac{s}{x})_q^\infty}\left[\frac{q^\alpha(q^{1-\alpha}-1)}{x(q-1)}\right] \\ &= \frac{\frac{q^\alpha}{x}[1-\alpha]_q}{(1-q^\alpha\frac{s}{x})_q^{2-\alpha}}. \end{aligned}$$

Using this relation and induction, one can easily see that

$$D_{q,s}^j\left(\frac{1}{(1-q^\alpha\frac{s}{x})_q^{1-\alpha}}\right)\Bigg|_{s=0} = \frac{q^{j\alpha}}{x^j}[1-\alpha]_q[2-\alpha]_q\cdots[j-\alpha]_q,$$

for any $j \ge 1$. Therefore, we have the q-Taylor expansion (see Definition 1)

$$\begin{aligned}\frac{1}{(1-q^{\alpha}\frac{s}{x})_q^{1-\alpha}} &= \sum_{j=0}^{\infty}\frac{[1-\alpha]_q[2-\alpha]_q\cdots[j-\alpha]_q}{[j]_q!}\left(\frac{q^{\alpha}s}{x}\right)^j \\ &= \sum_{j=0}^{\infty}\frac{(1-q^{1-\alpha})_q^j}{(1-q)_q^j}\left(\frac{q^{\alpha}s}{x}\right)^j \\ &= {}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} & q^{\beta} \\ & \end{matrix} ;q\ ;q^{\alpha}\tfrac{s}{x} \atop q^{\beta}\right], \qquad (2.11)\end{aligned}$$

and (2.10) is proved.

By using the same arguments as above we see that

$$\frac{1}{(1-q^{\alpha}t\frac{s}{x})_q^{1-\alpha}} = \sum_{n=0}^{\infty}\frac{(1-q^{1-\alpha})_q^n}{(1-q)_q^n}\left(t\frac{q^{\alpha}s}{x}\right)^n,$$

for $x \geq s$, $0 < t \leq 1$. Therefore

$$\begin{aligned}\int_0^1 \frac{t^{\beta-1}(1-qt)_q^{\gamma-\beta-1}}{(1-q^{\alpha}t\frac{s}{x})_q^{1-\alpha}}d_qt &= \sum_{n=0}^{\infty}\frac{(1-q^{1-\alpha})_q^n}{(1-q)_q^n}\left(\frac{q^{\alpha}s}{x}\right)^n\int_0^1 t^{\beta+n-1}(1-qt)_q^{\gamma-\beta-1}d_qt \\ &= \sum_{n=0}^{\infty}\frac{(1-q^{1-\alpha})_q^n}{(1-q)_q^n}\left(\frac{q^{\alpha}s}{x}\right)^n\frac{\Gamma_q(\beta+n)\Gamma_q(\gamma-\beta)}{\Gamma_q(\gamma+n)} \\ &= \frac{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}{\Gamma_q(\gamma)}\sum_{n=0}^{\infty}\frac{(1-q^{1-\alpha})_q^n(1-q^{\beta})_q^n}{(1-q)_q^n(1-q^{\gamma})_q^n}\left(\frac{q^{\alpha}s}{x}\right)^n \\ &= B_q(\beta,\gamma)\,{}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} \ q^{\beta} \\ q^{\gamma}\end{matrix};q\ ;q^{\alpha}\tfrac{s}{x}\right].\end{aligned}$$

and also (2.9) is proved. □

By Proposition 2.1, the integral (2.8) can be rewritten as

$$I_{q,\alpha}f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x {}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} \ q^{\beta} \\ q^{\beta}\end{matrix};q\ ;q^{\alpha}\tfrac{s}{x}\right] f(s)d_qs, \qquad \alpha\in\mathbb{R}^+,\ \beta\in\mathbb{R}.$$

More generally, we consider the q-analogue of $I_{\alpha}^{\gamma,\beta}$ (see (1.1))

$$I_{q,\alpha}^{\gamma,\beta}f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x {}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} \ q^{\beta} \\ q^{\gamma}\end{matrix};q\ ;q^{\alpha}\tfrac{s}{x}\right] f(s)d_qs, \qquad \alpha,\ \beta,\gamma\in\mathbb{R}^+.$$

Due to uniform convergence of the series ${}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} \ q \\ q^2\end{matrix};q\ ;q^{\alpha}\tfrac{s}{x}\right]$ for $0<\alpha<1$, we get that

$$\begin{aligned}\lim_{\alpha\to 0^+} {}_2\Phi_1\left[\begin{matrix} q^{1-\alpha} \ q \\ q^2\end{matrix};q\ ;q^{\alpha}\tfrac{s}{x}\right]\frac{s}{x} &= {}_2\Phi_1\left[\begin{matrix} q \ q \\ q^2\end{matrix};q\ ;\tfrac{s}{x}\right]\frac{s}{x} \\ &= \sum_{j=0}^{\infty}\frac{1-q}{1-q^{j+1}}\frac{s^{j+1}}{x^{j+1}} = \sum_{j=0}^{\infty}\frac{\left(\frac{s}{x}\right)^{j+1}}{[j+1]_q} = \sum_{j=1}^{\infty}\frac{(\frac{s}{x})^j}{[j]_q},\end{aligned}$$

which is the q-analogue of the Taylor series of the function $\ln \frac{x}{x-s}$ with $s<x$.

Definition 4. We define the q-analogue of the function $\ln \frac{x}{x-s}$, $0<s<x<\infty$, as follows:

$$\ln_q \frac{x}{x-s} := \sum_{j=1}^{\infty} \frac{(\frac{s}{x})^j}{[j]_q}.$$

Remark 5. We define the q-analog of (1.2) as follows:

$$\widehat{I}_q f(x) := \int_0^{qx} \ln_q \frac{x}{x-s} \frac{f(s)}{s} d_q s, \tag{2.12}$$

which is called the infinitesimal q-fractional integral operator.

Observe that:

$$\lim_{q\to 1} \widehat{I}_q f(x) = \int_0^{x} \ln \frac{x}{x-s} \frac{f(s)}{s} ds$$

.

Hence, from (2.12) we obtain the q-analogue of (1.3) in the following form:

$$\left(\int_0^\infty u^r(x)\left(\widehat{I}_q f(x)\right)^r d_q x\right)^{\frac{1}{r}} \le C\left(\int_0^\infty f^p(t) d_q t\right)^{\frac{1}{p}}, \quad \forall f(\cdot)\ge 0, \tag{2.13}$$

where $C>0$ independent of f.

In the q-integral we are allowed to change variables in the form $x=t\xi$ for $0<\xi<\infty$ (see [7]). So by making the substitution $t=qs$, and $d_q t = q d_q s$ inequality (2.13) becomes

$$\left(\int_0^\infty u^r(x)\left(\int_0^x s^{\gamma-1}\ln_q \frac{x}{x-qs}\widetilde{f}(s) d_q s\right)^r d_q x\right)^{\frac{1}{r}}$$
$$\le \widetilde{C}\left(\int_0^\infty \widetilde{f}^p(s) d_q s\right)^{\frac{1}{p}}, \quad \forall f(\cdot)\ge 0. \tag{2.14}$$

where $\widetilde{f}(s)=f(qs)$, $\widetilde{C}=q^{\gamma-\frac{1}{p}}C$.

Since inequality (2.13) holds if and only if inequality (2.14) holds, from now on we will investigate necessary and sufficient conditions the validity of inequality (2.14).

Notation. In the sequel, for any $p>1$ the conjugate number p' is defined by $p' := p/(p-1)$. Moreover, the symbol $M \ll K$ means that there exists $\alpha>0$ such that $M\le \alpha K$, where α is a constant which depend only on the numerical parameters such as p, q, r. If $M\ll K\ll M$, then we write $M\approx K$.

For the proof of our main theorems we will need the following well-known discrete weighted Hardy inequality proved by G. Bennett [6] (see also [12], p.58):

Theorem A. *Let $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ be non-negative sequences of real numbers and $1 < p \le r < \infty$. Then the inequality*

$$\left(\sum_{j=-\infty}^{\infty} \left(\sum_{i=j}^{\infty} f_i \right)^r u_j^r \right)^{\frac{1}{r}} \le C \left(\sum_{i=-\infty}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}}, \quad f \ge 0, i \in \mathbb{Z}, \tag{2.15}$$

with $C > 0$ independent of f_i, $i \in \mathbb{Z}$ holds if and only if

$$\mathfrak{B}_1 := \sup_{n \in Z} \left(\sum_{j=-\infty}^{n} u_j^r \right)^{\frac{1}{r}} \left(\sum_{i=n}^{\infty} v_i^{-p'} \right)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1}.$$

Moreover,$\mathfrak{B}_1 \approx C$, where C is the best constant in (2.15).

Theorem B. *Let $0 < r < p < \infty$ and $1 < p$. Then inequality (2.15) holds if and only if $\mathfrak{B}_2 < \infty$, where*

$$\mathfrak{B}_2 := \left(\sum_{k=-\infty}^{\infty} v_k^{-p'} \left(\sum_{i=-\infty}^{k} u_i^r \right)^{\frac{p}{p-r}} \left(\sum_{i=k}^{\infty} v_i^{-p'} \right)^{\frac{p(r-1)}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Moreover, $\mathfrak{B}_2 \approx C$, where C is the best constant in (2.15).

Also we need the following lemma ([5]):

Lemma A. *Let f, φ and g be nonnegative functions. Then*

$$\int_0^\infty \left(\int_0^\infty \mathcal{X}_{[z,\infty)}(t) f(t) d_q t \right)^\alpha \left(\int_0^\infty \mathcal{X}_{(0,z]}(x) g(x) d_q x \right)^\beta \varphi(z) d_q z$$
$$= (1-q)^{\alpha+\beta} \sum_{k=-\infty}^{\infty} \left[\left(\sum_{i=-\infty}^{k} q^i f(q^i) \right)^\alpha \left(\sum_{j=k}^{\infty} q^j g(q^j) \right)^\beta q^k \varphi(q^k) \right],$$

for $\alpha,\ \beta \in \mathbb{R}$.

3 Main results

Our main result reads:

Theorem 3.1. *Let $1 < p \le r < \infty$, $\gamma > \frac{1}{p}$. Then the inequality*

$$\left(\int_0^\infty u^r(x) \left(\int_0^x s^{\gamma-1} \ln_q \frac{x}{x-qs} f(s) d_q s \right)^r d_q x \right)^{\frac{1}{r}}$$
$$\le C \left(\int_0^\infty f^p(s) d_q s \right)^{\frac{1}{p}}, \quad \forall f(\cdot) \ge 0, \tag{3.1}$$

with $C>0$ independent of f holds if and only if $\boldsymbol{B}_1<\infty$, where

$$\boldsymbol{B}_1:=\sup_{x>0}x^{\gamma+\frac{1}{p'}}\left(\int_0^\infty \mathcal{X}_{[x,\infty)}(t)\frac{u^r(t)}{t^r}d_qt\right)^{\frac{1}{r}},$$

Moreover, $\boldsymbol{B}_1\approx C$, where C is the best constant in (3.1).

Theorem 3.2. *Let $0<r<p<\infty$, $1<p$ and $\gamma>\frac{1}{p}$. Then the inequality (3.1) holds if and only if $\boldsymbol{B}_2<\infty$, where*

$$\boldsymbol{B}_2:=\left(\int_0^\infty\left[x^{\gamma+\frac{1}{p'}}\left(\int_0^\infty \mathcal{X}_{[x,\infty)}(t)\frac{u^r(t)}{t^r}d_qt\right)^{\frac{1}{r}}\right]^{\frac{pr}{p-r}}d_qx\right)^{\frac{p-r}{pr}}.$$

Moreover, $\boldsymbol{B}_2\approx C$, where C is the best constant in (3.1).

Remark 6. By using formulas (2.4) and (2.5) in (3.1) we get that

$$\left(\sum_{j=-\infty}^{\infty}(1-q)q^ju^r(q^j)\left(\sum_{i=j}^{\infty}(1-q)q^{i\gamma}f(q^i)\ln_q\frac{1}{1-q^{i-j+1}}\right)^r\right)^{\frac{1}{r}}$$
$$\le C\left(\sum_{i=-\infty}^{\infty}(1-q)q^if^p(q^i)\right)^{\frac{1}{P}}.$$

Let

$$u_j^r=(1-q)^{1+\frac{r}{p'}}q^ju^r(q^j),\ f_i=(1-q)^{\frac{1}{p}}q^{\frac{i}{p}}f(q^i),\ a_{i,j}=\ln_q\frac{1}{1-q^{i-j+1}}. \tag{3.2}$$

Then we get that inequality (3.1) is equivalent to the discrete weighted Hardy-type inequality

$$\left(\sum_{j=-\infty}^{\infty}u_j^r\left(\sum_{i=j}^{\infty}q^{i(\gamma-\frac{1}{p})}f_ia_{i,j}\right)^r\right)^{\frac{1}{r}}\le C\left(\sum_{i=-\infty}^{\infty}f_i^p\right)^{\frac{1}{p}}. \tag{3.3}$$

Note that inequality (3.1) holds if and only if inequality (3.3) holds, so we will obtain the desired necessary and sufficient conditions for the validity of inequality (3.3).

Our next Lemmas give a characterization of the discrete Hardy-type inequality (3.3).

Lemma 3.1. *Let $1<p\le r<\infty,\ \gamma>\frac{1}{p}$. Then the inequality (3.3) holds if and only if $B_1<\infty$, where*

$$B_1:=\sup_{k\in\mathbb{Z}}\left(\sum_{i=k}^{\infty}q^{i(p'\gamma+1)}\right)^{\frac{1}{p'}}\left(\sum_{j=-\infty}^{k}q^{-jr}u_j^r\right)^{\frac{1}{r}}. \tag{3.4}$$

Moreover, $B_1\approx C$, where C is the best constant in (3.3).

Proof. Necessity. Let us assume that (3.3) holds with some $C > 0$. From (3.2) and Definition 4 we get that $q^{i+1}/q^j \le a_{i,j}$ for $j \le i$. Then

$$\sum_{j=-\infty}^{\infty} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma-\frac{1}{p})} f_i a_{i,j}\right)^r \ge q \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i\right)^r.$$

Moreover,

$$q\left(\sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i\right)^r\right)^{\frac{1}{r}} \le C\left(\sum_{j=-\infty}^{\infty} f_i^p\right)^{\frac{1}{p}}.$$

Hence, by Theorem A we obtain that

$$B_1 \ll C. \tag{3.5}$$

The proof of the necessity is complete.

Sufficiency. Let $B < \infty$ and $f \ge 0$ be arbitrary. We will show that inequality (3.3) holds.

We consider two cases separately: $0 < q \le \frac{1}{2}$ and $\frac{1}{2} < q < 1$.

1) Let $0 < q \le \frac{1}{2}$. Let $j \le k \le i$. Then from (3.2) and Definition 4 it follows that $a_{j,j} = \ln_q \frac{1}{1-q} \le \ln_q 2$ (we note that $\ln_q 2 := \sum\limits_{n=1}^{\infty} \frac{2^{-n}}{[n]_q}$), and

$$\begin{aligned} q^{-k}a_{k,j} - q^{-i}a_{i,j} &= q^{-k}\sum_{n=1}^{\infty} \frac{(qq^k/q^j)^n}{[n]_q} - q^{-i}\sum_{n=1}^{\infty} \frac{(qq^i/q^j)^n}{[n]_q} \\ &= \sum_{n=1}^{\infty} \frac{(q/q^j)^n}{[n]_q}\left(q^{k(n-1)} - q^{i(n-1)}\right) \ge 0, \end{aligned}$$

i.e.

$$q^{-i}a_{i,j} \le q^{-k}a_{k,j}, \tag{3.6}$$

for $j \le k \le i$.

Thus by (3.6) we have that

$$\begin{aligned} \sum_{j=-\infty}^{\infty} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma-\frac{1}{p})} f_i a_{i,j}\right)^r &= \sum_{j=-\infty}^{\infty} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} q^{-i} a_{i,j} f_i\right)^r \\ &\le \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r a_{j,j}^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i\right)^r \\ &\le (\ln_q 2)^r \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i\right)^r \\ &\ll \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i\right)^r. \end{aligned}$$

Hence, by Theorem A we obtain that

$$\left(\sum_{j=-\infty}^{\infty} q^{-jr}u_j^r\left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})}f_i\right)^r\right)^{\frac{1}{r}} \leq B_1\left(\sum_{j=-\infty}^{\infty} f_i^p\right)^{\frac{1}{p}},$$

which means that inequality (3.3) is valid and that $C \ll B_1$, where C is the best constant for which (3.3) holds.

2) Let $\frac{1}{2} < q < 1$. Then $\exists i_0 \in N$ such that $i_0 > 1$ and $q^{i_0} \leq \frac{1}{2} < q^{i_0-1}$. We assume that $\mathbb{Z} = \bigcup\limits_{k\in Z}[t_k+1, t_{k+1}]$ and $t_{k+1} - t_k = t_0$. Then the left hand side of (3.3) can be written as

$$\begin{aligned}\sum_{j=-\infty}^{\infty} u_j^r\left(\sum_{i=j}^{\infty} q^{i(\gamma-\frac{1}{p})}f_i a_{i,j}\right)^r &= \sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r\left(\sum_{i=j}^{\infty} q^{i(\gamma-\frac{1}{p})}f_i a_{i,j}\right)^r \\ &\approx \sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r\left(\sum_{i=j}^{t_{k+2}-1} q^{i(\gamma-\frac{1}{p})}f_i a_{i,j}\right)^r \\ &+ \sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r\left(\sum_{i=t_{k+2}}^{\infty} q^{i(\gamma-\frac{1}{p})}f_i a_{i,j}\right)^r \\ &= I_1 + I_2. \end{aligned} \tag{3.7}$$

To estimate I_1 we use Hölder's inequality. We find that

$$\begin{aligned} I_1 &\leq \sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r\left(\sum_{i=j}^{t_{k+2}-1} q^{ip'(\gamma-\frac{1}{p})}a_{i,j}^{p'}\right)^{\frac{r}{p'}}\left(\sum_{i=j}^{t_{k+2}-1} f_i^p\right)^{\frac{r}{p}} \\ &\leq \sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r\left(\sum_{i=j}^{\infty} q^{ip'(\gamma-\frac{1}{p})}a_{i,j}^{p'}\right)^{\frac{r}{p'}}\left(\sum_{i=t_k+1}^{t_{k+2}} f_i^p\right)^{\frac{r}{p}} \\ &= \sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r q^{jr(\gamma-\frac{1}{p})}\left(\sum_{i=0}^{\infty} q^{ip'(\gamma-\frac{1}{p})}a_{i,0}^{p'}\right)^{\frac{r}{p'}}\left(\sum_{i=t_k+1}^{t_{k+2}} f_i^p\right)^{\frac{r}{p}} \\ &= C_0^{\frac{r}{p'}}\sum_k\sum_{j=t_k+1}^{t_{k+1}} u_j^r q^{-jr}q^{jr(\gamma+\frac{1}{p'})}\left(\sum_{i=t_k+1}^{t_{k+2}} f_i^p\right)^{\frac{r}{p}}, \end{aligned} \tag{3.8}$$

where $C_0 := \sum\limits_{i=0}^{\infty} q^{ip'(\gamma-\frac{1}{p})}a_{i,0}^{p'}$.

Since

$$M := (1-q)C_0 = \int_0^1 x^{p'(\gamma-\frac{1}{p})}\left(\ln_q\frac{1}{1-qx}\right)^{p'} d_qx < \infty$$

and

$$q^{jr(\gamma+\frac{1}{p'})} \leq q^{(t_k+1)(\gamma+\frac{1}{p'})r} = q^{-(i_0-1)(\gamma+\frac{1}{p'})}q^{t_{k+1}r(\gamma+\frac{1}{p'})} \leq 2^{r(\gamma+\frac{1}{p'})}q^{t_{k+1}(p'\gamma+1)\frac{r}{p'}}, \tag{3.9}$$

for $t_k+1\le j$, we get that

$$\begin{aligned}
I_1 &\le 2^{r(\gamma+\frac{1}{p'})}M^{\frac{r}{p'}}[p'\gamma+1]_q^{\frac{r}{p'}}\sum_k\left(\sum_{i=t_k+1}^{t_{k+2}}f_i^p\right)^{\frac{r}{p}}\left(\frac{q^{t_{k+1}(p'\gamma+1)}}{1-q^{p'\gamma+1}}\right)^{\frac{r}{p'}}\sum_{j=-\infty}^{t_{k+1}}u_j^rq^{-jr}\\
&\ll \sum_k\left(\sum_{i=t_k+1}^{t_{k+2}}f_i^p\right)^{\frac{r}{p}}\left(\frac{q^{t_{k+1}(p'\gamma+1)}}{1-q^{p'\gamma+1}}\right)^{\frac{r}{p'}}\sum_{j=-\infty}^{t_{k+1}}u_j^rq^{-jr}\\
&= \sum_k\left(\sum_{i=t_k+1}^{t_{k+2}}f_i^p\right)^{\frac{r}{p}}\left[\left(\sum_{i=t_{k+1}}^{\infty}q^{ir(p'\gamma+1)}\right)^{\frac{1}{p'}}\left(\sum_{j=-\infty}^{t_{k+1}}u_j^rq^{-jr}\right)^{\frac{1}{r}}\right]^r\\
&\ll B_1^r\left(\sum_{i=-\infty}^{\infty}f_i^p\right)^{\frac{r}{p}}. \qquad (3.10)
\end{aligned}$$

Let $j\le k\le i$. Then from (3.2) and Definition 4 it follows that

$$\begin{aligned}
q^ka_{i,k}-q^ja_{i,j}\ge q^j\left(a_{i,k}-a_{i,j}\right) &= q^k\sum_{n=1}^{\infty}\frac{\left(q^{i+1}/q^k\right)^n}{[n]_q}-q^j\sum_{n=1}^{\infty}\frac{\left(q^{i+1}/q^j\right)^n}{[n]_q}\\
&= \sum_{n=1}^{\infty}\frac{q^{(i+1)n}}{[n]_q}\left(q^{k(1-n)}-q^{j(1-n)}\right)\ge 0,
\end{aligned}$$

i.e.

$$q^ja_{i,j}\le q^ka_{i,k}, \qquad (3.11)$$

for $j\le k\le i$.

Using (3.6) and (3.11) we find that

$$\frac{1}{q^{i-j}}a_{i,j}\le\frac{1}{q^{t_{k+1}-t_{k+2}}}\ln_q\frac{1}{1-q^{t_{k+1}-t_k}}=\frac{1}{q^{i_0}}\ln_q\frac{1}{1-q^{i_0}}\le 2\ln_q 2,$$

for $j\le t_{k+1}$ and $t_{k+2}\le i$.

Therefore,

$$\begin{aligned}
I_2 &= \sum_k\sum_{j=t_k+1}^{t_{k+1}}u_j^r\left(\sum_{i=t_{k+2}}^{\infty}q^{i(\gamma-\frac{1}{p})}f_ia_{i,j}\right)^r\\
&= \sum_k\sum_{j=t_k+1}^{t_{k+1}}q^{-jr}u_j^r\left(\sum_{i=t_{k+2}}^{\infty}q^{i(\gamma+\frac{1}{p'})}\frac{1}{q^{i-j}}a_{i,j}f_i\right)^r\\
&\le (2\ln_q 2)^{\frac{r}{p'}}\sum_k\sum_{j=t_k+1}^{t_{k+1}}q^{-jr}u_j^r\left(\sum_{i=t_{k+2}}^{\infty}q^{i(\gamma+\frac{1}{p'})}f_i\right)^r\\
&\ll \sum_{j=-\infty}^{\infty}q^{-jr}u_j^r\left(\sum_{i=j}^{\infty}q^{i(\gamma+\frac{1}{p'})}f_i\right)^r.
\end{aligned}$$

By using Theorem A we have that

$$I_2 \ll B_1^r \left(\sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{r}{p}}, \tag{3.12}$$

Thus, from (3.7), (3.10) and (3.12) it follows that inequality (3.3) is valid and we see that the best constant C in (3.3) is such that $C \ll B_1$, which together with (3.5) gives that $C \approx B_1$. □

Lemma 3.2. *Let $0 < r < p < \infty$ and $1 < p$. Then inequality (3.3) holds if and only if $B_2 < \infty$, where*

$$B_2 := \left(\sum_{k=-\infty}^{\infty} q^{k(p'\gamma+1)} \left(\sum_{i=-\infty}^{k} u_i^r q^{-ir} \right)^{\frac{p}{p-r}} \left(\sum_{i=k}^{\infty} q^{i(p'\gamma+1)} \right)^{\frac{p(r-1)}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Moreover, $B_2 \approx C$, where C is the best constant in (3.3).

Proof. In a similar way as in the proof of Lemma 3.1. by Theorem B we obtain that inequality (3.3) is valid and that $C \approx B_2$. where C is the best constant for which (3.3) holds for $0 < q \leq \frac{1}{2}$.

In case $\frac{1}{2} < q < 1$ the necessary part is due to Theorem B. Therefore,

$$B_2 \ll C. \tag{3.13}$$

To prove sufficiency we proceed as follows. Applying to (3.8) Hölder's inequality with the exponents $\frac{p}{p-r}$ and $\frac{p}{r}$ we obtain that

$$\begin{aligned} I_1 &\ll \sum_k \left(\sum_{i=t_k+1}^{t_{k+2}} f_i^p \right)^{\frac{r}{p}} q^{t_{k+1}(p'\gamma+1)\frac{r}{p'}} \sum_{j=-\infty}^{t_{k+1}} u_j^r q^{-jr} \\ &\leq \left(\sum_k q^{t_{k+1}(p'\gamma+1)\frac{r(p-1)}{p-r}} \left(\sum_{j=-\infty}^{t_{k+1}} u_j^r q^{-jr} \right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}} \left(\sum_k \sum_{i=t_k+1}^{t_{k+2}} f_i^p \right)^{\frac{r}{p}} \\ &\ll \widetilde{B}^{\frac{p-r}{p}} \left(\sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{r}{p}}. \end{aligned}$$

Since

$$\begin{aligned} \widetilde{B} &:= \sum_{i=-\infty}^{\infty} q^{i(p'\gamma+1)\frac{r(p-1)}{p-r}} \left(\sum_{j=-\infty}^{i} u_j^r q^{-jr} \right)^{\frac{p}{p-r}} \\ &\leq \sum_{i=-\infty}^{\infty} q^{i(p'\gamma+1)} \left(\frac{q^{i(p'\gamma+1)}}{1-q^{p'\gamma+1}} \right)^{\frac{p(r-1)}{p-r}} \left(\sum_{j=-\infty}^{i} u_j^r q^{-jr} \right)^{\frac{p}{p-r}} \\ &= B_2^{\frac{pr}{p-r}}, \end{aligned}$$

we have that

$$I_1 \ll B_2^r \left(\sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{r}{p}} \tag{3.14}$$

From (3.12) and Theorem B it follows that

$$\begin{aligned} I_2 &\ll \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i \right)^r \\ &\leq \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left(\sum_{i=j}^{\infty} q^{i(\gamma+\frac{1}{p'})} f_i \right)^r \\ &\leq B_2^r \left(\sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{r}{p}}. \end{aligned} \tag{3.15}$$

Thus, from (3.14) and (3.15) it follows that $C \ll B_2$ which means that the inequality (3.3) is valid, which together with (3.13) gives $B_2 \approx C$. □

Lemma 3.3. *Let $\gamma > \frac{1}{p}$, and $B_1 < \infty$. Then*

$$\alpha \boldsymbol{B}_1 = \sup_{k\in\mathbb{Z}} \left(\sum_{i=k}^{\infty} q^{i(p'\gamma+1)} \right)^{\frac{1}{p'}} \left(\sum_{j=-\infty}^{k} q^{j(1-r)} u^r(q^j) \right)^{\frac{1}{r}}, \tag{3.16}$$

for $r > 0$, where $\alpha = [p'\gamma+1]_q^{-\frac{1}{p'}} (1-q)^{-\frac{1}{r}-\frac{1}{p'}}$.

Proof. Let $\gamma > \frac{1}{p}$. By using (2.7) we obtain that

$$I(x) = x^{\gamma+\frac{1}{p'}} \left(\int_0^{\infty} \mathcal{X}_{[x,\infty)}(t) t^{-r} u^r(t) d_q t \right)^{\frac{1}{r}} = (1-q)^{\frac{1}{r}} x^{\gamma+\frac{1}{p'}} \left(\sum_{q^j \geq x} q^{(1-r)j} u^r(q^j) \right)^{\frac{1}{r}},$$

for $\forall r > 0$. Then

$$I(x) = (1-q)^{\frac{1}{p'}+\frac{1}{r}} [p'\gamma+1]_q^{\frac{1}{p'}} \left(\sum_{i=k}^{\infty} q^{i(p'\gamma+1)} \right)^{\frac{1}{p'}} \left(\sum_{j=-\infty}^{k} q^{(1-r)i} u^r(q^i) \right)^{\frac{1}{r}},$$

for $x = q^k, \forall k \in \mathbb{Z}$. Moreover,

$$I(x) = (1-q)^{\frac{1}{p'}+\frac{1}{r}} [p'\gamma+1]_q^{\frac{1}{p'}} \left(\sum_{i=k}^{\infty} q^{i(p'\gamma+1)} \right)^{\frac{1}{p'}} \left(\sum_{j=-\infty}^{k-1} q^{(1-r)i} u^r(q^i) \right)^{\frac{1}{r}},$$

for $q^k < x < q^{k-1}$. Hence

$$\sup_{q^k < x \leq q^{k-1}} I(x) = (1-q)^{\frac{1}{p'}+\frac{1}{r}} [p'\gamma+1]_q^{\frac{1}{p'}} \left(\sum_{i=k}^{\infty} q^{i(p'\gamma+1)} \right)^{\frac{1}{p'}} \left(\sum_{j=-\infty}^{k} q^{(1-r)i} u^r(q^i) \right)^{\frac{1}{r}},$$

and

$$\alpha\mathbf{B}_1 = \alpha \sup_{k\in Z} \sup_{q^k<x\leq q^{k-1}} I(x) = \sup_{k\in Z} \left(\sum_{i=k}^{\infty} q^{i(p'\gamma+1)}\right)^{\frac{1}{p'}} \left(\sum_{j=-\infty}^{k} q^{(1-r)i} u^r(q^i)\right)^{\frac{1}{r}}.$$

We have proved that (3.16) holds. □

Next, we prove Theorem 3.1.

Proof of Theorem 3.1. First we note that inequality (3.3) is equivalent to inequality (3.1). Moreover, by Lemma 3.1 inequality (3.1) holds if and only if $B_1 < \infty$. From (3.2) and Lemma 3.3 we have that $B_1 = \alpha\mathbf{B}_1$. which means that $\mathbf{B}_1 \approx C$ and inequality (3.1) holds if and only if $\mathbf{B}_1 < \infty$. □

Proof of Theorem 3.2. In a similar way as in the proof of Theorem 3.1, by Lemma 3.2 we have that inequality (3.1) holds if and only if $B_2 < \infty$. From (3.2) and Lemma A we have that

$$\begin{aligned}
B_2^{\frac{pr}{p-r}} &= (1-q)\sum_{k=-\infty}^{\infty} q^{k(p'\gamma+1)} \left((1-q)\sum_{i=-\infty}^{k} u^r(q^i)q^{-ir}\right)^{\frac{p}{p-r}} \\
&\times \left((1-q)\sum_{i=k}^{\infty} q^{i(p'\gamma+1)}\right)^{\frac{p(r-1)}{p-r}} \\
&= \int_0^\infty x^{p'\gamma+1} \left(\int_0^\infty \mathcal{X}_{[x,\infty)}(t)\frac{u^r(t)}{t^r}d_qt\right)^{\frac{p}{p-r}} \left(\int_0^\infty \mathcal{X}_{(0,x]}(t)s^{p'\gamma}d_qs\right)^{\frac{p(r-1)}{p-r}} d_qx \\
&= [p'\gamma+1]_q^{-\frac{pr}{p-r}} \int_0^\infty \left(x^{\gamma+\frac{1}{p'}} \left(\int_0^\infty \mathcal{X}_{[x,\infty)}(t)\frac{u^r(t)}{t^r}d_qt\right)^{\frac{1}{r}}\right)^{\frac{pr}{p-r}} d_qx \\
&\ll \mathbf{B}_2,
\end{aligned}$$

which means that $\mathbf{B}_2 \approx C$ and inequality (3.1) holds if and only if $\mathbf{B}_2 < \infty$. The proof is complete. □

Acknowledgments

The author thank Professor Ryskul Oinarov (L.N. Gumilyev Eurasian National University, Kazakhstan) and Lars-Erik Persson (Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden) for good advices which have improved the final version of this paper.
This work was supported by Scientific Committee of Ministry of Education and Science of the Republic of Kazakhstan, grant no. 5495/GF4. It was also supported by the Russian Sientific Foundation (project RFFI 16-31-50042).

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Received: 20.11.2015