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NURZHAN BOKAYEV

(to the 60th birthday)

On January 5, 2016 was the 60th birthday of Doctor of Physical-Mathematical Sciences (1996), Professor Nurzhan Adilkhanovich Bokayev. Professor Bokayev is the head of the department "Higher Mathematics" of the L.N. Gumilyov Eurasian National University (since 2009), the Vice-President of Mathematical Society of the Turkic World (since 2014), and a member of the Editorial Board of our journal.

N.A. Bokayevwas born in the Urnek village, Karabalyk district, Kostanay region. He graduated from the E.A. Buketov Karaganda State University in 1977 and the M.V. Lomonosov Moscow State University's full-time postgraduate study in 1984.

Scientific works of Professor Bokayev are devoted to studying problems of the theory of functions, in particular of the theory of orthogonal series.

He proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems; investigated Besov-type function spaces with respect to the Price bases; studied the Price - and Haar-type p-adic operators; introduced new classes of Faber-Schauder-type systems of functions and spaces of multivariable functions of bounded p-variation and of bounded p-fluctuation, obtained estimates for the best approximation of functions in these spaces by polynomials with respect to the Walsh and Haar systems, established weighted integrability conditions of the sum of multiple trigonometric series and series with respect to multiplicative systems, found the embedding criterion for the Nikol'skii-Besov spaces with respect to multiplicative bases and the coefficient criterion for belonging of functions to such spaces.

His scientific results have made essential contribution to the theory of series with respect to the Walsh and Haar systems and multiplicative systems.

N.A. Bokayev has published more than 150 scientific papers. Under his supervision 8 dissertations have been defended: 4 candidate of sciences dissertations and 4 PhD dissertations.

The Editorial Board of the Eurasian Mathematical Journal congratulates Nurzhan Adilkhanovich Bokayev on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

The EMJ has been included in the Emerging Sources Citation Index

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On behalf of the Editorial Board of the EMJ
V.I. Burenkov, T.V. Tararykova, A.M. Temirkhanova

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ON THE EXISTENCE OF A RESOLVENT AND SEPARABILITY FOR A CLASS OF SINGULAR HYPERBOLIC TYPE DIFFERENTIAL OPERATORS ON AN UNBOUNDED DOMAIN

M. Muratbekov, M. Otelbaev

Communicated by K.N. Ospanov

Key words: hyperbolic type operator, resolvent, separability, unbounded domain.

AMS Mathematics Subject Classification: 35M10.

Abstract. In the present work the existence of a resolvent and separability for a class of hyperbolic type operators with increasing coefficients in an unbounded domain are proved.

1 Introduction

Let $\Omega = \{(x, y) : -\infty < x < \infty, -1 < y < 1\}$. Consider the differential operator

$$L_0 u = u_{xx} - u_{yy} + a(x)u_x + c(x)u \tag{1.1}$$

initially defined on the set $C_0^\infty(\overline{\Omega})$, which consists of all infinitely differentiable functions satisfying the condition $u(x, -1) = u(x, 1) = 0$ and compactly supported with respect to the variable x. The coefficients $a(x)$ and $c(x)$ are continuous functions in $\mathbb{R} = (-\infty, \infty)$.

Note that the coefficients $a(x), c(x)$ of operator (1.1) can be unbounded functions.

Differential operators of hyperbolic type on $\mathbb{R}^n$ were considered in [13], where the coefficients of the operator were continuous and unbounded function in $\mathbb{R}^n$.

In this paper we study the issues on the existence of a resolvent and on separability of operator (1.1)

Definition 1. We call the operator L separable if the following estimate

$$\|u_{xx} - u_{yy}\|_2 + \|a(x)u_x\|_2 + \|c(x)u\|_2 \le c(\|Lu\|_2 + \|u\|_2)$$

holds for all $u \in D(L)$, where $c > 0$ is constant which does not depend on u.

The term "separability" was introduced in the fundamental papers of W.N. Everitt and M. Giertz [17-20], where they investigated the separability of the Sturm-Liouville operator. Later issues of separability of differential operators were studied by M. Otelbaev [14-15], K.Kh. Boymatov [1-2], and their disciples. The main results obtained in these papers are based on the method of localization of resolvents of differential operators and the various options, which had been proposed by the author of [14]. In the works of the above authors separability was studied only for elliptic and pseudo-differential operators. A very comprehensive bibliography is contained in [14,15,1-7].

Operators of hyperbolic and mixed types in the case of an unbounded domain with growing and oscillating coefficients were considered in [8-11]. In these works the studied operators were initially defined on the set of infinitely differentiable functions and the coefficients were periodic with respect to the variable x and compactly supported with respect to y in the domain $\Omega = \{(x,y) : -\pi < x < \pi, -\infty < y < \infty\}$. In contrast to operator (1.1) the coefficients at $u_x(x,y)$ and $u(x,y)$ of the studied operator depended only on y.

A completely different situation arises when investigating operator (1.1). In particular it is not anymore possible to use the method of separation of variables from [8-11].

Denote by $K(\tau, b)$ the class of the coefficients satisfying the conditions:

i) $|a(x)| \geq \delta_0 > 0$, $c(x) \geq \delta > 0$ are continuous functions in $\mathbb{R}$;

ii) $c_0 c(x) \leq a^2(x) \leq c_1 c(x)$ for all $x \in \mathbb{R}$, $c_0 > 0$ and $c_1 > 0$ are constants;

iii) $|a(x)-a(t)|^2+|c(x)-c(t)| \leq \tau c(t)$ for all $x, t \in \mathbb{R}$ such that $|x-t| \leq bd(t)$, $d(t) = \frac{1}{[c(t)]^{1/2}}$, $b > 0$, $\tau > 0$.

Theorem 1.1. *Let $a(x), c(x) \in K(\tau, b)$. Then there exist numbers τ_0 and b_0 such that, for all $\tau \in (0, \tau_0)$ and $b > b_0$ the closure L of the operator $L_0 u = u_{xx} - u_{yy} + a(x)u_x + c(x)u$ with the domain $D(L_0) = C_0^\infty(\overline{\Omega})$ exists in $L_2(\Omega)$.*

Theorem 1.2. *Let $a(x), c(x) \in K(\tau, b)$. Then there exist numbers τ_0 and b_0 such that, for all $\tau \in (0, \tau_0)$ and $b > b_0$ the operator L is continuously invertible operator in $L_2(\Omega)$.*

Theorem 1.3. *Let the conditions of Theorem 1.2 hold. Then the operator L is separable.*

It is easy to check that all conditions of Theorems 1.1-1.3 hold for the operator $L_0 u = u_{xx} - u_{yy} + (|x| + 32)u_x + (4x^2 + 4^6)u$. Therefore this operator has a closure is continuously invertible and separable.

First we will analyze some important properties of the resolvent of an operator with constant coefficients.

2 The properties of the resolvent of a differential operator with constant coefficients

Consider the operator

$$L_j u = u_{xx} - u_{yy} + a(x_j)u_x + c(x_j)u \tag{2.1}$$

defined on the set $C_0^\infty(\bar{\Omega})$, where $x_j \in \mathbb{R}$, $j = 1, 2, ...$ (their special choice will be made later).

Evidently, the operator L_j admits a closure in $L_2(\Omega)$ which will be denoted again by L_j.

Theorem 2.1. *Let conditions i)-ii) hold. Then the operator L_j is continuously invertible in $L_2(\Omega)$.*

Theorem 2.2. *Let conditions i)-ii) hold. Then the following estimates hold:*

$$\|L_j^{-1}\|_{2\to 2} \leq \frac{c}{c(x_j)}; \tag{2.2}$$

$$\|D_x L_j^{-1}\|_{2\to 2} \leq \frac{c}{|a(x_j)|}; \tag{2.3}$$

$$\|D_y L_j^{-1}\|_{2\to 2} \leq \frac{c}{\sqrt{c(x_j)}}, \tag{2.4}$$

where, from now on, $c > 0$ is a positive number (in general, different in different places), and $\|\cdot\|_2$ is the norm in the space $L_2(\Omega)$.

We need to consider some lemmas for the proof of these theorems.

2.1 Auxiliary statements and estimates in dimension one

Consider the operator

$$l_{t,j}u = -u^{''}(y) + (-t^2 + ita(x_j) + c(x_j))u(y) \ \ (-\infty < t < \infty)$$

defined on $C_0^2[-1,1]$, which consists of all infinitely differentiable functions on $[-1,1]$ satisfying the condition $u(x,-1) = u(x,1) = 0$.

The operator $l_{t,j}$ admits a closure in $L_2(-1,1)$ which will be denoted by $l_{t,j}$ again.

Lemma 2.1. *Let conditions i)-ii) hold. Then:*

a) $\|l_{t,j}u\|_2 \geq |a(x_j)||t|\|u\|_2, \ \ u \in D(l_{t,j}), \ \ -\infty < t < \infty,$

b) $c\|l_{t,j}u\|_2 \geq c(x_j)\|u\|_2, \ \ u \in D(l_{t,j}), \ \ -\infty < t < \infty,$

c) $\frac{c}{\sqrt{c(x_j)}}\|l_{t,j}u\|_2 \geq \|u'\|_2, \ \ u \in D(l_{t,j}), \ \ -\infty < t < \infty,$

where $D(l_{t,j})$ is the domain of the closed operator.

Lemma 2.2. *Let conditions i)-ii) hold. Then the operator $l_{t,j}$ is continuously invertible in $L_2(-1,1)$, and*

$$\|l_{t,j}^{-1}\|_{2\to 2} \leq \frac{c}{c(x_j)}.$$

To prove these lemmas we consider the following differential operator:

$$(l_{t,j,\gamma} + \lambda E)u = -u^{''}(y) + (-t^2 + it(a(x_j) + \gamma)) + c(x_j) + \lambda)u(y) \tag{2.5}$$

defined on the set $C_0^2[-1,1]$, where $\gamma a(x) > 0$ for all $x \in \mathbb{R}, \lambda \geq 0$.

The closure of $l_{t,j,\gamma}$ is also denoted by $l_{t,j,\gamma}$ and further referring to expression (2.5) we will consider this closure.

Lemma 2.3. *Let conditions i)-ii) hold and $\lambda \geq 0$. Then:*

a) $\|(l_{t,j,\gamma} + \lambda E)u\|_2 \geq (|a(x_j)| + |\gamma|)|t|\|u\|_2, \;\; u \in D(l_{t,j,\gamma}), \;\; -\infty < t < \infty,$

b) $c\|(l_{t,j,\gamma} + \lambda E)u\|_2 \geq (c(x_j) + \lambda)\|u\|_2, \;\; u \in D(l_{t,j,\gamma}), \;\; -\infty < t < \infty,$

c) $\frac{c}{\sqrt{c(x_j)+\lambda}}\|(l_{t,j,\gamma} + \lambda E)u\|_2 \geq \|u'\|_2, \;\; u \in D(l_{t,j,\gamma}), \;\; -\infty < t < \infty,$

where $D(l_{t,j,\gamma})$ is the domain of the closed operator.

Proof. Let $u \in C_0^2[-1,1]$. Integrating by parts the expression $< (l_{t,j,\gamma} + \lambda E)u, u >$, and taking into account that terms outside the integral vanish due to the condition $u(-1) = u(1) = 0$, we obtain

$$| < (l_{t,j,\gamma} + \lambda E)u, u > | = \left| \int_{-1}^{1} \left[|u'|^2 + (-t^2 + it(a(x_j) + \gamma) + c(x_j) + \lambda)|u|^2 \right] dy \right|. \tag{2.6}$$

Hence, using the properties of complex numbers, we obtain

$$| < (l_{t,j,\gamma} + \lambda E)u, u > | \geq \left| \int_{-1}^{1} it(a(x_j) + \gamma)|u|^2 dy \right| \geq |t|(|a(x_j)| + |\gamma|)\|u\|_2^2, \tag{2.7}$$

where we have taken into account that $\gamma a(x_j) > 0$. From (2.7) and using the Cauchy-Schwarz inequality, we have

$$|(l_{t,j,\gamma} + \lambda E)u| \geq |t|(|a(x_j)| + |\gamma|)\|u\|_2. \tag{2.8}$$

By the closedness of $l_{t,j,\gamma} + \lambda E$, this inequality is valid for all $u \in D(l_{t,j,\gamma})$. The part $a)$ of Lemma 2.3 is proved.

Now, we prove the part $b)$ of Lemma 2.3. From (2.6), using the Cauchy-Schwarz inequality with the parameter $\varepsilon > 0$, we have

$$\frac{1}{2(c(x_j) + \lambda)}\|(l_{t,j,\gamma} + \lambda E)u\|_2^2 + \frac{c(x_j) + \lambda}{2}\|u\|_2^2 \geq$$
$$\geq \int_{-1}^{1} \left[|u'|^2 + (c(x_j) + \lambda)|u|^2 \right] dy - \int_{-1}^{1} |t|^2 |k(y)||u|^2 dy, \tag{2.9}$$

where $\varepsilon = c(x_j) + \lambda$. Going back to (2.8), squaring, and multiplying both sides by $\frac{\tilde{c}}{2(c(x_j)+\lambda)}$, we obtain

$$\frac{\tilde{c}}{2(c(x_j) + \lambda)}\|(l_{t,j,\gamma} + \lambda E)u\|_2^2 \geq \frac{\tilde{c}|t|^2(|a(x_j)| + |\gamma|)^2}{2(c(x_j) + \lambda)}\|u\|_2^2, \tag{2.10}$$

where $\tilde{c} > 0$ is a constant. From (2.9) and (2.10) it follows that

$$\frac{c_2}{2(c(x_j) + \lambda)}\|(l_{t,j,\gamma} + \lambda E)u\|_2^2 \geq \int_{-1}^{1} \left[|u'|^2 + \left[(c(x_j) + \lambda) - \frac{c(x_j) + \lambda}{2} \right] |u|^2 \right] dy +$$
$$+ \int_{-1}^{1} |t|^2 \left[\frac{\tilde{c}|t|^2(a(x_j) + \gamma)^2}{2(c(x_j) + \lambda)} - 1 \right] |u|^2 dy, \tag{2.11}$$

where $c_2 = \tilde{c} + 1$. Hence taking into account condition *ii*) and choosing γ and $\tilde{c}$ so that $\frac{\tilde{c}|t|^2(a(x_j)+\gamma)^2}{2(c(x_j)+\lambda)} - 1 \geq 0$, we obtain

$$\frac{c}{c(x_j)+\lambda}\|(l_{t,j,\gamma} + \lambda E)u\|_2^2 \geq \int_{-1}^{1} \left[|u'|^2 + (c(x_j)+\lambda)|u|^2\right] dy. \tag{2.12}$$

From (2.12) it follows that

$$c\|(l_{t,j,\gamma} + \lambda E)u\|_2 \geq (c(x_j)+\lambda)\|u\|_2,$$

So, the part *b*) of Lemma 2.3 is proved. The part *c*) of Lemma 2.3 follows from (2.12). □

Lemma 2.4. *Let conditions i)-ii) hold and* $\lambda \geq 0$. *Then the operator* $l_{t,j,\gamma} + \lambda E$ *is continuously invertible in* $L_2(-1,1)$, *and*

$$\|(l_{t,j,\gamma} + \lambda E)^{-1}\|_{2\to 2} \leq \frac{c}{c(x_j)+\lambda}$$

Proof. From the part *b*) of Lemma 2.3 it follows that the operator $(l_{t,j,\gamma} + \lambda E)^{-1}$ exists and is bounded on $R(l_{t,j,\gamma} + \lambda E)$. It remains to show that $R(l_{t,j,\gamma} + \lambda E) \equiv L_2(-1,1)$. Assume the contrary. Let $R(l_{t,j,\gamma} + \lambda E) \neq L_2(-1,1)$. Then there exists a function $v \in L_2(-1,1)$, $v \neq 0$ such that

$< (l_{t,j,\gamma} + \lambda E)u, v >= 0$ for all $u \in D(l_{t,j,\gamma})$.

This means that

$< u, (l_{t,j,\gamma} + \lambda E)^* v >= 0$ for all $u \in D(l_{t,j,\gamma})$.

Hence, in the sense of distributions, we have

$(l_{t,j,\gamma} + \lambda E)^* v = -v'' + (-t^2 - it(a(x_j)+\gamma) + c(x_j) + \lambda)v = 0.$

By direct calculations we can verify that $v'' \in L_2(-1,1)$.
Further, integrating by parts, we have

$$0 =< u, (l_{t,j,\gamma} + \lambda E)^* v >= u'\bar{v} \mid_{-1}^{1} + < (l_{t,j,\gamma} + \lambda E)u, v >, \ \ u \in D(l_{t,j,\gamma}).$$

By assumption,$< (l_{t,j,\gamma} + \lambda E)u, v >= 0$, therefore $\bar{v} \mid_{-1}^{1} = 0$. Hence, by the arbitrariness of $u(y)$, it follows that $v(-1) = v(1) = 0$. Taking into account this equality, integrating by parts the expression $| < (l_{t,j,\gamma} + \lambda E)^* v, v > |$ and using condition *i*) we obtain the following inequality

$$\|(l_{t,j,\gamma} + \lambda E)^* v\|_2 \geq c\|v\|_2,$$

where $c = c(\delta, \delta_0)$. From the last inequality it follows that $v = 0$. □

Consider the problem

$$(l_{t,j} + \lambda E)u = -u''(y) + (-t^2 + it(a(x_j) + c(x_j) + \lambda)u(y) = f, \tag{2.13}$$

$$u(-1) = u(1) = 0, \tag{2.14}$$

where $f \in L_2(-1,1)$.

By a solution of problem (2.13)-(2.14) we mean a function $u \in L_2(-1,1)$ for which there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^2[-1,1]$ such that

$$\|u_n - u\|_2 \to 0, \quad \|(l_{t,j} + \lambda E)u_n - f\|_2 \to 0 \quad \text{as} \quad n \to \infty,$$

where $\|\cdot\|_2$ is the $L_2(-1,1)$- norm.

Lemma 2.5. *Let $\lambda \geq 0$. Then the operator $l_{t,j} + \lambda E$ is continuously invertible, and for the inverse operator $(l_{t,j} + \lambda E)^{-1}$ the equality*

$$(l_{t,j} + \lambda E)^{-1} f = (l_{t,j,\gamma} + \lambda E)^{-1}(E - A_\gamma)^{-1} f, f \in L_2[-1,1], \tag{2.15}$$

holds, where $\|A_\gamma\|_{2\to 2} < 1$.

Proof. Problem (2.13) - (2.14) is equivalent to the equation $v - A_\gamma v = f$, where

$$v = (l_{t,j,\gamma} + \lambda E)u, \quad A_\gamma = it\gamma(l_{t,j,\gamma} + \lambda E)^{-1}.$$

From the part $a)$ of Lemma 2.3 it follows that

$$\|A_\gamma v\|_2 \leq \frac{\cdot|\gamma|}{(\delta_0 + |\gamma|)}\|v\|_2.$$

Hence $\|A_\gamma\|_{2\to 2} < 1$ and

$$u = (l_{t,j} + \lambda E)^{-1} f = (l_{t,j,\gamma} + \lambda E)^{-1}(E - A_\gamma)^{-1} f.$$

In the case $t = 0$ the operator $l_{0,j} + \lambda E$ is self adjoint (see [16], V.3.6) and the estimate $\|(l_{0,j} + \lambda E)u\|_2 \geq (c(x_j) + \lambda)\|u\|_2$ holds. Hence, it follows that the operator $l_{0,j} + \lambda E$ has a bounded inverse $(l_{0,j} + \lambda E)^{-1}$ defined on the whole $L_2[-1,1]$. □

Now, Lemma 2.1 and Lemma 2.2 follow from lemmas 2.3-2.5.

2.2 Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Consider the problem

$$L'_j u = u_{xx} - u_{yy} - a(x_j)u_x + c(x_j)u = f \in C_0^\infty(\overline{\Omega}),$$

$$u(x,-1) = u(x,1) = 0,$$

where the operator L'_j is the formal adjoint operator of L_j.

Applying the Fourier transform with respect to the variable x we obtain

$$l'_{t,j}\tilde{u} = -\tilde{u}'' + (-t^2 - ita(x_j) + c(x_j))\tilde{u} = \tilde{f}(t,y),$$

$$\tilde{u}(-1) = \tilde{u}(1) = 0$$

where $\tilde{u}, \tilde{f}$ are the Fourier transforms with respect to the variable x.

It is easy to show that there exists the inverse operator $l'^{-1}_{t,j}$ defined on $L_2(-1,1)$. To show this we repeat the calculations and reasoning used in the proof of Lemma 2.1 [10]. Hence, by using properties of the Fourier transform we obtain

$$u(x,y) = L'^{-1}_j f = F^{-1}_{t\to x} l'^{-1}_{t,j} \tilde{f}. \tag{2.16}$$

Lemma 2.6. *Let conditions i)-ii) hold. Then*

$$D(L_j) \subseteq D(L_j'^*)$$

where $D(L_j)$ *and* $D(L_j'^*)$ *are the domains of the operators* L_j *and* $L_j'^*$*, respectively.*

Proof. According to the definition of the adjoint operator the equality

$$\langle L_j'u, v\rangle = \langle u, L_j'^*v\rangle$$

holds for all $u(x,y) \in D(L_j')$, $v(x,y) \in D(L_j'^*)$.

If we prove that this equality also holds for all $v \in D(L_j)$ then the lemma is proved.

Let $u_n(x,y) \in C_0^\infty(\overline{\Omega})$ and $u_n \to u \in D(L_j')$, $v_n(x,y) \in C_0^\infty(\overline{\Omega})$ and $v_n \to v \in D(L_j)$. Then the equality

$$< L_j'u_n, v_n > = < u_n, L_jv_n > \tag{2.17}$$

holds for all $u_n, v_n \in C_0^\infty(\overline{\Omega})$.

The validity of this equality can be verified by integrating by parts and taking the boundary conditions into account.

Now, passing to the limit in (2.17)

$$< L_j'u, v > = < u, L_jv > .$$

□

Lemma 2.7. *Let conditions i)-ii) hold. Then* $Ker(L_j) = \{0\}$.

Proof. From (2.16) it follows that the range of the operator L_j' coincides with the whole space $L_2(\Omega)$, i.e.

$$R(L_j') = L_2(\Omega). \tag{2.18}$$

From the general theory of linear operators it is well known that

$$L_2(\Omega) = R(L_j') \dot{+} Ker L_j'^*,$$

where $\dot{+}$ denotes the orthogonal sum. Thus from (2.18) we obtain $Ker L_j'^* = \{0\}$.

Hence, using Lemma 2.6 we obtain $Ker(L_j) = \{0\}$. □

Now we show the existence of the inverse operator L_j^{-1} of the operator L_j.

To do this, consider the problem:

$$L_ju = u_{xx} - u_{yy} + a(x_j)u_x + c(x_j)u = f \in C_0^\infty(\Omega), \tag{2.19}$$

$$u(x,-1) = u(x,1) = 0.$$

Applying the Fourier transform with respect to the variable x we obtain

$$l_{t,j}\tilde{u} = -\tilde{u}'' + (-t^2 + ita(x_j) + c(x_j))\tilde{u} = \tilde{f}(t,y),$$

$$\tilde{u}(-1) = \tilde{u}(1) = 0,$$

where

$$\tilde{u}(t,y) = F_{x\to t}u(x,y) = \frac{1}{\sqrt{2\pi}}\int_R u(x,y)e^{-ixt}dx$$

$$\tilde{f}(t,y) = F_{x\to t}f(x,y) = \frac{1}{\sqrt{2\pi}}\int_R f(x,y)e^{-ixt}dx$$

By Lemma 2.2 it follows that

$$\tilde{u} = l_{t,j}^{-1}\tilde{f}.$$

Next, using the Fourier transform $F_{t\to x}^{-1}$ we obtain

$$u(x,y) = L_j^{-1}f = F_{t\to x}^{-1}l_{t,j}^{-1}\tilde{f}. \tag{2.20}$$

This equality holds for all $f \in L_2(\Omega)$, by virtue of the continuity of the operator $l_{t,j}^{-1}$ and of the Fourier transform. Hence by Lemma 2.7 it follows that the continuous operator L_j^{-1} defined in $L_2(\Omega)$ exists. □

Proof of Theorem 2.2. From (2.20) using the properties of the Fourier transform we obtain

$$\|u\|_2^2 = \|L_j^{-1}f\|_2^2 = \|F_{t\to x}^{-1}l_{t,j}^{-1}\tilde{f}\|_2^2 = \int_{-\infty}^{+\infty}\left(\int_{-1}^{1}|l_{t,j}^{-1}\tilde{f}(t,y)|^2dy\right)dt \le \sup_{t\in\mathbb{R}}\|l_{t,j}^{-1}\|_{2\to2}^2\|f\|_2^2.$$

Hence by Lemma 2.2 we obtain

$$\|L_j^{-1}\|_{2\to2}^2 \le \frac{c^2}{c^2(x_j)}. \tag{2.21}$$

Inequality (2.2) is proved.

Next

$$D_xL_j^{-1}f(x,y) = \frac{\partial}{\partial x}F_{t\to x}^{-1}l_{t,j}^{-1}\tilde{f}(t,y) = F_{t\to x}^{-1}(it)l_{t,j}^{-1}\tilde{f}(t,y).$$

Hence

$$\|D_xL_j^{-1}f\|_2^2 \le \sup_{t\in\mathbb{R}}\|itl_{t,j}^{-1}\|_{2\to2}^2\|f\|_2^2.$$

By virtue of the part $a)$ of Lemma 2.1 from the last inequality we obtain

$$\|D_xL_j^{-1}f\|_2^2 \le \frac{1}{|a(x_j)|^2}\|f\|_2^2,$$

which proves inequality (2.3).

Similarly, and using the part $c)$ of Lemma 2.1 we obtain

$$\|D_yL_j^{-1}f\|_2^2 \le \frac{c^2}{c(x_j)}\|f\|_2^2.$$

□

3 Proofs of main theorems

To prove the main theorems, in addition to the lemmas given above, we will use the following statement on coverings of de Guzman-Besicovitch type (see [12,15]).

Lemma 3.1. *Let conditions i)-iii) hold. Then there exists a covering such that:*

a) $\sum\limits_{\{j\}} \varphi_j^2 \equiv 1,\ \ \varphi_j \in C_0^\infty(\Delta_j),\ \ \bigcup\limits_{\{j\}} \Delta_j = \mathbb{R},\ \ j \in Z;$

b) $\|D_x^\alpha \varphi_j\|_{C(\mathbb{R})} \le \frac{c}{b^\alpha d^\alpha(x_j)}$,

where $b > 0, c > 0, \alpha = 0,1,2,$ $d(x_j)$ *is the length of the interval* $\Delta_j = (x_j - \frac{d(x_j)}{2}, x_j + \frac{d(x_j)}{2})$;

c) every set Δ_j *intersects at most* ξ *sets of the family* $\{\Delta_j\}_{j=-\infty}^{j=+\infty}$, *where* ξ *is some constant.*

Remark 4. Note that by using conditions $i) - iii)$ it is possible to build the system of functions $\{\varphi_j(\cdot)\}_{j=-\infty}^{j=+\infty}$ in such way that any $x \in \mathbb{R}$ can belong to at most three segments of the system of segments $\{supp\varphi_j(\cdot)\}$. This condition is assumed to be satisfied in the following.

We define the operator

$$Kf = \sum_{\{j\}} \varphi_j L_j^{-1} \varphi_j f,\ \ f(x,y) \in C_0^\infty(\overline{\Omega}).$$

We can easily verify that the operator K is bounded, has a closure in $L_2(\Omega)$, and $Kf \subseteq D(L)$ for all $f(x,y) \in C_0^\infty(\overline{\Omega})$.

Applying the operator L to the operator Kf we obtain

$$LKf = f + Af + Bf,\ \ f(x,y) \in C_0^\infty(\overline{\Omega}),$$

where

$$Af = \sum_{\{j\}} \varphi_j(a(x) - a(x_j))(L_j^{-1}\varphi_j f)_x + \sum_{\{j\}} \varphi_j(c(x) - c(x_j))L_j^{-1}\varphi_j f, \tag{3.1}$$

$$Bf = \sum_{\{j\}} (\varphi_j)_x a(x) L_j^{-1}\varphi_j f + \sum_{\{j\}} (\varphi_j)_{xx} L_j^{-1}\varphi_j f + 2\sum_{\{j\}} (\varphi_j)_x (L_j^{-1}\varphi_j f)_x, \tag{3.2}$$

i.e. the following lemma holds.

Lemma 3.2. *Let conditions i)-iii) hold. Then the equality*

$$LKf = [E + A + B]f \tag{3.3}$$

holds for all functions $f \in C_0^\infty(\overline{\Omega})$, *where the operators* A, B *are defined by (3.1)-(3.2).*

Lemma 3.3. *Let $a(x), c(x) \in K(\tau, b)$. Then there exist numbers τ_0 and b_0 such that, for all $\tau \in (0, \tau_0)$ and $b > b_0$ the operators A and B are bounded and the inequality*

$$\|A + B\|_{2\to 2} < 1$$

holds.

Lemma 3.4. *Let the assumptions of Lemma 3.3 hold. Then*

$$L_{np}^{-1} = K[E + A + B]^{-1}, \tag{3.4}$$

where L_{np}^{-1} is the right inverse operator of the operator L.

Proof. Using Lemma 3.3 and (3.3), we obtain the representation (3.4). □

Proof of Lemma 3.3. We estimate the norm of the operator A in the space $L_2(\Omega)$. To show this we estimate each term separately. We will first estimate the norm of the first term of equation (3.1). According to Theorem 2.2 and Remark 1 we have

$$\left\| \sum_{\{j\}} \varphi_j (a(x) - a(x_j)) D_x L_j^{-1} \varphi_j f \right\|_2^2 \leq 9 \sum_{\{j\}} \left\| \varphi_j (a(x) - a(x_j)) D_x (L_j^{-1} \varphi_j f) \right\|_2^2$$

or

$$\left\| \sum_{\{j\}} \varphi_j (a(x) - a(x_j)) D_x L_j^{-1} \varphi_j f \right\|_2^2 \leq 9 \sup_{\{j\}} \sup_{x \in \overline{\Delta}_j} \frac{|a(x) - a(x_j)|^2}{|a(x_j)|^2} \|f\|_2^2. \tag{3.5}$$

Here the equality $\| \sum_{\{j\}} \varphi_j f \|_2^2 = \|f\|_2^2$ was taken into account.

Now, we will estimate the norm of the second term. Taking into account Theorem 2.2 and repeating the computations used in the proof of inequality (3.5), we can see that

$$\left\| \sum_{\{j\}} \varphi_j (c(x) - c(x_j)) L_j^{-1} \varphi_j f(x, y) \right\|_2^2 \leq 9 \sup_{\{j\}} \sup_{x \in \overline{\Delta}_j} \frac{|c(x) - c(x_j)|^2}{c^2(x_j)} \|f\|_2^2. \tag{3.6}$$

Taking into account conditions $i) - iii)$ from inequalities (3.5) and (3.6) we obtain

$$\|A\|_{2\to 2}^2 \leq c(\tau + \tau^2). \tag{3.7}$$

We must choose $\tau_0 = c(\tau + \tau^2) < \frac{1}{2}$ for the validity of Lemma 3.3.

Now, we will estimate the norm of the operator B. Consider the first term. According to Theorem 2.2 and Lemma 3.1 we obtain

$$\left\| \sum_{\{j\}} (\varphi_j)_x a(x) L_j^{-1} \varphi_j f(x, y) \right\|_2^2 \leq 9 \sum_{\{j\}} \sup_{x \in \overline{\Delta}_j} \frac{c^2 |a(x)|^2}{b^2 d^2(x_j)} \left\| L_j^{-1} \right\|_2^2 \|\varphi_j f\|_2^2,$$

where $c>0$ is the constant from Lemma 3.1. Hence, according to (2.21) and taking into account conditions $i)-iii)$ we obtain

$$\left\|\sum_{\{j\}}(\varphi_j)_x a(x)L_j^{-1}\varphi_j f(x,y)\right\|_2^2 \le 9\sup_{\{j\}}\sup_{x\in\overline{\Delta}_j}\frac{c^2|a(x)|^2}{b^2d^2(x_j)(c(x_j))^2}\|f\|_2^2 \le \frac{c(\tau+1)}{b^2}. \quad (3.8)$$

Now we consider the norm of the second term of the operator B, i.e.

$$\left\|\sum_{\{j\}}(\varphi_j)_{xx}L_j^{-1}\varphi_j f(x,y)\right\|_2^2 \le 9\sup_{\{j\}}\sup_{x\in\overline{\Delta}_j}\frac{c^4}{b^4d^4(x_j)(c(x_j))^2}\|f\|_2^2.$$

Hence, using conditions $i)-iii)$ we obtain

$$\left\|\sum_{\{j\}}(\varphi_j)_{xx}a(x)L_j^{-1}\varphi_j f\right\|_2^2 \le \frac{c}{b^4}\|f\|_2^2, \quad (3.9)$$

where c is the constant in Lemma 3.1.

Similarly, we obtain

$$\left\|\sum_{\{j\}}(\varphi_j)_x D_x L_j^{-1}\varphi_j f\right\|_2^2 \le c\sup_{\{j\}}\sup_{x\in\overline{\Delta}_j}\frac{c^2}{b^2d^2(x_j)|a(x_j)|^2}\|f\|_2^2 \le \frac{c}{b^2}\|f\|_2^2. \quad (3.10)$$

for the third term of the operator B.

From estimates (3.8)-(3.10) it follows that

$$\|B\|_2^2 \le \left(\frac{c(\tau+1)}{b^2}+\frac{c}{b^4}+\frac{c}{b^2}\right) \le c\left(\frac{\tau+1}{b^2}+\frac{1}{b^4}+\frac{1}{b^2}\right). \quad (3.11)$$

For the validity of Lemma 3.3 we must choose b so that $\|B\|_{2\to 2}<\frac{1}{2}$. Now Lemma 3.3 follows from (3.7) and (3.11). □

Let

$$\tilde a(x)=\sum_{\{j\}}a(x_j)\varphi_j^2,\quad \tilde c(x)=\sum_{\{j\}}c(x_j)\varphi_j^2;$$

$$\tilde L u = u_{xx}-u_{yy}+\tilde a(x)u_x+\tilde c(x)u,\quad u\in C_0^\infty(\overline{\Omega});$$

$$\tilde L' u = u_{xx}-u_{yy}-\tilde a(x)u_x-(\tilde a(x))_x u+\tilde c(x)u,\quad u\in C_0^\infty(\overline{\Omega}),$$

where $\tilde L'$ is the formal adjoint operator. Note that for functions $u,v\in C_0^\infty(\overline{\Omega})$ the equality

$$<\tilde L u, v> = < u, \tilde L' v>$$

holds.

The operators $\tilde L$ and $\tilde L'$ admit closures in the space $L_2(\Omega)$ which also will be denoted by $\tilde L$, $\tilde L'$ respectively.

We introduce the operator

$$M^\sharp f = \sum_{\{j\}} \varphi_j L_j'^{-1} \varphi_j f, \quad f(x,y) \in C_0^\infty(\overline{\Omega}),$$

where $L_j'^{-1}$ is the inverse operator to the operator L_j',

$$L_j' u = u_{xx} - u_{yy} - a(x_j)u_x + c(x_j)u, u \in D(L_j').$$

Lemma 3.5. *Let conditions i)-iii) hold. Then the following equalities hold for all functions $f(x,y) \in C_0^\infty(\overline{\Omega})$*

$$\tilde{L}Kf = [E + A_1 + B_1]f, \tag{3.12}$$

$$\tilde{L}'M^\sharp f = [E + A_2 + B_2]f, \tag{3.13}$$

where

$$Kf = \sum_{\{j\}} \varphi_j L_j^{-1} \varphi_j f,$$

$$A_1 f = \sum_{\{j\}} \varphi_j(\tilde{a}(x) - a(x_j))[L_j^{-1}\varphi_j f]_x + \sum_{\{j\}} \varphi_j(\tilde{c}(x) - c(x_j))L_j^{-1}\varphi_j f,$$

$$B_1 f = \sum_{\{j\}} (\varphi_j)_x \tilde{a}(x) L_j^{-1}\varphi_j f + \sum_{\{j\}} (\varphi_j)_{xx} L_j^{-1}\varphi_j f + 2\sum_{\{j\}} (\varphi_j)_x [L_j^{-1}\varphi_j f]_x,$$

$$A_2 f = \sum_{\{j\}} \varphi_j(a(x_j) - \tilde{a}(x))[L_j'^{-1}\varphi_j f]_x + \sum_{\{j\}} \varphi_j(\tilde{c}(x) - c(x_j))L_j'^{-1}\varphi_j f,$$

$$B_2 f = \sum_{\{j\}} (\varphi_j)_x \tilde{a}(x) L_j'^{-1}\varphi_j f - \sum_{\{j\}} \varphi_j(\tilde{a}(x))_x L_j'^{-1}\varphi_j f.$$

Here it was taken into account that $(\varphi_j)_{yy} = 0, (\varphi_j)_y = 0$.

Proof. Lemma 3.5 is proved exactly in the same way as Lemma 3.2. □

Lemma 3.6. *Let $a, c \in K(\tau, b)$. Then there exist numbers τ_0 and b_0 such that, for all $\tau \in (0, \tau_0)$ and $b > b_0$ the operators $A_i + B_i$, $i = 1, 2$, are bounded and for them the inequality*

$$\|A_i + B_i\|_{2\to 2} < \frac{1}{2}, \quad i = 1, 2,$$

holds.

Lemma 3.7. *Let the assumptions of Lemma 3.6 hold. Then the equalities*

$$\tilde{L}_{np}^{-1} = K[E + A_1 + B_1]^{-1}, \tag{3.14}$$

$$\tilde{L}_{np}'^{-1} = M^\sharp[E + A_2 + B_2]^{-1}, \tag{3.15}$$

hold, where $\tilde{L}_{np}^{-1}, \tilde{L}_{np}'^{-1}$ are the right inverse operators of $\tilde{L}, \tilde{L}'$ respectively.

Lemma 3.7 is proved exactly in the same way as Lemma 3.4.

Proof of Lemma 3.6. We estimate the norm of the operator A_1 in $L_2(\Omega)$. To do this, we estimate each term separately. Using Remark of Lemma 3.1 and the fact that at most three functions φ_j are nonzero on $supp\varphi_j$, we obtain

$$\left\|\sum_{\{j\}} \varphi_j(\tilde{a}(x) - a(x_j))D_x L_j^{-1}\varphi_j f\right\|_2^2 \le 9\sum_{\{j\}} \sup_{x\in\overline{\Delta}_j} |\tilde{a}(x) - a(x_j)|^2 \left\|D_x L_j^{-1}\right\|_2^2 \|\varphi_j f\|_2^2 . \tag{3.16}$$

According to Lemma 3.1 we have $\sum\limits_{i=j-1}^{j+1} \varphi_i^2(x) \equiv 1$, $x \in \overline{\Delta}_j$. Taking this into account and the conditions $i) - iii)$ we obtain

$$|\tilde{a}(x) - a(x_j)| \le c\sqrt{\tau(\tau+2)}|a(x_j)|. \tag{3.17}$$

Taking into account the conditions $i) - iii)$ and directly calculating we verify the validity the following inequalities

$$\frac{1}{2(\tau+1)} \le \frac{c(x)}{c(t)} \le 2(\tau+1) \;\; for \;\; |x-t| \le b\cdot d(t),$$

$$\frac{c_0}{c_1}\frac{1}{2(\tau+1)} \le \frac{a^2(x)}{a^2(t)} \le \frac{c_0}{c_1}\cdot 2(\tau+1) \;\; for \;\; |x-t| \le b\cdot d(t). \tag{3.18}$$

With inequalities (3.17), (3.18) and (2.3), from (3.16) we obtain

$$\left\|\sum_{\{j\}} \varphi_j(\tilde{a}(x) - a(x_j))D_x L_j^{-1}\varphi_j f\right\|_2^2$$

$$\le 9\frac{c\tau(2+\tau)|a(x_j)|^2}{|a(x_j)|^2}\|f\|_2^2 \le 9c\tau(2+\tau)\|f\|_2^2 . \tag{3.19}$$

Exactly in the same way, repeating the computations used in the proof of inequality (3.19), we obtain

$$\left\|\sum_{\{j\}} \varphi_j(\tilde{c}(x) - c(x_j))L_j^{-1}\varphi_j f\right\|_2^2 \le 9c\tau^2(2+\tau)\|f\|_2^2 . \tag{3.20}$$

Finally we obtain from (3.19)-(3.20)

$$\|A_1\|_{2\to2}^2 \le 9c(\tau(2+\tau) + \tau^2(2+\tau)). \tag{3.21}$$

We estimate the norm of the first term of the operator B_1:

$$\left\|\sum_{\{j\}} (\varphi_j)_x \tilde{a}(x) L_j^{-1}\varphi_j f\right\|_2^2 \le 9\sup_{\{j\}}\left(\sup_{x\in\overline{\Delta}_j} \frac{c^2|\tilde{a}(x)|^2}{b^2 d^2(x_j)}\right)\frac{1}{(c(x_j))^2}\|f\|_2^2 . \tag{3.22}$$

On the other hand we have:

$$|\tilde{a}(x)| = \left|\sum_{i=j-1}^{j+1} a(x_i)\varphi_i\right| \leq |a(x_{j-1})| + |a(x_j)| + |a(x_{j+1})|,$$

for $x \in \overline{\Delta}_j$. Hence according to inequality (3.18) and condition $ii)$ we have:

$$|\tilde{a}(x)| \leq (4\tau + 5)|a(x_j)|. \tag{3.23}$$

Hence, from (3.20) by (3.23) we obtain

$$\left\|\sum_{\{j\}} (\varphi_j)_x \tilde{a}(x) L_j^{-1} \varphi_j f\right\|_2^2 \leq \frac{c(4\tau+5)^2}{b^2}. \tag{3.24}$$

Next, using estimate (2.2) and Lemma 3.1 and direct calculations we have:

$$\left\|\sum_{\{j\}} (\varphi_j)_{xx} L_j^{-1} \varphi_j f\right\|_2^2 \leq 12 \sup_{\{j\}} \frac{c^4}{b^4 d^4(x_j)} \cdot \frac{1}{(c(x_j))^2} \|f\|_2^2 \leq 12 \frac{c}{b^4} \|f\|_2^2. \tag{3.25}$$

Next, using Theorem 2.2, Lemma 3.1 and conditions $i) - iii)$ we obtain

$$\left\|2\sum_{\{j\}} (\varphi_j)_x L_j^{-1} \varphi_j f\right\|_2^2 \leq 48 \sup_{\{j\}} \frac{c^2}{b^2 d^2(x_j)} \cdot \frac{1}{a^2(x_j)} \|f\|_2^2 \leq \frac{c}{b^2} \|f\|_2^2. \tag{3.26}$$

From inequalities (3.24)-(3.26) it follows that

$$\|B\|_{2\to 2}^2 \leq \frac{c(4\tau+5)^2}{b^2} + \frac{12c}{b^4} + \frac{c}{b^2}, \tag{3.27}$$

Now we must choose τ and b so that

$$\|A_1 + B_1\|_{2\to 2} \leq \|A\|_{2\to 2} + \|B\|_{2\to 2} < \frac{1}{2}. \tag{3.28}$$

for the validity of Lemma 3.6.

Repeating the computations used in the proof of inequality (3.28), we obtain

$$\|A_2 + B_2\|_{2\to 2} < \frac{1}{2}. \tag{3.29}$$

□

Lemma 3.8. *Let the assumptions of Lemma 3.6 be satisfied. Then the equality*

$$\tilde{L}^{-1} = K[E + A_1 + B_1]^{-1}) \tag{3.30}$$

holds, where $\tilde{L}^{-1}$ *is the inverse operator of* $\tilde{L}$.

Proof. To prove Lemma 3.8, we will show $Ker\tilde{L} = \{0\}$. It is well known from the general theory of linear operators that

$$L_2(\Omega) = R(\tilde{L})\dot{+}Ker\tilde{L}^*,;\ L_2(\Omega) = R(\tilde{L}')\dot{+}Ker(\tilde{L}')^*.$$

From equalities (3.14), (3.15) it follows that

$$Ker\tilde{L}^* \equiv \{0\},;\ Ker(\tilde{L}')^* \equiv \{0\}. \tag{3.31}$$

Since $D(\tilde{L}) \subseteq D((\tilde{L}')^*)$, then from equality (3.31) it follows

$$Ker(\tilde{L}) \equiv \{0\}. \tag{3.32}$$

Taking into account (3.32) from (3.14) we have

$$\tilde{L}^{-1} = K[E + A_1 + B_1]^{-1}.$$

□

Lemma 3.9. *Let the assumptions of Lemma 3.8 be satisfied. Then $KerK = \{0\}$.*

Proof. Since $Ker[E + A_1 + B_1] \equiv \{0\}$, then from (3.30) and (3.32) it follows that $KerK = \{0\}$. □

Lemma 3.10. *Let the assumptions of Theorem 1.1 holds. Then the operator L admits a closure.*

Proof. Let $u_n \to 0,\ \ Lu_n \to v(v \in L_2(\Omega),\ \ u_n \in C_0^\infty(\overline{\Omega}),\ \ n = 1, 2, ...)$. By Lemma 3.9 it follows that u_n is represented as $u_n = Kv_n,\ \ v_n \in L_2(\Omega)$ (since $C_0^\infty(\overline{\Omega}) \subset D(\tilde{L}) = R(K)$). We have

$$Lu_n = LKv_n = [E + A + B]v_n \to v\ \ for\ \ n \to \infty,$$

hence

$$v_n \to [E + A + B]^{-1}v,\ \ Kv_n \to K[E + A + B]^{-1}v.$$

From $u_n = Kv_n \to 0$, we obtain $K[E + A + B]^{-1}v = 0$. Hence, we conclude that $v = 0$. □

Lemma 3.11. *Let the assumptions of Lemma 3.8 holds. Then $KerL \equiv \{0\}$.*

Proof. Let $Lu = 0,\ \ u \in D(L),\ \ u \neq 0$. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ such that

$$u_n \to u,\ \ Lu_n \to Lu,\ \ u_n \in C_0^\infty(\overline{\Omega}) \subset D(L).$$

Now using the representation $u_n = Kv_n,\ \ v_n \in L_2(\Omega)$, we obtain

$$Lu_n = LKv_n = [E + A + B]v_n \to 0 \tag{3.33}$$

as $n \to \infty$. Since $Ker[E + A + B] \equiv \{0\}$, then from (3.33) it follows that $v_n \to 0$. Hence and from the representation $u_n = Kv_n$ we obtain $u_n \to 0$. Therefore $u = 0$. □

Proofs of Theorems 1.1 and 1.2. The proofs of both theorems follow by Lemmas 3.4, 3.10 and 3.11. □

Proof of Theorem 1.3. By virtue of (3.4) and Lemma 3.11 we can conclude the operator $c(x)(L+\lambda E)^{-1}$ is bounded (or unbounded) with the operator $c(x)K(E+A+B)^{-1}$. Hence, repeating the computations used in the proof of lemmas 3.3 and 3.4, using the inequality (3.18) we obtain

$$\|c(x)L^{-1}\|_{2\to 2} \le c_1 < \infty,$$

where $c_1 > 0$ is constant.

Similarly, we obtain

$$\|u(x)D_x L^{-1}\|_{2\to 2} \le c_2 < \infty,$$

where $c_2 > 0$ is constant.

By virtue of these inequalities we obtain

$$\|u_{xx} - u_{yy}\|_2 = \|Lu - a(x)u_x - c(x)u\|_2 \le \|Lu\|_2 + \|a(x)u_x\|_2 + \|c(x)u\|_2 \le$$

$$\le \|Lu\|_2 + \|a(x)D_x L^{-1}Lu\|_2 + \|c(x)L^{-1}Lu\|_2 \le c_3\|Lu\|_2,$$

where$c_3 > 0$ is constant which does not depending on $u(x, y)$.

Hence, it follows that

$$\|u_{xx} - u_{yy}\|_2 + \|a(x)u_x\|_2 + \|c(x)u\|_2 \le c\|Lu\|_2,$$

where $c > 0$ is constant, $u(x, y) \in D(L)$. □

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