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From time to time the EMJ publishes survey papers.

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Kazakhstan

NURZHAN BOKAYEV

(to the 60th birthday)

On January 5, 2016 was the 60th birthday of Doctor of Physical-Mathematical Sciences (1996), Professor Nurzhan Adilkhanovich Bokayev. Professor Bokayev is the head of the department "Higher Mathematics" of the L.N. Gumilyov Eurasian National University (since 2009), the Vice-President of Mathematical Society of the Turkic World (since 2014), and a member of the Editorial Board of our journal.

N.A. Bokayevwas born in the Urnek village, Karabalyk district, Kostanay region. He graduated from the E.A. Buketov Karaganda State University in 1977 and the M.V. Lomonosov Moscow State University's full-time postgraduate study in 1984.

Scientific works of Professor Bokayev are devoted to studying problems of the theory of functions, in particular of the theory of orthogonal series.

He proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems; investigated Besov-type function spaces with respect to the Price bases; studied the Price - and Haar-type p-adic operators; introduced new classes of Faber-Schauder-type systems of functions and spaces of multivariable functions of bounded p-variation and of bounded p-fluctuation, obtained estimates for the best approximation of functions in these spaces by polynomials with respect to the Walsh and Haar systems, established weighted integrability conditions of the sum of multiple trigonometric series and series with respect to multiplicative systems, found the embedding criterion for the Nikol'skii-Besov spaces with respect to multiplicative bases and the coefficient criterion for belonging of functions to such spaces.

His scientific results have made essential contribution to the theory of series with respect to the Walsh and Haar systems and multiplicative systems.

N.A. Bokayev has published more than 150 scientific papers. Under his supervision 8 dissertations have been defended: 4 candidate of sciences dissertations and 4 PhD dissertations.

The Editorial Board of the Eurasian Mathematical Journal congratulates Nurzhan Adilkhanovich Bokayev on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

The EMJ has been included in the Emerging Sources Citation Index

This year, Thomson Reuters is launching the Emerging Sources Citation Index (ESCI), which will extend the universe of publications in Web of Science to include high-quality, peer-reviewed publications of regional importance and in emerging scientific fields. ESCI will also make content important to funders, key opinion leaders, and evaluators visible in Web of Science Core Collection even if it has not yet demonstrated citation impact on an international audience.

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On behalf of the Editorial Board of the EMJ
V.I. Burenkov, T.V. Tararykova, A.M. Temirkhanova

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ON A CERTAIN INTEGRAL OPERATOR ACTING ON FUNCTIONS DEFINED ON THE DYADIC GROUP

A.I. Rubinstein

Communicated by M.L. Goldman

Dedicated to the 80^{th} anniversary of Professor V.A. Skvorzov.

Key words: integral operator, dyadic group, modulus of continuity.

AMS Mathematics Subject Classification: 47G10.

Abstract. A certain integral operator acting on functions defined on the dyadic group is studied in this article.

1 Introduction

Let $G = \{x = (x_1, x_2, \dots),\ x_k \in \{0; 1\}\}$ be the Abelian group with the operation "$\dotplus$" defined by

$$x \dotplus y = (x_1, x_2, \dots) \dotplus (y_1, y_2, \dots) = z = (z_1, z_2, \dots),$$

where $z_k = x_k + y_k \pmod 2$.

As is well known, the dyadic group G may be represented on $[0; 1)$ by means of the relation $x \to \sum\limits_{k\in\mathbb{N}} 2^{-k} x_k$, with the exclusion of the dyadic rational numbers of the interval and those elements of G that have infinite "tail" of zeros or the units of length. This representation is unique.

It is obvious that the subset

$$U_{k-1} = \{\, x = (\underbrace{0, \dots, 0}_{k-1}, x_k, \dots) \,\}, \quad k \in \mathbb{N}$$

of G is a subgroup of G and

$$G = U_0 \supset U_1 \supset \dots, \quad \bigcap_{k\in\mathbb{N}} U_k = \{0\}.$$

This chain of subgroups determines a topology of G, being the basic system of neighbourhoods of the zero of G. On the group G one can define, in a standard way, the translation $\dotplus$ invariant normalized Lebesgue-Haar measure and the Lebesgue integral with respect this measure (see [1]). In addition, we have

$$\mu(U_{k-1}) = 2^{-(k-1)}, \quad k \in \mathbb{N}.$$

Let $f : G \to \mathbb{R}$. The sequence

$$\omega^{(p)}(f) = \{\omega_k^{(p)}(f)\} = \{\sup_{h\in U_k} ||f(x \dot{+} h) - f(x)||_{L_p(G)};\ k = 0, 1, \dots ,\ 1 \leqslant p \leqslant \infty\} \searrow 0$$

is called the modulus of continuity of the function f in $L_p(G)$ (see, e.g. [2]).

As is established in [2], [3], [4], for arbitrary $\omega = \{\omega_k\} \searrow 0$ and arbitrary $p \in [1; +\infty]$ one can find a function $f \in L_p(G)$ such that

$$\omega_k^{(p)}(f) = \omega_k, \quad k = 0, 1, \dots .$$

In [5] the operator has been defined

$$(Kf)(x) \ = \ - \lim_{m\to\infty} \int\limits_{G\setminus U_m} (f(x \dot{+} t) - f(x))\, K(t)\, d\mu(t)\,, \tag{1.1}$$

where for $t \neq 0$

$$K(t) = 2^n \quad \text{for} \quad t \in U_{n-1} \setminus U_n\,, \quad n \in \mathbb{N}.$$

It is obvious that operator (1.1) appears as a certain analogue of the conjugate operator.

As is established in [5]

$$(Kw_n)(x) \ = \ (k + 2)\, w_n(x)$$

for $2^k \leqslant n < 2^{k+1}$, $k = 0, 1, \dots ,$ where $\{w_n(x)\}$ is the Walsh-Paley system.

It is natural question: when the condition

$$\omega_n^{(1)}(f) \ = \ O(\omega_n)\,, \quad n \in \mathbb{N}$$

implies the condition

$$Kf \in L(G)\,.$$

We shall prove the following assertion.

Proposition 1.1. *(i) The condition* $n\omega_n \to 0$ *is necessary for the inclusion*

$$Kf \in L(G) \tag{1.2}$$

for all functions f *satisfying the condition* $\omega_n^{(1)}(f) \ = \ O(\omega_n)$.
(ii) The condition

$$\sum_{n\in\mathbb{N}} n^2\, \omega_n \ < \ \infty$$

is sufficient for (1.2).

If the function f is considered on $[0; 1]$ then

$$\omega_n^{(1)}(f) \ \sim \ \sup_{0 \leqslant t \leqslant 2^{-n}} \int\limits_0^1 |f(x + t) - f(x)|\, dt \ = \ \omega^{(1)}(f; 2^{-n})\,.$$

For $\{\omega_n\} \searrow 0$ we will consider the function

$$f(x) \;=\; \frac{1}{2} \sum_{n\geqslant 0} \omega_n \, T_n(x)\,, \tag{1.3}$$

where

$$T_n(x) \;=\; \begin{cases} 2^n & \text{if} \quad x \in U_{n+1}\,, \\ -2^n & \text{if} \quad x \in U_n \setminus U_{n+1}\,, \\ 0 & \text{if} \quad x \in G \setminus U_n\,. \end{cases}$$

It is clear, that series (1.3) has the finite sum for $x \neq 0$. Moreover,

$$f(x) \;=\; \sum_{k=1}^{n} 2^{k-1}(\omega_{k-1} - \omega_k) \quad \text{for} \quad x \in U_n \setminus U_{n+1} \tag{1.4}$$

and

$$\int\limits_G |f(x)| \, d\mu(x) \;=\; \sum_{n\in\mathbb{N}} \sum_{k=1}^{n} 2^{k-1}(\omega_{k-1} - \omega_k)\, 2^{-(n+1)} \;<\; \infty\,,$$

whenewer $f \in L(G)$.

Let us calculate $\omega_n^{(1)}(f)$. We have

$$\omega_n^{(1)}(f) = \sup_{h\in U_n} \int\limits_G |f(x \dot{+} h) - f(x)| \, d\mu(x) = \sup_{k\geqslant n} \; \sup_{h\in U_k\setminus U_{k+1}} \int\limits_G |f(x \dot{+} h) - f(x)| \, d\mu(x)$$

$$= \sup_{k\geqslant n} \; \sup_{h\in U_k\setminus U_{k+1}} \sum_{j=0}^{2^k-1} \; \int\limits_{U_k \dot{+} j\cdot 2^{-k}} |f(x \dot{+} h) - f(x)| \, d\mu(x)\,.$$

By (1.4) the function f is constant on $U_k \dot{+} j\cdot 2^{-k}$ for $j \neq 0$. One has $x \dot{+} h \in U_k \dot{+} j\cdot 2^{-k}$ for $x \in U_k \dot{+} j \cdot 2^{-k}$. Hence

$$\begin{aligned} \omega_n^{(1)}(f) \;=\; & \sup_{k\geqslant n} \; \sup_{h\in U_k\setminus U_{k+1}} \Biggl(\; \int\limits_{U_{k+1}} |f(x \dot{+} h) - f(x)| \, d\mu(x) \\ & + \int\limits_{U_k\setminus U_{k+1}} |f(x \dot{+} h) - f(x)| d\mu(x) \Biggr) = \sup_{k\geqslant n} \; \sup_{h\in U_k\setminus U_{k+1}} \Biggl(\Bigl| \sum_{\nu=1}^{k} 2^{\nu-1}(\omega_{\nu-1} - \omega_\nu) \Bigr| \\ & + \int\limits_{U_k\setminus U_{k+1}} \Bigl| f(x \dot{+} h) - \sum_{\nu=1}^{k} 2^{\nu-1}(\omega_{\nu-1} - \omega_\nu) \Bigr| \, d\mu(x) \Biggr). \end{aligned} \tag{1.5}$$

The translation $x \dot{+} h$ preserves the measure, and if $x, h \in U_k \setminus U_{k+1}$ then $\{x \dot{+} h\} = U_{k+1} \bigcup (U_k \setminus U_{k+1})$. But $f(x) = \mathit{const}$ on U_{k+1} and $U_k \setminus U_{k+1}$. Therefore, by (1.5) one has

$$\begin{aligned}
\omega_n^{(1)}(f) &= 2\sup_{k\geqslant n}\int\limits_{U_{k+1}} |\,2^{\nu-1}(\omega_{\nu-1}-\omega_\nu)-f(x)\,|\,d\mu(x)\\
&= 2\sup_{k\geqslant n}\sum_{s\geqslant k+1}\int\limits_{U_s\setminus U_{s+1}}\Big|\sum_{\nu=1}^{k}2^{\nu-1}(\omega_{\nu-1}-\omega_\nu)-f(x)\Big|\,d\mu(x)\\
&= 2\sup_{k\geqslant n}\sum_{s\geqslant k+1}\Big|\sum_{\nu=1}^{k}2^{\nu-1}(\omega_{\nu-1}-\omega_\nu)\,-\,\sum_{\nu=1}^{s}2^{\nu-1}(\omega_{\nu-1}-\omega_\nu)\,2^{-(s+1)}\Big|\\
&= 2\sup_{k\geqslant n}\sum_{s\geqslant k+1}\Big|\sum_{\nu=k+1}^{s}2^{\nu-1}(\omega_{\nu-1}-\omega_\nu)\Big|\,2^{-(s+1)}\\
&= 2\sum_{s\geqslant n+1}\left(\sum_{\nu\geqslant n+1}^{s}2^{\nu-1}(\omega_{\nu-1}-\omega_\nu)\right)2^{-(s+1)}\,. \qquad (1.6)
\end{aligned}$$

From (1.3) we find

$$T_n(x) = \sum_{k=2^n}^{2^{n+1}-1} w_k(x)$$

($\{w_k(x)\}$ is the Walsh-Paley system).

Thus by (1.6) we have

$$\omega_n^{(1)}(f) = \sum_{\nu\geqslant n+1} 2^\nu(\omega_{\nu-1}-\omega_\nu)\sum_{s\geqslant\nu}2^{-(s+1)} = \omega_n$$

and

$$\omega_n^{(1)}(Kf) = (n+2)\,\omega_n\,.$$

The point (i) in Proposition 1.1 is proved.

Let

$$c_0 + \sum_{n\geqslant 0}\sum_{k=2^n}^{2^{n+1}-1} c_k\,w_k(x)$$

be the Fourier-Walsh-Paley series for the function f. Then

$$\begin{aligned}
(Kf)(x) &\sim \sum_{n\geqslant 0}(n+2)\sum_{k=2^n}^{2^{n+1}-1}c_k w_k(x) = \sum_{n\geqslant 0}(n+2)\Big(S_{2^{n+1}-1}(f;\,x)-S_{2^n-1}(f;\,x)\Big)\\
&= \sum_{n\geqslant 0}(n+2)\Big((S_{2^{n+1}-1}(f;\,x)-f(x))-(S_{2^n-1}(f;\,x)-f(x)\Big), \qquad (1.7)
\end{aligned}$$

where $S_n(f;\,x)$ is the partial sum of the Fourier-Walsh-Paley series for f.

It follows from (1.7) that

$$||Kf||_{L(G)} \leqslant c \sum_{n\in\mathbb{N}} n \, ||S_{2^n}(f;\, x) - f(x)||_{L(G)} \,. \tag{1.8}$$

By [6], p. 45 one has

$$\begin{aligned} |S_{2^k}(f;\, x) - f(x)| &= \Big| \int\limits_G D_{2^k}(t)(f(x \dotplus t) - f(x)) \, d\mu(t) \Big| \\ &= 2^k \Big| \int_{U_k} (f(x \dotplus t) - f(x)) \, d\mu(t) \Big| \leqslant 2^k \, \omega_n^{(1)}(f) \leqslant C \cdot n \, \omega_n^{(1)}(f) \end{aligned}$$

for $2^k \leqslant n < 2^{k+1}$.

By using (1.8) we find that

$$Kf \in L(G) \,,$$

if

$$\sum_{n\in\mathbb{N}} n^2 \, \omega_n^{(1)}(f) < \infty \,.$$

Proposition 1.1 is completely proved.

In conclusion, we recall the following statement proved in [7].

Proposition 1.2. *If the function* $a(y)[\log_2 y] > 0$ *is convex down for* $y > 1$,

$$\lim_{y\to\infty} a(y)[\log_2 y] = 0 \,,$$

and

$$y(a(y)[\log_2 y] - a(y+1)[\log_2(y+1)]) \downarrow , \quad y \cdot a(y) \uparrow ,$$

then

$$(Kf)(x) < C_1 \left\{ \frac{1}{x} \, a\left(\frac{1}{x}\right) \left[\log_2 \frac{1}{x}\right] + \int\limits_1^{1/x} t \Big(a(t) \, [\log_2 t] - a(t+1) \, [\log_2(t+1)] \Big) \, dt \right\};$$

$$(Kf)(x) > C_2 \int\limits_1^{1/x} t \Big(a(t) \, [\log_2 t] - a(t+1) \, [\log_2(t+1)] \Big) \, dt \,;$$

$$\varlimsup_{x\to+0} \frac{(Kf)(x)}{\frac{1}{x} \, a\left(\frac{1}{x}\right) \left[\log_2 \frac{1}{x}\right] + \int\limits_1^{1/x} t \Big(a(t) \, [\log_2 t] - a(t+1) \, [\log_2(t+1)] \Big) \, dt} > C_3 \,;$$

$$\varliminf_{x\to+0} \frac{(Kf)(x)}{\int\limits_1^{1/x} t \Big(a(t) \, [\log_2 t] - a(t+1) \, [\log_2(t+1)] \Big) \, dt} < C_4 \,,$$

where

$$(Kf)(x) = \sum_{n\in\mathbb{N}} a(n) \, [\log_2 n] \, w_n(x),$$

and C_1, C_2, C_3, C_4 *are some positive constants.*

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Aleksander Iosifovich Rubinstein
Department of Higher Mathematics
National Research Nuclear University (MEPHI)
31 Kashirskoe Ave.
115409 Moscow, Russia
and
Department of Higher Mathematics and Physics
Moscow State Forest University
1 First Institutskaya St.
141005 Mytischi, Moscow region, Russia
E-mail: rubinstein_aleksandr@mail.ru

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