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NURZHAN BOKAYEV

(to the 60th birthday)

On January 5, 2016 was the 60th birthday of Doctor of Physical-Mathematical Sciences (1996), Professor Nurzhan Adilkhanovich Bokayev. Professor Bokayev is the head of the department "Higher Mathematics" of the L.N. Gumilyov Eurasian National University (since 2009), the Vice-President of Mathematical Society of the Turkic World (since 2014), and a member of the Editorial Board of our journal.

N.A. Bokayevwas born in the Urnek village, Karabalyk district, Kostanay region. He graduated from the E.A. Buketov Karaganda State University in 1977 and the M.V. Lomonosov Moscow State University's full-time postgraduate study in 1984.

Scientific works of Professor Bokayev are devoted to studying problems of the theory of functions, in particular of the theory of orthogonal series.

He proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems; investigated Besov-type function spaces with respect to the Price bases; studied the Price - and Haar-type p-adic operators; introduced new classes of Faber-Schauder-type systems of functions and spaces of multivariable functions of bounded p-variation and of bounded p-fluctuation, obtained estimates for the best approximation of functions in these spaces by polynomials with respect to the Walsh and Haar systems, established weighted integrability conditions of the sum of multiple trigonometric series and series with respect to multiplicative systems, found the embedding criterion for the Nikol'skii-Besov spaces with respect to multiplicative bases and the coefficient criterion for belonging of functions to such spaces.

His scientific results have made essential contribution to the theory of series with respect to the Walsh and Haar systems and multiplicative systems.

N.A. Bokayev has published more than 150 scientific papers. Under his supervision 8 dissertations have been defended: 4 candidate of sciences dissertations and 4 PhD dissertations.

The Editorial Board of the Eurasian Mathematical Journal congratulates Nurzhan Adilkhanovich Bokayev on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

The EMJ has been included in the Emerging Sources Citation Index

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On behalf of the Editorial Board of the EMJ
V.I. Burenkov, T.V. Tararykova, A.M. Temirkhanova

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INEQUALITIES BETWEEN THE NORMS OF A FUNCTION AND ITS DERIVATIVES

A. S. Kochurov

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Key words: inequalities for derivatives, necessary conditions for an extremum, Weierstrass formula, Euler equation.

AMS Mathematics Subject Classification: 26D10.

Abstract. The paper is devoted to the problem of finding the maximum of the norm $\|x\|_q$ with the constraints $\|x\|_p = \eta$, $\|\dot{x}\|_r = \sigma$, $x(0) = a$, $a, \sigma, \eta > 0$, for functions $x \in L_p(\mathbb{R}_-)$ with derivatives $\dot{x} \in L_r(\mathbb{R}_-)$, $0 < p \leqslant q < \infty$, $r > 1$. The arguments employed are based on the standard machinery of the calculus of variations

1 Introduction

Kolmogorov inequalities for derivatives (or Landau–Kolmogorov inequalities) on the line or on the half-line are inequalities of the form

$$\|x^{(k)}(\cdot)\|_{L_q(T)} \leqslant K \cdot \|x(\cdot)\|^{\alpha}_{L_p(T)} \|x^{(n)}(\cdot)\|^{\beta}_{L_r(T)}, \qquad x(\cdot) \in \mathcal{W}^n_{p,r}(T),$$

where T is $\mathbb{R}$ or $\mathbb{R}_-$, n, k, $k < n$, are nonnegative integers, $\mathcal{W}^n_{p,r}(T)$ is the class of functions $x(\cdot)$ in $L_p(T)$ whose derivatives $x^{(n-1)}(\cdot)$ of order $(n-1)$ are locally absolutely continuous on T and the nth derivatives $x^{(n)}(\cdot)$ belong to the space $L_r(T)$, α, β are positive numbers, $\alpha + \beta = 1$. The smallest possible constant $K > 0$ in this inequality is the solution of the extremal problem

$$\Big(\int_T |x^{(k)}|^q \, dt\Big)^{1/q} \to \sup, \qquad \int_T |x|^p \, dt \leqslant 1, \quad \int_T |x^{(n)}|^r \, dt \leqslant 1, \tag{1.1}$$

over all functions $x(\cdot) \in \mathcal{W}^n_{p,r}(T)$. This constant is called the Kolmogorov constant (or the Landau–Kolmogorov constant). The first problem of this kind, for $p = q = r = \infty$, $n = 2$, $k = 1$, was solved by Landau (1913) on the half-line ([7]) and by Hadamard (1914) on the line. Hadamard's result was extended in 1937 by Kolmogorov [5], who found the exact value in (1.1) for $p = q = r = \infty$, $T = \mathbb{R}$ with all possible n, k. At present, no exact solution of (1.1) is known in the general case, it is only known for particular values of p, q, r, T, n and k. Such partial solutions with arbitrary k and n were found by G. H. Hardy, J. E. Littlewood, G. Pólya, E. Stein, L.V. Taikov, Yu. I. Lyubich, N. P. Kuptsov, V. N. Gabushin. Besides, there is a number of studies in which the exact solutions were found for small n (mostly for $n = 2$) for some particular values of the remaining parameters p, q, r, T and k. The available exact constants K and means of finding them are given in [12], [2], [9], [3], [13], [10] [6], [8], [4]. Exact solutions of (1.1) have great importance for various problems of the recovery of functionals (see, for example, [8]); they also appear in the problem of the best approximation to the differentiation operator [10].

Proofs and surveys of available results on inequalities for derivatives may also be found in [3], [6].

In 1941 Sz.-Nagy ([11], see also [3]) found the solution of (1.1) for all possible $p, q > 0$, $r \geqslant 1$ and $T = \mathbb{R}$, $n = 1$, $k = 0$. For $r > 1$ we shall obtain this and similar results using the standard machinery of the calculus of variations.

2 Main results

Let $p, q \in (0, \infty)$, $r \in (1, \infty)$ a, η, $\sigma > 0$, and let $p \leqslant q$. On the half-line $\mathbb{R}_-$ we consider the problem of the calculus of variations

$$-\int_{\mathbb{R}_-} |x|^q \, dt \to \inf \quad \int_{\mathbb{R}_-} |x|^p \, dt = \eta^p, \ \int_{\mathbb{R}_-} |\dot{x}|^r \, dt = \sigma^r \tag{2.1}$$

(here $\dot{x}(\cdot)$ denotes the derivative in t of a function $x(\cdot)$) and the isoperimetric problem

$$-\int_{\mathbb{R}_-} |x|^q \, dt \to \inf \quad \int_{\mathbb{R}_-} |x|^p \, dt = \eta^p, \ \int_{\mathbb{R}_-} |\dot{x}|^r \, dt = \sigma^r, \ x(0) = a. \tag{2.2}$$

If $p = q$, then problems (2.1) and (2.2) degenerate: the value of (2.1) is $(-\eta^q)$, while in (2.2) it may happen that there will be no admissible functions, but if there are admissible functions, then the value of (2.2) is also $(-\eta^q)$.

Throughout it will be assumed that $p < q$. Note that η, σ are fixed, hence $a > 0$ in (2.2) cannot be arbitrarily large — it is bounded from above by the value of the problem

$$x(0) \to \sup \quad \int_{\mathbb{R}_-} |x|^p \, dt = \eta^p, \ \int_{\mathbb{R}_-} |\dot{x}|^r \, dt = \sigma^r, \tag{2.3}$$

which is equal to

$$(1-s)^{s-1} \sigma^{1-s} \eta^s, \quad s = (1 + r'/p)^{-1}, \quad 1/r + 1/r' = 1$$

(this fact will be considered separately from (2.1) and (2.2)):

Lemma 2.1. (see [11]). *Let $0 < p < \infty$, $1 < r < \infty$. Then the solution of problems* (2.3), (2.30) *is equal to* $(1-s)^{s-1} \sigma^{1-s} \eta^s$.

The proof of Lemma 2.1 will be given later.

Consider the following problem of the calculus of variations

$$\mathcal{J}_0(x(\cdot)) \to \text{extr}, \quad \mathcal{J}_i(x(\cdot)) = 0, \ 1 \leqslant i \leqslant m, \tag{2.4}$$

where

$$\mathcal{J}_i(x(\cdot)) = \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t)) \, dt + \ell_i(x(t_0), x(t_1)),$$

$L_i : \mathbb{R}^{2n+1} \to \mathbb{R}$, $\ell_i : \mathbb{R}^{2n} \to \mathbb{R}$, $0 \leqslant i \leqslant m$. Its Lagrange function reads as

$$\mathcal{L}(x, \overline{\lambda}) = \sum_{i=0}^{m} \lambda_i \mathcal{J}_i(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt + \ell(x(t_0), x(t_1))$$

$$L = \sum_{i=0}^{m} \lambda_i L_i\,, \;\; \ell = \sum_{i=0}^{m} \lambda_i \ell_i.$$

Let $x_1(\cdot) \in C^1[t_0, t_1]$ and let $Q(x_1)$ be the set of points $\tau \in [t_0, t_1]$ for any of which there is a neighbourhood $U = U(t, x_1) \subset \mathbb{R}^{2n+1}$ of the point $(\tau, x_1(\tau), \dot{x}_1(\tau))^T$ at which both the functions $L_i(t, x, \dot{x})$, $i = 0, \dots, m$, and their partial derivatives in x and in $\dot{x}$ are defined and continuous (this definition implies that $Q(x_1)$ is an open subset of $[t_0, t_1]$). We will use the following statement

Proposition 1. (see [1]). *If a function $\widehat{x}(\cdot) \in C^1[t_0, t_1]$ is a local extremum for problem (2.4) in the space $C^1([t_0, t_1])$, then there exists a vector $\overline{\lambda} = (\lambda_0, \dots, \lambda_m) \neq 0$ of Lagrange multipliers such that, at the points of the set $Q(\widehat{x})$, the function $\widehat{L}_{\dot{x}}$ is continuously differentiable in t and the Euler equation holds*

$$-\frac{d}{dt}\widehat{L}_{\dot{x}}(t) + \widehat{L}_x(t) = 0,$$

where $\widehat{L}_{\dot{x}}(t) = L_{\dot{x}}(t, \widehat{x}(t), \dot{\widehat{x}}(t))$, $\widehat{L}_x(t) = L_x(t, \widehat{x}(t), \dot{\widehat{x}}(t))$.

If $t_j \in Q(\widehat{x})$ for $j = 0$ (or $j = 1$) and if at the point $\widehat{x}(t_j)$ all the functions $\ell_i(\cdot, \widehat{x}(t_1))$ for $j = 0$ (or all $\ell_i(\widehat{x}(t_0), \cdot)$ for $j = 1$), $i = 0, \dots, m$, are continuously differentiable, then the transversality condition holds

$$\widehat{L}_{\dot{x}}(t_j) = (-1)^j \ell_{x(t_j)}(\widehat{x}(t_0), \widehat{x}(t_1)).$$

The proof of Proposition 1 will be given later.

To solve problem (2.2) we construct a special function which is admissible for (2.2) and verify that it minimizes the problem.

We let $L = -\lambda_0 |x|^q + B|x|^p + A|\dot{x}|^r$ denote the Lagrangian of (2.2), where (λ_0, A, B) are the Lagrange multipliers. A solution $\widehat{x}(\cdot)$ of problem (2.2) in the space $C^1(\mathbb{R}_-)$ can be found from the necessary conditions for an extremum, in accordance with which (see Proposition 1) for $\widehat{x}(t)$ at each point $t \in \mathbb{R}_-$ either the Euler equation $-\frac{d}{dt}L_{\dot{x}}(t, \widehat{x}(t), \dot{\widehat{x}}(t)) + L_x(t, \widehat{x}(t), \dot{\widehat{x}}(t)) = 0$ holds or $\widehat{x}(t) = 0$. The Lagrangian L does not explicitly depend on t, and hence (see [1]) the Euler equation has the first integral (the 'energy' integral):

$$\lambda_0 |x|^q - B|x|^p + A(r-1)|\dot{x}|^r = \text{const}\,.$$

If const $= 0$, then this equation is clearly satisfied by the identically zero function. We shall assume that the solution of (2.2) is sought among the functions which tend to zero together with its first derivative as $t \to -\infty$. Hence,

$$|x|^q - B|x|^p + A(r-1)|\dot{x}|^r = 0 \tag{2.5}$$

(we assume that $\lambda_0 = 1$, $A > 0$, $B > 0$). Equation (2.5) will be considered on the entire real line; a continuously differentiable solution thereof satisfying the constraints in (2.2) will be sought in the form of a function $\widehat{x}(\cdot)$ with support in $(m_-, m_+) \subset \mathbb{R}$, which is strictly increasing on $(m_-, \tau]$ and strictly decreasing on $[\tau, m_+)$. Note that $\tau \in (m_-, m_+)$ may be positive or negative and that for m_-, m_+ it may happen that $m_- = -\infty$, $m_+ = +\infty$ (a possible occurrence of two intervals of monotonicity for $\widehat{x}$ on $(-\infty, 0)$ is due to the fact that with fixed parameters η, σ and sufficiently small value of a in (2.2) there are no monotone solutions of (2.5) on $(-\infty, 0)$).

From the character of variation of $\widehat{x}(\cdot)$ one may change, on each of the intervals (m_-, τ), $(\tau, 0)$, to a new independent variable $\widehat{x}$ and assume that on each of these intervals

$$\dot{\widehat{x}} = \dot{\widehat{x}}(\widehat{x}), \quad t = t(\widehat{x}).$$

Such a change enables one to skip finding the explicit form of the function $\widehat{x}(t)$ and be confined with its parametric representation $\dot{\widehat{x}}(\widehat{x})$, which follows from (2.5): since τ is a point of maximum of the function $\widehat{x}$, from (2.5) and the properties of $\widehat{x}$ it follows that $\widehat{x}(\tau) = B^{1/(q-p)}$, $\widehat{x}(m_-) = 0$, and moreover,

$$dt = \left(\frac{B\widehat{x}^p - \widehat{x}^q}{A(r-1)}\right)^{-1/r} d\widehat{x}, \quad \tau - t = \int_{\widehat{x}(t)}^{B^{1/(q-p)}} \left(\frac{Bz^p - z^q}{A(r-1)}\right)^{-1/r} dz, \tag{2.6}$$

on (m_-, τ) and

$$dt = \operatorname{sgn}\tau \left(\frac{B\widehat{x}^p - \widehat{x}^q}{A(r-1)}\right)^{-1/r} dx, \quad \tau - t = \operatorname{sgn}\tau \int_{\widehat{x}(t)}^{B^{1/(q-p)}} \left(\frac{Bz^p - z^q}{A(r-1)}\right)^{-1/r} dz,$$

on $(\tau, 0)$.

From (2.6) it is clear that the satisfaction (and failure) of the equalities $m_- = -\infty$, $m_+ = +\infty$ depends on the convergence of the integral

$$\int_0^1 z^{-p/r} dz.$$

Accordingly, if the integral diverges ($p \geqslant r$), then $m_- = -\infty$, $m_+ = +\infty$; if it converges ($0 < p < r$), then $m_- > -\infty$, $m_+ < +\infty$.

To find the Lagrange multipliers A, B and the value of $\operatorname{sgn}\tau$ there are two equations, which result from eliminating the variable t in the isoperimetric constraints in (2.2):

$$\left(\int_0^{B^{1/(q-p)}} + \operatorname{sgn}\tau \int_{B^{1/(q-p)}}^a\right) z^p \left(\frac{Bz^p - z^q}{A(r-1)}\right)^{-1/r} dz = \eta^p,$$

$$\left(\int_0^{B^{1/(q-p)}} + \operatorname{sgn}\tau \int_{B^{1/(q-p)}}^a\right) \left(\frac{Bz^p - z^q}{A(r-1)}\right)^{1-1/r} dz = \sigma^r.$$

Below it will be shown that from these equations one may find the unknowns A, B, $\operatorname{sgn}\tau$, and that the value of problem (2.2) is equal to

$$\mathcal{J} = \mathcal{J}(a, \eta, \sigma) = -\left(\int_0^{B^{1/(q-p)}} + \operatorname{sgn}\tau \int_{B^{1/(q-p)}}^a\right) z^q \left(\frac{Bz^p - z^q}{A(r-1)}\right)^{-1/r} dz. \tag{2.7}$$

Theorem 2.1. *Let* $0 < p < q < \infty$, $r \in (1, \infty)$, $1/r + 1/r' = 1$, $s = (1 + r'/p)^{-1}$, a, η, $\sigma > 0$. *If* $a \leqslant (1-s)^{s-1} \sigma^{1-s} \eta^s$, *then the value of problem* (2.2) *is* $\mathcal{J}(a, \eta, \sigma)$. *If* $a > (1-s)^{s-1} \sigma^{1-s} \eta^s$, *then problem* (2.2) *has no admissible functions.*

Proof. If $a > (1-s)^{s-1} \sigma^{1-s} \eta^s$, then the conclusion of Theorem 2.1 follows from Lemma 2.1. Let $a \leqslant (1-s)^{s-1} \sigma^{1-s} \eta^s$ and let $x(\cdot)$ be a function which is continuously differentiable on $\mathbb{R}_-$, admissible for (2.2), is supported in $[-l, 0]$, and which, for any $t \in (-l, 0]$, satisfies the strict inequality $x(t) > 0$ and the strict inequality (see (2.3))

$$x(t) < (1-s)^{s-1} \beta(t)^{(1-s)/r} \alpha(t)^{s/p}, \quad s = (1 + r'/p)^{-1}, \quad 1/r + 1/r' = 1, \tag{2.8}$$

where

$$\alpha(t) := \int_{-l}^{t} |x(z)|^p \, dz, \quad \beta(t) := \int_{-l}^{t} |\dot{x}(z)|^r \, dz.$$

Lemma 2.2. *Let* $r > 1$, $q > p > 0$. *Then, for* $t \in (-l, 0]$, *there exist continuously differentiable functions* $C(t) > 0$, $D(t) > 0$ *and a continuous function* $s(t)$ *satisfying relations*

$$\left(\int_0^{D^{1/(q-p)}} + \operatorname{sgn}(s-t) \int_{D^{1/(q-p)}}^{x(t)} \right) z^p \left(\frac{Dz^p - z^q}{C(r-1)} \right)^{-1/r} dz = \alpha(t),$$

$$\left(\int_0^{D^{1/(q-p)}} + \operatorname{sgn}(s-t) \int_{D^{1/(q-p)}}^{x(t)} \right) \left(\frac{Dz^p - z^q}{C(r-1)} \right)^{1-1/r} dz = \beta(t), \tag{2.9}$$

$$|s-t| = \int_{x(t)}^{D^{1/(q-p)}} \left(\frac{Dz^p - z^q}{C(r-1)} \right)^{-1/r} dz.$$

Moreover, the condition $s(t) = t$ *is equivalent to the equality* $x(t) = D(t)^{1/(q-p)}$ *and the expression* $x(t) \cdot D(t)^{1/(p-q)}$ *depends only on* $\alpha(t)^{r-1} \beta(t) \, x(t)^{-pr-r+p}$.

As a result, the values $A = C(0)$, $B = D(0)$, $\tau = s(0)$, and hence, the function $\widehat{x}(\cdot)$ are well-defined.

Example. Let $\nu > 1$, $r > 1$, $q > p > 0$, $y(t) = \varepsilon \cdot (t+l)^\nu$, $t \in [-l, 0]$, $\varepsilon, l > 0$. Let us find the functions $C(t)$ and $D(t)$ with this $y(\cdot)$:

$$\alpha(t) = \int_{-l}^{t} |y(z)|^p \, dz = \frac{\varepsilon^p (t+l)^{\nu p + 1}}{\nu p + 1}, \quad \beta(t) = \int_{-l}^{t} |\dot{y}(z)|^r \, dz = \frac{(\nu \, \varepsilon)^r (t+l)^{(\nu-1)r+1}}{(\nu-1)r+1},$$

$$\frac{\alpha^{r-1}(t) \cdot \beta(t)}{y(t)^{pr+r-p}} = \frac{\nu^r}{(\nu p+1)^{r-1}((\nu-1)r+1)}.$$

Hence, $D(t) = \gamma^{p-q} \cdot (t+l)^{\nu(q-p)}$, $\gamma = \text{const}$. From (2.9) we have

$$(C \cdot \beta)^{1/r'} \beta^{1/r} = \text{const} \cdot D^{\frac{1}{q-p}(\frac{q}{r'}+1)},$$

that is $(C\cdot\beta)^{1/r'} = \text{const}\cdot(t+l)^{\nu(\frac{q}{r'}+1)-\nu+\frac{1}{r'}}$, $C\cdot\beta = \text{const}\cdot(t+l)^{\nu q+1}$, $C(t) = \text{const}\cdot(t+l)^{\nu q-(\nu-1)r}$.

Thus, for any $\nu > 1$ and $y(t) = \varepsilon\cdot(t+l)^{\nu}$, the functions $D(\cdot)$, $C(\cdot)\beta(\cdot)$ are continuously differentiable on the half-line $\mathbb{R}_-$ and are supported in the interval $[-l,0]$. The function $C(\cdot)$ has the same properties if $\nu > 1$ is sufficiently close to 1.

Lemma 2.3. *Let, in addition, $x(\cdot)$ satisfy the conditions*:

$$\lim_{t\to -l} D(t)\cdot\alpha(t) = 0, \quad \lim_{t\to -l} C(t)\cdot\beta(t) = 0, \tag{2.10}$$

where the functions $C(\cdot)$, $D(\cdot)$ and $s(\cdot)$ are defined in Lemma 2.2. Then,

$$\int_{\mathbb{R}_-} (\widehat{x}^q - x^q)\,dt = \tag{2.11}$$

$$= \int_{[-l,0]} C\cdot\left(|\dot{x}|^r - \left(\frac{D\cdot x^p - x^q}{C\cdot(r-1)}\right) - r\,\mathrm{sgn}(s-t)\left(\frac{D\cdot x^p - x^q}{C\cdot(r-1)}\right)^{1-1/r}\right.$$
$$\left.\times\left(\dot{x} - \mathrm{sgn}(s-t)\left(\frac{D\cdot x^p - x^q}{C\cdot(r-1)}\right)^{1/r}\right)\right)dt$$

(this equality is a corollary of the Weierstrass identity for the isoperimetric problem, see [1]).

Proofs of Lemma 2.2 and Lemma 2.3 will be given later.

The right-hand side in (2.11) is nonnegative, because $C(\cdot)$ is positive and the function $|\dot{x}|^r$ is convex in $\dot{x}$. Hence,

$$\int_{\mathbb{R}_-} (\widehat{x}^q - x^q)\,dt \geqslant 0 \tag{2.12}$$

for functions $x(\cdot)$ satisfying (2.8), (2.10). In the remaining part of the proof of the theorem inequality (2.12) is extended to arbitrary admissible functions.

We shall assume that the function $x(\cdot)$ is continuously differentiable, admissible for (2.2), has support in $[-l,0]$, and, for all $t\in(-l,0]$, satisfies the inequality $x(t)\geqslant 0$, while inequality (2.12) does not hold.

Consider an auxiliary continuously differentiable function $y(t)$, which vanishes identically on $(-\infty,-1]$, is equal to 1 on $[0,+\infty)$, is equal to $(t+1)^{\nu}$, $\nu > 1$, on $[-1,-1/2]$, and which strictly increases on $(-1,0]$. We choose ν so that $y(\cdot)$ does not satisfy the Euler equation for problem (2.3)

$$\frac{d}{dt}\left(\dot{y}^{\,r-1}\right) = \lambda\cdot y^{\,p-1}$$

(see Lemma 2.1) on $[-1,-1/2]$ for any $\lambda\in\mathbb{R}$. According to the example after Lemma 2.2 and Proposition 1, the function

$$x_\varepsilon(t) = x(t) + \varepsilon\cdot y((t+l)/\varepsilon)$$

satisfies for any $\varepsilon > 0$ all the assumptions in (2.8), (2.10). Hence, for $z(\cdot) = x_\varepsilon(\cdot)$,

$$-\mathcal{J}(z(0), \|z\|_p^p, \|\dot{z}\|_r^r) - \int_{\mathbb{R}_-} \big|z(t)\big|^q \, dt \geqslant 0, \tag{2.13}$$

which contradicts the continuity (see Lemma 2.2) of $\mathcal{J}$ in $a > 0$, $\eta > 0$, $\sigma > 0$ and the fact that (2.12) does not hold with $x(\cdot)$.

Assume that $x(\cdot)$ is an absolutely continuous function which is admissible for (2.2) and for which (2.12) does not hold. On $\mathbb{R}_-$ we consider an absolutely continuous function $z_1(\cdot)$, which is supported in $[-l, 0]$, agrees with $x(\cdot)$ on $[-l+\varepsilon, 0]$, $l > \varepsilon > 0$, and is linear function on $[-l, -l+\varepsilon]$. Hence, using the continuity of the integral, one may find l, ε so that (2.13) would not be satisfied for $z(\cdot) = z_1(\cdot)$.

Similarly, approximating the function $\dot{z}_1(\cdot)$ within $\varepsilon_1 > 0$ in the metric of $L_r(\mathbb{R}_-)$ by a continuous function $\dot{z}_2(\cdot)$ supported in $[-l_1, 0]$, $l_1 > 0$, we obtain a function $z_2(\cdot)$, which is a continuously differentiable on $\mathbb{R}_-$, has support in $[-l_1, 0]$), and for which $\|z_1 - z_2\|_{C(\mathbb{R}_-)} = o(1)$ (as $\varepsilon_1 \to 0$). Hence, one may choose a sufficiently small $\varepsilon_1 > 0$ so that (2.13) would not hold for $z(\cdot) = z_2(\cdot)$.

Let δ_i, $i \in I$, be the intervals on which the function $z_2(\cdot)$ has constant sign. If the number of such intervals is finite, then we set $z_3(t) = |z_2(t)|$, $t \in \mathbb{R}_-$. If the number of such intervals in infinite,

$$\{t \in [-l_1, 0] \,:\, z_2(t) \neq 0\} = \bigcup\nolimits_{i=1}^{\infty} \delta_i \,, \quad \delta_i \cap \delta_j = \varnothing \,, \; i \neq j,$$

then, given $k \in \mathbb{N}$, we put

$$z_3(t) = |z_2(t)| \,, \quad t \in \delta_i \,, \; i = 1, \dots, k \,, \quad z_3(t) = 0 \,, \quad t \in \mathbb{R}_- \setminus \cup_{i=1}^k \delta_i.$$

For a function $z(\cdot) = z_3(\cdot)$ (with sufficiently large $k \in \mathbb{N}$ in the second case) inequality (2.13) is not satisfied and this function is nonnegative, continuous and has compact support. Besides, it is continuously differentiable at all points of $\mathbb{R}_-$, except for, possibly, at a finite number of points, where its derivative has jump discontinuities. For $\varepsilon > 0$, we replace $\dot{z}_3(\cdot)$ by the continuous function $\dot{z}_4(\cdot)$, which on small intervals around each point of discontinuity of $\dot{z}_3(\cdot)$ is a linear function and which coincides with $\dot{z}_3(\cdot)$ at the remaining points of $\mathbb{R}_-$ so that

$$\|\dot{z}_4 - \dot{z}_3\|_r \leqslant \varepsilon, \quad \|z_4 - z_3\|_{C(\mathbb{R}_-)} \leqslant \varepsilon$$

(here $z_4(\cdot)$ is such a primitive of $\dot{z}_4(\cdot)$ which vanishes at all points $t \in \mathbb{R}_-$ lying sufficiently far from the origin). If $[-l_\varepsilon, 0]$ is the support of $z_4(\cdot)$, then for

$$z(t) = z_4(t) + \varepsilon \cdot y((t + l_\varepsilon)/\varepsilon)$$

inequality (2.13) is satisfied, but as $\varepsilon \to 0^+$ this contradicts the continuity of $\mathcal{J}$ in $a, \eta, \sigma > 0$ and the fact that, for $z_3(\cdot)$, inequality (2.12) is not satisfied. This contradiction shows that (2.12) holds for all functions admissible for (2.2). □

Proof of Lemma 2.3. Let $x(\cdot)$ be a function which is continuously differentiable on $\mathbb{R}_-$, admissible for (2.2), has support in $[-l, 0]$, and such that, for all $t \in (-l, 0]$, it

satisfies the inequality $x(t) > 0$ and inequality (2.8). By Lemma 2.2, on $(-l, 0]$ there exist continuously differentiable functions $C(t), D(t) > 0$ and a continuous function $s(t)$, $t \in (-l, 0]$, obeying the system of equations (2.9). Moreover, both the function $\widehat{x}(\cdot)$ and the norm $\|\widehat{x}\|_{L_q(\mathbb{R}_-)}$ are defined.
Opening the brackets in the right-hand side of (2.11) and cancelling, we have

$$\int_{\mathbb{R}_-} \widehat{x}^q\,dt = \int_{[-l,0]} \Big(C|\dot{x}|^r + Dx^p - r\,\mathrm{sgn}(s-t)C\dot{x}\left(\frac{Dx^p - x^q}{C(r-1)}\right)^{1-1/r}\Big)\,dt. \qquad (2.14)$$

Let us check (2.14):

$$\int_{\mathbb{R}_-} \widehat{x}^q\,dt \overset{(2.6)}{=} \left(\int_0^{B^{1/(q-p)}} + \mathrm{sgn}\,\tau \int_{B^{1/(q-p)}}^{a}\right) z^q \left(\frac{Bz^p - z^q}{A(r-1)}\right)^{-1/r} dz$$

$$\overset{(\mathrm{def})}{=} \left(\int_0^{D(0)^{1/(q-p)}} + \mathrm{sgn}(s(0)-0) \int_{D(0)^{1/(q-p)}}^{x(0)}\right) z^q \left(\frac{D(0)z^p - z^q}{C(0)(r-1)}\right)^{-1/r} dz$$

$$\overset{(2.9)}{=} D(0)\int_{-l}^{0} |x(z)|^p\,dz - C(0)(r-1)\int_{-l}^{0} |\dot{x}(z)|^r\,dz$$

$$\overset{(2.10)}{=} \int_{-l}^{0} d\left(D(t)\int_{-l}^{t} |x(z)|^p\,dz - C(t)(r-1)\int_{-l}^{t} |\dot{x}(z)|^r\,dz\right)$$

$$= \int_{-l}^{0} \Big(D(t)|x(t)|^p - C(t)(r-1)|\dot{x}(t)|^r\Big)\,dt$$

$$+ \int_{-l}^{0} \left(\int_{-l}^{t} |x(z)|^p\,dz\right) d(D(t)) - (r-1)\int_{-l}^{0}\left(\int_{-l}^{t} |\dot{x}(z)|^r\,dz\right) d(C(t)).$$

Let J be the sum of integrals with respect to $d(D(t))$ and $d(C(t))$ in the last expression. We calculate J using the fact that for $t > -l$ the functions $D(t) > 0$ and $C(t) > 0$ are continuously differentiable in t.

Let $t \in (-l, 0]$. Assume that either $s(t) \neq t$ or $s(t) = t$ holds identically in some neighbourhood of t (in the latter case, by Lemma 2.2, $D(t)^{1/(q-p)} = x(t)$ in this neighbourhood). Hence,

$$|\dot{x}(t)|^r = \left(\int_{-l}^{t} |\dot{x}(z)|^r\,dz\right)'$$

$$\overset{(2.9)}{=} \left(\left(\int_0^{D(t)^{1/(q-p)}} + \mathrm{sgn}(s(t)-t)\int_{D(t)^{1/(q-p)}}^{x(t)}\right)\left(\frac{D(t)z^p - z^q}{C(t)(r-1)}\right)^{1-1/r} dz\right)'$$

$$= \mathrm{sgn}(s(t)-t)\left(\frac{D(t)x^p(t) - x^q(t)}{C(t)(r-1)}\right)^{1-1/r} \dot{x}(t)$$

$$+ \frac{r-1}{r}\left(\int_0^{D(t)^{1/(q-p)}} + \mathrm{sgn}(s(t)-t)\int_{D(t)^{1/(q-p)}}^{x(t)}\right)\left(\frac{Dz^p - z^q}{C(r-1)}\right)^{-1/r}$$

$$\times \left(\left(\frac{D}{C(r-1)}\right)' z^p - \left(\frac{1}{C(r-1)}\right)' z^q\right) dz \overset{(2.9)}{=} \mathrm{sgn}(s(t)-t)\left(\frac{Dx^p - x^q}{C(r-1)}\right)^{1-1/r} \dot{x}$$

$$+\frac{1}{rC}\left(D'(t)\int_{-l}^{t}|x(z)|^p\,dz - C'(t)(r-1)\int_{-l}^{t}|\dot{x}(z)|^r\,dz\right).$$

In the remaining points t, at which $s(t)=t$, the equality

$$|\dot{x}(t)|^r = \operatorname{sgn}(s(t)-t)\left(\frac{Dx^p - x^q}{C(r-1)}\right)^{1-1/r}\dot{x}$$

$$+\frac{1}{rC}\left(D'(t)\int_{-l}^{t}|x(z)|^p\,dz - C'(t)(r-1)\int_{-l}^{t}|\dot{x}(z)|^r\,dz\right)$$

also holds, because its right- and left-hand sides are continuous in t. Hence,

$$J = \int_{-l}^{0} rC(t)|\dot{x}(t)|^r\,dt - \int_{-l}^{0}\Big(rC(t)\operatorname{sgn}(s(t)-t)\left(\frac{Dx^p - x^q}{C(r-1)}\right)^{1-1/r}\dot{x}\Big)\,dt$$

Finally, using this expression for J, we obtain (2.14). □

Proof of Lemma 2.2. Let $x(\cdot)$ be a function which is continuously differentiable on $\mathbb{R}_-$, admissible for (2.2), supported in $[-l,0]$, and which satisfies the inequalities $x(t)>0$ and (2.8) for all $t\in(-l,0]$.

Before proceeding with the general case in Lemma 2.2, we partially examine the case $q=4$, $p=r=2$, for which the quantities C, D and s from (2.9) can be found explicitly:

$$\left(\int_0^{\sqrt{D}} + \operatorname{sgn}(s-t)\int_{\sqrt{D}}^{x}\right) z^2\left(\frac{Dz^2 - z^4}{C}\right)^{-1/2} dz = \sqrt{C}(\sqrt{D} - g\sqrt{D-x^2}) = \alpha,$$

$$\left(\int_0^{\sqrt{D}} + \operatorname{sgn}(s-t)\int_{\sqrt{D}}^{x}\right)\left(\frac{Dz^2 - z^4}{C}\right)^{1/2} dz = \frac{\sqrt{D^3} - g\sqrt{(D-x^2)^3}}{3\sqrt{C}} = \beta,$$

$$|s-t| = \int_x^{\sqrt{D}}\left(\frac{Dz^2 - z^4}{C}\right)^{-1/2} dz,$$

where we put $g=\operatorname{sgn}(s-t)$, $x=x(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, $t>-l$. Note that in this case the quantities $\alpha,\beta>0$, $x\geqslant 0$ are related by inequality (2.3):

$$x^4 < 4\alpha\beta.$$

Excluding the unknown C from the first two equations, we get

$$(\sqrt{D} - g\sqrt{D-x^2})(\sqrt{D^3} - g\sqrt{(D-x^2)^3}) = 3\alpha\beta.$$

Next, replacing D by $\gamma = D - x^2/2$, we get

$$(\sqrt{\gamma + x^2/2} - g\sqrt{\gamma - x^2/2})(\sqrt{(\gamma + x^2/2)^3} - g\sqrt{(\gamma - x^2/2)^3}) = 3\alpha\beta$$

which gives, by expanding the brackets,

$$2\gamma^2 + x^4/2 - g\sqrt{\gamma^2 - x^4/4}\cdot 2\gamma = 3\alpha\beta, \quad 2\gamma^2 + x^4/2 - 3\alpha\beta = g\sqrt{\gamma^2 - x^4/4}\cdot 2\gamma. \quad (2.15)$$

If $\gamma^2 - x^4/4 \neq 0$, then $g = \operatorname{sgn}(2\gamma^2 + x^4/2 - 3\alpha\beta)$, which gives, after squaring,

$$4\gamma^2(x^4/2 - 3\alpha\beta) + (x^4/2 - 3\alpha\beta)^2 = (-x^4)\cdot\gamma^2,$$

$$(3\alpha\beta - x^4/2)^2 = 3(4\alpha\beta - x^4)\gamma^2.$$

Hence,

$$D - \frac{x^2}{2} = \gamma = \frac{3\alpha\beta - x^4/2}{\sqrt{3(4\alpha\beta - x^4)}}, \tag{2.16}$$

$$2\gamma^2 + x^4/2 - 3\alpha\beta = \frac{2(3\alpha\beta - x^4/2)^2}{3(4\alpha\beta - x^4)} + x^4/2 - 3\alpha\beta = (3\alpha\beta - x^4/2)\left(\frac{2(3\alpha\beta - x^4/2)}{3(4\alpha\beta - x^4)} - 1\right)$$

$$= \frac{(6\alpha\beta - x^4)(x^4 - 3\alpha\beta)}{3(4\alpha\beta - x^4)},$$

$$g = \operatorname{sgn}(2\gamma^2 + x^4/2 - 3\alpha\beta) = \operatorname{sgn}(x^4 - 3\alpha\beta), \tag{2.17}$$

$$\sqrt{C} = \frac{\alpha}{(\sqrt{D} - g\sqrt{D - x^2})}.$$

Consequently, using (2.15) and (2.16), we get

$$g\sqrt{D - x^2} = g\sqrt{(\gamma^2 - x^4/4)/D} = \frac{x^4 - 3\alpha\beta}{\sqrt{3D(4\alpha\beta - x^4)}}.$$

After simplifications, we have

$$C = \frac{\alpha^2}{\sqrt{3(4\alpha\beta - x^4)}}.$$

If the equality $\gamma^2 - x^4/4 = 0$ holds in (2.15), then one will not be able to find g in a unique way, while for D and C, we have

$$D - \frac{x^2}{2} = \gamma = \frac{x^2}{2}, \quad D = x^2, \quad x^4 = 3\alpha\beta, \quad C = \frac{\alpha^2}{x^2} = \frac{\alpha^2}{\sqrt{3\alpha\beta}}.$$

In this case, we set $g = 1$.

Let us now proceed with the proof of Lemma 2.2. Let $t \in (-l, 0]$. We exclude the unknown C from the first two equalities in (2.9) and replace the variable $z \to zD^{1/(q-p)}$ under the integral sign, hence we get

$$\frac{1}{\gamma^{pr+r-p}}\left(\left(\int_0^1 + g\int_1^\gamma\right)(z^p - z^q)^{\frac{r-1}{r}}\,dz\right)\left(\left(\int_0^1 + g\int_1^\gamma\right)\frac{z^p}{\sqrt[r]{z^p - z^q}}\,dz\right)^{r-1} = \frac{\alpha^{r-1}\beta}{x(t)^{pr+r-p}}, \tag{2.18}$$

where $\gamma = x(t)D^{\frac{1}{p-q}} \in (0,1]$, $g = \operatorname{sgn}(s - t)$. For the general equation

$$\frac{1}{\gamma^{pr+r-p}}\left(\left(\int_0^1 + g\int_1^\gamma\right)\frac{z^p}{\sqrt[r]{z^p - z^q}}\,dz\right)^{r-1}\left(\left(\int_0^1 + g\int_1^\gamma\right)(z^p - z^q)^{\frac{r-1}{r}}\,dz\right) = y \tag{2.19}$$

we claim that $\gamma = \gamma(y)$ depends smoothly on y.

To this aim we rewrite the left-hand side of (2.19) with $g > 0$ as

$$\frac{1}{\gamma^{pr+r-p}}\left(\int_0^\gamma \frac{z^p}{\sqrt[r]{z^p - z^q}}\,dz\right)^{r-1}\left(\int_0^\gamma (z^p - z^q)^{\frac{r-1}{r}}\,dz\right)$$

and prove that it is strictly monotone increasing in $\gamma \in (0,1]$. Changing the variable $z \to z^{1/(1+p/r')}$ in the integrals and setting $k = r'(q-p)/(p+r') > 0$, $w = \gamma^{1+p/r'}$, we find that the left-hand side of (2.19) with $g > 0$ differs only by a constant positive factor from

$$\frac{1}{w^r}\left(\int_0^w (1-z^k)^{-1/r}\,dz\right)^{r-1}\left(\int_0^w (1-z^k)^{1/r'}\,dz\right). \tag{2.20}$$

Let us transform (2.20). We have

$$\int_0^w (1-z^k)^{1/r'}\,dz = \int_0^w (1-z^k)^{-1/r}\,dz - \int_0^w z^{k-1}z(1-z^k)^{-1/r}\,dz$$
$$= \int_0^w (1-z^k)^{-1/r}\,dz + \frac{r'}{k}w(1-w^k)^{1/r'} - \frac{r'}{k}\int_0^w (1-z^k)^{1/r'}\,dz,$$

and hence

$$(1+\frac{r'}{k})\int_0^w (1-z^k)^{1/r'}\,dz = \int_0^w (1-z^k)^{-1/r}\,dz + \frac{r'}{k}w(1-w^k)^{1/r'}.$$

Therefore, the verification of monotonicity of (2.20) reduces to the verification of monotonicity in $w \in (0,1)$,

$$b^r + \frac{r'}{k}(1-c)^{1/r'}b^{r-1}, \quad b := \frac{1}{w}\int_0^w (1-z^k)^{-1/r}\,dz, \quad c := w^k.$$

The derivative of this function in w is as follows:

$$\Big(rb^{r-1}\left(-b + (1-c)^{-1/r}\right) - (1-c)^{-1/r}cb^{r-1}$$
$$+\frac{r'(r-1)}{k}(1-c)^{1/r'}b^{r-2}\left(-b+(1-c)^{-1/r}\right)\Big)\frac{1}{w}$$
$$= \left(-rb^r + (1-c)^{-1/r}b^{r-1}(r - c - \frac{r}{k}(1-c)) + \frac{r}{k}b^{r-2}(1-c)^{1-2/r}\right)\frac{1}{w}, \tag{2.21}$$

and its sign agrees with the sign of

$$\varphi(c) = -rb^2(1-c)^{1/r} + b(r - c - \frac{r}{k}(1-c)) + \frac{r}{k}(1-c)^{1-1/r}. \tag{2.22}$$

From the explicit representation it is seen that $\varphi \in C(0,1]$ and $\varphi(1) = b(r-1) > 0$. In a small neighbourhood of $c = 0$ we have,

$$b = 1 + \frac{c}{r+rk} + \frac{c^2(1+r)}{2(1+2k)r^2} + o(c^2), \quad (1-c)^{1/r} = 1 - \frac{c}{r} - \frac{c^2}{2rr'} + o(c^2),$$

$$(1-c)^{1-1/r} = 1 - \frac{c}{r'} - \frac{c^2}{2rr'} + o(c^2).$$

Substituting into (2.22), this gives

$$\varphi(c) = \frac{k^2(k/r'+1)}{(1+k)^2(1+2k)}\, c^2 + o(c^2) > 0,$$

correspondingly, the function φ is positive on $(0, c_0)$ if $c_0 > 0$ is sufficiently small. Assume now that φ vanishes at some point $c \in (0,1)$. Let us check that in this case the derivative of φ at this point is positive, which will yield a contradiction, because this condition cannot be satisfied for all $c \in (0,1)$ for which $\varphi(c) = 0$.

The derivative of the mapping $\varphi(c)$ in w reads as ($c = w^k$)

$$-2rb(1-c)^{1/r}\frac{-b+(1-c)^{-1/r}}{w} + b^2(1-c)^{-1+1/r}\frac{kc}{w} + (r-c-\frac{r}{k}(1-c))\frac{-b+(1-c)^{-1/r}}{w}$$
$$-b(k-r)\frac{c}{w} - (r-1)(1-c)^{-1/r}\frac{c}{w}.$$

As above, its sign agrees with that of

$$b^2(1-c)^{\frac{2}{r}}(2r+\frac{kc}{1-c}) - b(1-c)^{\frac{1}{r}}(2r+r-c-\frac{r}{k}(1-c)+(k-r)c) + (r-c-\frac{r}{k}(1-c)-(r-1)c).$$

We set $d := b(1-c)^{1/r} > 0$, $m := 1-c \in (0,1)$, and write the equation $\varphi(c)(1-c)^{1/r} = 0$ and the expression of the same sign as the derivative of the function φ at the point c in terms of d, m:

$$-rd^2 + d(r-1+m-\frac{r}{k}m) + \frac{r}{k}m = 0 \quad \Leftrightarrow \quad d(dr-r+1) = m(dk+r-dr)/k, \tag{2.23}$$

$$\begin{aligned} d^2(k+(2r-k)m) - dm(2r+rm-1+m-\frac{r}{k}m+k-km) + r(1-1/k)m^2 \\ = d^2(k+(2r-k)m) - dm(2r-1+k) + (dk+r-dr)(1-1/k)m^2 \\ \overset{(2.23)}{=} d\Big(d(k+(2r-k)m) - m(2r-1+k) + (dr-r+1)(k-1)m\Big) \\ = d\Big(dk - mdk + 2rm(d-1) + r(d-1)(k-1)m\Big) \\ \overset{(2.23)}{=} d\Big(mr(1-d) + dkr(1-d) + 2rm(d-1) + r(d-1)(k-1)m\Big) \\ = dkr(1-d)(d-m). \end{aligned} \tag{2.24}$$

If one assumes that

$$dk + r - dr \leqslant 0$$

in the equation (2.23), then we would also have $dr - r + 1 \leqslant 0$, a contradiction. Hence, $dk + r - dr > 0$ and (2.23) implies that

$$m = \frac{dk(dr-r+1)}{dk+r-dr},$$

the sign of $(1-d)(d-m)$ from (2.24) agreeing with the sign of

$$(1-d)(dk+r-dr-k(dr-r+1)) = (1-d)^2(r+kr-k) > 0,$$

because, if $d = 1$, then $m = 1$, a contradiction with the condition $m \in (0,1)$.

Thus, for $g > 0$ the expression on the left of (2.19) has continuous and strictly positive derivative in $\gamma \in (0,1)$. This derivative tends to $+\infty$ as γ approaches 1 from the left (see (2.21)). If γ varies over $(0,1]$, the left-hand side of (2.19) increases from $(1+p/r')^{-r}$ to

$$I = \left(\int_0^1 \frac{z^p}{\sqrt[r]{z^p - z^q}}\, dz\right)^{r-1} \int_0^1 (z^p - z^q)^{\frac{r-1}{r}}\, dz$$

(the limit as $\gamma \to +0$ of the left-hand side of (2.19) is $(1+p/r')^{-r}$, inasmuch as

$$\int_0^\gamma \frac{z^p}{\sqrt[r]{z^p - z^q}}\, dz = (1+o(1)) \cdot \int_0^\gamma z^{p/r'}\, dz = \frac{\gamma^{1+p/r'} \cdot (1+o(1))}{1+p/r'}$$

as $\gamma \to +0$).

If now $g < 0$, then each of the cofactors on the left of (2.19) decreases continuously and strictly monotonically in $\gamma \in (0,1)$, so that their product decreases from $+\infty$ to I, the derivative of the product does not vanish and tends to $-\infty$ as γ approaches 1 on the left. All these facts can be obtained by a direct calculation of the derivative.

As a result, the left-hand side of (2.19) attains all values from $(1+p/r')^{-r}$ to $+\infty$, and for $y \in \big((1+p/r')^{-r}, \infty\big)$ the dependence $\gamma = \gamma(y)$ is continuously differentiable, it has two intervals of monotonicity and a point of strong maximum $\gamma(y) = 1$ with $y = I$, at which (2.19), $g = 1$ with $y < I$, $g = -1$ with $y > I$; for $y = I$ one cannot find $g \in \{-1, +1\}$ uniquely, and so we set $g = 1$ by definition.

On the other hand, by the inequality for derivatives (see (2.3)), the number $(1+\frac{p}{r'})^{-r}$ is the least possible in the right-hand side of (2.18).

Thus, for any

$$y(t) = \frac{\alpha^{r-1}\beta}{x(t)^{pr+r-p}}, \quad x(t) > 0, \tag{2.25}$$

the values $\gamma(y(t)) \in (0,1)$, $g(y(t)) \in \{-1, +1\}$ and

$$D(t) = \big(x(t)/\gamma(y(t))\big)^{q-p} > 0 \tag{2.26}$$

in (2.18) are uniquely defined; moreover, $D(t)$ depends on t in a continuously differentiable way. Using (2.9) we may hence uniquely recover the continuously differentiable function $C(t) > 0$ and the function $s(t)$. $\square$

Let us now examine problem (2.1).

Theorem 2.2. (see [11]). *Let $0 < p < q < \infty$, $r \in (1,\infty)$, η, $\sigma > 0$. Then the value of problem (2.1) is*

$$\widehat{\mathcal{J}} = \widehat{\mathcal{J}}(\eta,\sigma) = -\frac{p+r'}{q-p} B\Big(\frac{s}{q-p}, 1/r'\Big)^{\frac{p-q}{s}} \Big(\sigma^{-r}\frac{1}{q+r'}\Big)^{\frac{p-q}{sr}} \Big(\eta^{-p}\frac{1}{q-p}\Big)^{\frac{-qr-r+q}{sr}},$$

$s = 1 + p/r'$, where $B\big(\frac{s}{q-p}, 1/r'\big)$ is the Beta function at the point $\big(\frac{s}{q-p}, 1/r'\big)$.

Proof. Let $a, \eta, \sigma > 0$. We fix η, σ. As in the proof of Lemma 2.2, we exclude A from (2.7) and change the variable $z \to zB^{1/(q-p)}$ under the integral signs

$$\left(\frac{a}{\gamma}\right)^{\frac{pr+r-p}{r-1}} \left(\eta^{-p}\left(\int_0^1 + g\int_1^\gamma\right)\frac{z^p}{\sqrt[r]{z^p-z^q}}\,dz\right)$$

$$\times\left(\sigma^{-r}\left(\int_0^1 + g\int_1^\gamma\right)(z^p-z^q)^{\frac{r-1}{r}}\,dz\right)^{\frac{1}{r-1}} = 1,$$

$$\mathcal{J}(a,\eta,\sigma) = -\left(\frac{a}{\gamma}\right)^{\frac{qr+r-q}{r-1}} \left(\left(\int_0^1 + g\int_1^\gamma\right)\frac{z^q}{\sqrt[r]{z^p-z^q}}\,dz\right)$$

$$\times\left(\sigma^{-r}\left(\int_0^1 + g\int_1^\gamma\right)(z^p-z^q)^{\frac{r-1}{r}}\,dz\right)^{\frac{1}{r-1}},$$

where $\gamma = aB^{\frac{1}{p-q}} \in (0,1]$, $g = \operatorname{sgn}(\tau)$. Expressing a from the first equality and substituting in the expression for $\mathcal{J}(a,\eta,\sigma)$, gives

$$\mathcal{J} = \mathcal{J}(a,\eta,\sigma) = -F\cdot\Big(\sigma^{-r}(E-F)\Big)^{\frac{p-q}{pr+r-p}}\Big(\eta^{-p}E\Big)^{\frac{-qr-r+q}{pr+r-p}},$$

$$E = \left(\int_0^1 + g\int_1^\gamma\right)\frac{z^p}{\sqrt[r]{z^p-z^q}}\,dz, \quad F = \left(\int_0^1 + g\int_1^\gamma\right)\frac{z^q}{\sqrt[r]{z^p-z^q}}\,dz.$$

Let us find the infimum of $\mathcal{J}(a,\eta,\sigma)$ over all possible values of a, g. We note that if $g = 1$ or $g = -1$ and $\gamma \in (0,1)$, then the function $\mathcal{J}(a,\eta,\sigma)$ at the point (a,η,σ), which corresponds to (γ, g), may not have a local minimum in a, for otherwise the function $\widehat{x}(\cdot) = \widehat{x}(a,\eta,\sigma)(\cdot)$, which is continuously differentiable on $\mathbb{R}_-$, would deliver a local minimum in problem (2.1). However, in this case the transversality condition for the function $L(\cdot)$ from (2.2) must be satisfied on $\widehat{x}(\cdot)$ (see Proposition 1):

$$L_{\dot{x}}(0,\widehat{x}(0),\dot{\widehat{x}}(0)) = 0 \quad\Leftrightarrow\quad A\,r\operatorname{sgn}\dot{\widehat{x}}(0)\,|\dot{\widehat{x}}(0)|^{r-1} = 0 \quad\Leftrightarrow\quad \dot{\widehat{x}}(0) = 0.$$

The function $\widehat{x}(\cdot)$ satisfies (2.5), and hence, $a = \widehat{x}(0) = B^{1/(q-p)}$; that is, $\gamma = 1$ with this a, which contradicts the assumption $\gamma \in (0,1)$.

Let $g = 1$ and $\gamma_0 \in (0,1)$. For convenience, instead of examining $\mathcal{J}$ we shall consider the value $\ln|\mathcal{J}|$: the sign of the increment of $\ln|\mathcal{J}|$ between the values at 1 and at γ_0 agrees with the sign of the derivative $\ln|\mathcal{J}|$, which is as follows:

$$\frac{\gamma^q}{F}(\gamma^p-\gamma^q)^{-1/r} + \frac{p-q}{pr+r-p}\cdot\frac{(\gamma^p-\gamma^q)^{1/r'}}{E-F} + \frac{-qr-r+q}{pr+r-p}\cdot\frac{\gamma^p}{E}(\gamma^p-\gamma^q)^{-1/r}$$

at the intermediate point $\gamma \in (\gamma_0, 1)$. This sign also agrees with the sign of

$$\frac{\gamma^q}{F} + \frac{p-q}{pr+r-p}\cdot\frac{\gamma^p-\gamma^q}{E-F} + \frac{-qr-r+q}{pr+r-p}\cdot\frac{\gamma^p}{E}. \tag{2.27}$$

This expression will vanish for $\gamma = 1$, because in this case $E = I_1$, $F = I_2$, $E - F = I_1 - I_2$,

$$I_1 = \int_0^1 \frac{z^p}{\sqrt[r]{z^p-z^q}}\,dz = \frac{1}{q-p}\int_0^1 z^{\frac{1+p/r'}{q-p}-1}(1-z)^{-1/r}\,dz = \frac{1}{q-p}B\Big(\frac{1+p/r'}{q-p}, 1/r'\Big),$$

$$I_1 - I_2 = \int_0^1 (z^p - z^q)^{1/r'}\,dz = \frac{1}{q-p} B\Big(\frac{1+p/r'}{q-p}, 1/r'+1\Big) = \frac{1}{q+r'} B\Big(\frac{1+p/r'}{q-p}, 1/r'\Big),$$

$$I_2 = \int_0^1 \frac{z^q}{\sqrt[r]{z^p - z^q}}\,dz = \Big(\frac{1}{q-p} - \frac{1}{q+r'}\Big) B\Big(\frac{1+p/r'}{q-p}, 1/r'\Big),$$

where $B(\cdot,\cdot)$ is the beta function. So, (2.27) with $\gamma = 1$ becomes

$$\frac{1}{I_2} + \frac{-qr - r + q}{pr + r - p} \cdot \frac{1}{I_1} = 0.$$

Hence, the sign of the expression in (2.27) agrees with the opposite sign of

$$q\frac{\gamma^{q-1}}{F} - \frac{\gamma^{2q}}{F^2}(\gamma^p - \gamma^q)^{-1/r} + \frac{p-q}{pr+r-p} \cdot \Big(\frac{p\gamma^{p-1} - q\gamma^{q-1}}{E-F} - \frac{(\gamma^p - \gamma^q)^{1+1/r'}}{(E-F)^2}\Big)$$

$$+ \frac{-qr - r + q}{pr + r - p} \cdot \Big(p\frac{\gamma^{p-1}}{E} - \frac{\gamma^{2p}}{E^2}(\gamma^p - \gamma^q)^{-1/r}\Big),$$

while the principal significant terms in this sum are as follows:

$$-(\gamma^p - \gamma^q)^{-1/r}\Big(\frac{\gamma^{2q}}{F^2} + \frac{-qr - r + q}{pr + r - p} \cdot \frac{\gamma^{2p}}{E^2}\Big).$$

Substituting $\gamma = 1$ into the bracketed expression, we find that

$$\frac{1}{I_2^2} + \frac{-qr - r + q}{pr + r - p} \cdot \frac{1}{I_1^2} = I_1^2\Big(\frac{(q+r')^2}{(p+r')^2} - \frac{q+r'}{p+r'}\Big) > 0.$$

Hence, if γ_0 is sufficiently close to 1, then the sign of the derivative of the expression in (2.27) is negative, while this expression is positive, because it equals the product of the derivative at an intermediate point and the variation of the argument $(\gamma - 1) < 0$. Hence, and since both $\mathcal{J}(\cdot)$ and $\ln|\mathcal{J}|$ have no local minima at the points a that correspond to $\gamma \in (0,1)$, it follows that for $a > 0$, $a \sim (\gamma, g)$, $g = 1$ and $\gamma \in (0,1]$, the least possible value for $\mathcal{J}(a, \eta, \sigma)$ is attained for $\gamma = 1$; it is as follows:

$$\widehat{\mathcal{J}} = -I_2 \cdot \Big(\sigma^{-r}(I_1 - I_2)\Big)^{\frac{p-q}{pr+r-p}} \Big(\eta^{-p} I_1\Big)^{\frac{-qr-r+q}{pr+r-p}}.$$

Using the representation of I_1, I_2 in terms of the Beta function, $\widehat{\mathcal{J}}$ can be written as

$$\widehat{\mathcal{J}} = -\frac{p+r'}{q-p} B\Big(\frac{1+p/r'}{q-p}, 1/r'\Big)^{\frac{p-q}{1+p/r'}} \Big(\sigma^{-r}\frac{1}{q+r'}\Big)^{\frac{p-q}{pr+r-p}} \Big(\eta^{-p}\frac{1}{q-p}\Big)^{\frac{-qr-r+q}{pr+r-p}}.$$

If $g = -1$ and $\gamma \in (0,1)$ tends to zero, then

$$\mathcal{J} = -2\,I_2 \cdot \Big(2\sigma^{-r}(I_1 - I_2)\Big)^{\frac{p-q}{pr+r-p}} \Big(2\eta^{-p} I_1\Big)^{\frac{-qr-r+q}{pr+r-p}} (1 + \bar{o}(1)) \tag{2.28}$$

and therefore, $\widehat{\mathcal{J}} < \mathcal{J}$ for small values of γ. Again using the fact that $\mathcal{J}(\cdot)$ has no local minima at the points a corresponding to $\gamma \in (0,1)$, it follows that, for $a > 0$,

$a \sim (\gamma, g)$, $g = -1$ and $\gamma \in [0, 1]$, the least value of $\mathcal{J}(a, \eta, \sigma)$ is attained on $\widehat{\mathcal{J}}$. Note that the function $\widehat{\mathcal{J}} = \widehat{\mathcal{J}}(\eta, \sigma)$ depends on $\eta > 0$, $\sigma > 0$ in a continuously differentiable way.

It remains to consider the case of an admissible function $x(\cdot)$ for (2.1), for which $x(0) = 0$. Assume that for such a function

$$-\int_{\mathbb{R}_-} |x|^q \, dt < \widehat{\mathcal{J}}. \tag{2.29}$$

Without loss of generality we may assume that $x(\cdot)$ does not vanish identically on any interval $[\tau, 0]$, $\tau < 0$. Let $\tau < 0$ be such that $x(\tau) \neq 0$. By the above,

$$-\int_{-\infty}^{\tau} |x|^q \, dt \geqslant \widehat{\mathcal{J}}\Big(\|x\|^p_{L_p((-\infty,\tau])}, \|\dot{x}\|^r_{L_r((-\infty,\tau])}\Big),$$

which contradicts (2.29) and the continuity of $\widehat{\mathcal{J}}(\eta, \sigma)$, because $\tau < 0$ can be taken arbitrarily close to zero. □

Corollary 2.1. (see [11]). *The quantity*

$$2\, I_2 \cdot \Big(2\sigma^{-r}(I_1 - I_2)\Big)^{\frac{p-q}{pr+r-p}} \Big(2\eta^{-p} I_1\Big)^{\frac{-qr-r+q}{pr+r-p}}$$

(*see* (2.28)) *is the value of problem* (1.1) *with* $p, q > 0$, $r > 1$ *and* $T = \mathbb{R}$, $n = 1$, $k = 0$.

Let $0 < p < \infty$, $1 < r < \infty$, $\eta, \sigma > 0$. Let us now examine problem (2.3), whose value agrees with that of the problem

$$x(0) \to \sup \quad \int_{\mathbb{R}_-} |x|^p \, dt \leqslant \eta^p, \ \int_{\mathbb{R}_-} |\dot{x}|^r \, dt \leqslant \sigma^r. \tag{2.30}$$

We set

$$\widehat{x}(t) = \begin{cases} p^{1/p}\sigma^{1/p}\eta^{1/r'} \cdot e^{\sigma t/\sigma} & \text{for } p = r, \\ \left(\frac{p+r'}{r'}\right)^{\frac{r'}{p+r'}} \sigma^{1-s}\eta^s \left(1 - \frac{p-r}{pr-p+r} a_1 t\right)_+^{\frac{r}{r-p}} & \text{for } p \neq r, \end{cases}$$

where $(u)_+ := \max\{u, 0\}$ and

$$s = (1 + r'/p)^{-1}, \quad 1/r + 1/r' = 1, \quad a_1 = \left(\frac{\sigma}{\eta(1-s)}\right)^{r's}.$$

Proof of Lemma 2.1. If $p \geqslant 1$, then problem (2.30) is convex, and in this case the verification of the conclusions of Lemma 2.1 presents no challenge: for $p > 1$ the necessary conditions for an extremum are such that, for the function $\widehat{x}$, the Euler equations $-\frac{d}{dt}L_{\dot{x}} + L_x = 0$ for the Lagrangian $L = L(t, x, \dot{x}) = \lambda|x|^p + \mu|\dot{x}|^r$, the transversality condition at the origin $\mu r|\dot{x}(0)|^{r-1} \operatorname{sgn} \dot{x}(0) = 1$, the complementary slackness conditions

$$\lambda\Big(\int_{\mathbb{R}_-} |x|^p \, dt - \eta^p\Big) = 0, \quad \mu\Big(\int_{\mathbb{R}_-} |\dot{x}|^r \, dt - \sigma^r\Big) = 0,$$

and the nonnegative conditions for $\lambda, \mu \geqslant 0$ ([1]),

$$\lambda = (r-1)\sigma^r \eta^{-p}\mu, \quad \mu = \frac{1}{r}\left((1-s)\sigma^{-r(1+1/p)}\eta\right)^s,$$

must all hold. Moreover, the necessary conditions for an extremum imply that, for any compactly supported continuously differentiable function $x(\cdot)$, the equality (the principal identity) holds:

$$x(0) = p\lambda \int_{\mathbb{R}_-} (\widehat{x})^{p-1} \cdot x\,dt + r\mu \int_{\mathbb{R}_-} (\widehat{x}\,')^{r-1} \cdot x'\,dt.$$

This condition can be also verified directly by integration by parts. Hence, if the function $x(\cdot)$ satisfies in addition the constraints of problem (2.30), then

$$|x(0)| \leqslant p\lambda \Big|\int_{\mathbb{R}_-} (\widehat{x})^{p-1} \cdot x\,dt\Big| + r\mu \Big|\int_{\mathbb{R}_-} (\widehat{x}\,')^{r-1} \cdot x'\,dt\Big|$$

$$\leqslant p\lambda\eta\Big(\int_{\mathbb{R}_-} (\widehat{x})^p\,dt\Big)^{1/p'} + r\mu\sigma\Big(\int_{\mathbb{R}_-} (\widehat{x}\,')^r\,dt\Big)^{1/r'}$$

$$= p\lambda\eta^p + r\mu\sigma^r = r\mu\sigma^r(p/r'+1) = (1-s)^{s-1}\sigma^{1-s}\eta^s.$$

Hence, for any compactly supported continuously differentiable function which is admissible for (2.30),

$$|x(0)| \leqslant (1-s)^{s-1}\sigma^{1-s}\eta^s. \tag{2.31}$$

Since the expression $(1-s)^{s-1}\sigma^{1-s}\eta^s$ in the right-hand side of (2.31) depends on $\eta, \sigma > 0$ continuously, it follows that (2.31) is also satisfied for an arbitrary function which is admissible for (2.30). Inequality (2.30) is sharp: it becomes an equality for the function $\widehat{x}(\cdot)$.

The case $p = 1$ is a limiting case for $p > 1$: if $x(\cdot)$ is an arbitrary continuously differentiable function with compact support on $\mathbb{R}_-$, then, for any $p > 1$,

$$|x(0)| \leqslant (1-s)^{s-1}\Big(\int_{\mathbb{R}_-} |x|^p\,dt\Big)^{s/p}\Big(\int_{\mathbb{R}_-} |\dot{x}|^r\,dt\Big)^{(1-s)/r}, \quad s = (1+r'/p)^{-1}.$$

This continuity in $p \geqslant 1$ implies that this inequality also holds for $x(\cdot)$ and with $p = 1$. As in the case $p > 1$, this secures the assertion of the lemma with $p = 1$.

In the case $p < 1$, considering the transformations $x(t) \to \alpha x(\beta t)$ with $\alpha, \beta > 0$, we transform (2.3) to the problem

$$\int_{\mathbb{R}_-} |y|^p\,dt \to \inf \quad \int_{\mathbb{R}_-} |\dot{y}|^r\,dt = w^r,\ y(0) = a. \tag{2.32}$$

Let $L = \lambda_0|y|^p + A|\dot{y}|^r$, (λ_0, A) be the Lagrange multipliers. The solution to (2.32) may be found from the necessary conditions for an extremum, which (by Proposition 1) are related to the first integral for the Euler equations [1] (the 'energy' integral):

$$-\lambda_0|y|^p + A(r-1)|\dot{y}|^r = \text{const}$$

and to the function $x \equiv 0$. Thus, the solution to (2.32) will be sought among the functions that tend to zero together with their first derivative as $t \to -\infty$; we shall also assume that $\lambda_0 = 1$, $A > 0$. Hence,

$$-|y|^p + A(r-1)|\dot{y}|^r = 0. \tag{2.33}$$

A monotone solution of (2.33) satisfies the equation $\dot{y} = ((r-1)A)^{-1/r} y^{p/r}$; this is

$$\widehat{y}(t) = \Big(a^{\frac{r-p}{r}} + \frac{(1-p/r)\cdot t}{((r-1)A)^{1/r}}\Big)_+^{\frac{r}{r-p}}.$$

In order to find A it remains to invoke the isoperimetric constraint from (2.32). The optimality of the solution thus obtained will be proved via the Weierstrass formula in the isoperimetric problem (see [1]). We choose an admissible continuously differentiable function $y(t)$, supported in $[-l, 0]$, and assuming only positive values on $(-l, 0]$. For any $t \in (-l, 0]$, we define $\lambda = \lambda(t) > 0$, $C = C(t) > 0$, so as to have $(y = y(t))$

$$\Big(\lambda + \frac{(1-p/r)\cdot t}{((r-1)C)^{1/r}}\Big)_+^{\frac{r}{r-p}} = y, \qquad \frac{1}{(r-1)C}\int\limits_{-\infty}^{t}\Big(\lambda + \frac{(1-p/r)\cdot \tau}{((r-1)C)^{1/r}}\Big)_+^{\frac{pr}{r-p}} d\tau$$

$$= \sigma = \sigma(t) := \int\limits_{-l}^{t} |\dot{y}(\tau)|^r \, d\tau$$

or (after integration)

$$\lambda + \frac{(1-p/r)\cdot t}{((r-1)C)^{1/r}} = y^{\frac{r-p}{r}}, \qquad \frac{1}{((r-1)C)^{1/r'}}\Big(\lambda + \frac{(1-p/r)\cdot t}{((r-1)C)^{1/r}}\Big)^{\frac{pr}{r-p}+1}\frac{1}{1+p/r'} = \sigma.$$

Hence,

$$\frac{1}{((r-1)C)^{1/r'}}\,\frac{y^{1+p/r'}}{1+p/r'} = \sigma, \qquad (r-1)C = \frac{y^{p+r'}\sigma^{-r'}}{(1+p/r')^{r'}}. \tag{2.34}$$

We set

$$u = u(t) := \Big(\frac{y^p}{(r-1)C}\Big)^{1/r}.$$

We assume, in addition, that the function $y(\cdot)$ satisfies the condition

$$\lim_{t\to -l+0} C(t)^{1/r}\cdot y(t)^{1+p/r'} = 0 \tag{2.35}$$

(this condition is satisfied for the functions from the example after Lemma 2.2). For $y(\cdot)$ satisfying (2.35), we write down the Weierstrass identity and check it:

$$\int_{\mathbb{R}_-}(y^p - \widehat{y}^p)\,dt = \int_{-l}^{0}\big(y^p + C|\dot{y}|^r - y^p - Cu^r - (\dot{y}-u)Cru^{r-1}\big)\,dt$$

$$= \int_{-l}^{0} C\cdot\big(|\dot{y}|^r - u^r - ru^{r-1}(\dot{y}-u)\big)\,dt \geqslant 0.$$

After cancelling and substituting $u(\cdot)$, we see that it is required to prove the equality

$$\int_{\mathbb{R}_-} (-\widehat{y}^p)\,dt = \int_{-l}^{0} \big(C|\dot y|^r - r\dot y C^{1/r} y^{p/r'} (r-1)^{-1/r'}\big)\,dt. \tag{2.36}$$

We have $dt = ((r-1)A)^{1/r}\widehat{y}^{\,-p/r}\,d\widehat{y}$ for $t \in \mathbb{R}_-$ lying in the interior of the support of $\widehat{y}(\cdot)$, and hence

$$\int_{\mathbb{R}_-} (-\widehat{y}^p)\,dt = \int_0^a (-z^p)((r-1)A)^{1/r} z^{-p/r}\,dz \overset{\text{(def)}}{=} -((r-1)C(0))^{1/r}\int_0^{y(0)} z^{p/r'}\,dz$$

$$\overset{(2.35)}{=} -\int_{-l}^{0} \frac{d}{dt}\Big(((r-1)C(t))^{1/r}\int_0^{y(t)} z^{p/r'}\,dz\Big)\,dt$$

$$= -\int_{-l}^{0}\Big(((r-1)C(t))^{1/r} y(t)^{p/r'}\dot y(t) + \frac{1}{r}((r-1)C(t))^{-1/r'}(r-1)\dot C(t)\frac{y(t)^{1+p/r'}}{1+p/r'}\Big)\,dt.$$

From (2.34) it follows that, for $t \in (-l, 0]$,

$$(r-1)\dot C(t) = (p+r')(r-1)C(t)\,\frac{\dot y(t)}{y(t)} - r'(r-1)C(t)\frac{|\dot y(t)|^r}{\sigma}.$$

Using this representation for $(r-1)\dot C(t)$, we find that

$$\int_{\mathbb{R}_-} (-\widehat{y}^p)\,dt = -\int_{-l}^{0}\Big(((r-1)C(t))^{1/r} y(t)^{p/r'}\dot y(t) + \frac{p+r'}{r}((r-1)C(t))^{1/r}\,\frac{\dot y(t)\,y(t)^{p/r'}}{1+p/r'}$$

$$-\frac{1}{r}\big((r-1)C(t)\big)^{-1/r'} r'(r-1)C(t)\frac{|\dot y(t)|^r}{\sigma}\frac{y(t)^{1+p/r'}}{1+p/r'}\Big)\,dt$$

$$\overset{(2.34)}{=} -\int_{-l}^{0}\Big(r'((r-1)C(t))^{1/r} y(t)^{p/r'}\dot y(t) \; - \frac{\sigma}{r} r'(r-1)C(t)\frac{|\dot y(t)|^r}{\sigma}\Big)\,dt,$$

which proves (2.36). Thus, in problem (2.32), among the admissible continuously differentiable functions $y(\cdot)$, with finite support on $[-l, 0]$, which are strictly positive on $(-l, 0]$ and satisfy (2.35), the smallest value of $\|y\|_p$ is attained for $\widehat{y}(\cdot)$.

Hence, as in the proof of Theorem 2.1, it follows that $\widehat{y}(\cdot)$ is a solution to (2.32). Again considering the transformations $\widehat{y}(t) \to \alpha\widehat{y}(\beta t)$ with $\alpha, \beta > 0$, we prove that $\widehat{x}(\cdot)$ is a solution to (2.3) and (2.30). □

Proof of Proposition 1. For the transversality condition the desired conclusion will be obtained at a point t_0 with $t_0 \in Q(\widehat{x})$ and when at $\widehat{x}(t_0)$ all the functions $\ell_i(\cdot, \widehat{x}(t_1))$, $i = 0, \dots, m$, are continuously differentiable. Let H denote the linear subspace of the space $C^1([t_0, t_1], \mathbb{R}^n)$ consisting of all functions $v(\cdot)$, $v(t_1) = 0$, supported in $Q(\widehat{x})$. For the mapping $\Lambda(\cdot)$ from H into $\mathbb{R}^{m+1}$,

$$\Lambda v(\cdot) = (\mathcal{J}_0'(\widehat{x}(\cdot))[v(\cdot)], \dots, \mathcal{J}_m'(\widehat{x}(\cdot))[v(\cdot)])^T, \; v(\cdot) \in H,$$

$$\mathcal{J}_i'(\widehat{x}(\cdot))[v(\cdot)] = \int_{\operatorname{supp}(v)} \Big((\widehat{L}_i)_{\dot x}(t)\cdot \dot v(t) + (\widehat{L}_i)_x(t)\cdot v(t)\Big)dt + \ell_{x(t_0)}(\widehat{x}(t_0), \widehat{x}(t_1))\cdot v(t_0)$$

(here $\mathrm{supp}\,(v)$ is the support a function $v(\cdot)\big)$, one of the following two cases is possible: the range of Λ is a proper subspace of $\mathbb{R}^{m+1}$ or is the whole space $\mathbb{R}^{m+1}$. In the first case, there exists a nonzero vector $y \in \mathbb{R}^{m+1}$ orthogonal to this subspace. Let $\overline{\lambda} = (\lambda_0, \ldots, \lambda_m) = y^T$. Then $\overline{\lambda} \cdot \Lambda v(\cdot) = \langle y, \Lambda v(\cdot)\rangle = \int_{\mathrm{supp}\,(v)} (\widehat{L}_{\dot{x}}(t) \cdot \dot{v}(t) + \widehat{L}_x(t) \cdot v(t))dt + \ell_{x(t_0)}(\widehat{x}(t_0), \widehat{x}(t_1)) \cdot v(t_0) = 0$ for any $v(\cdot) \in H$.

Let $t_2 \in Q(\widehat{x})$ and let $U = U(t_2, \widehat{x}) \subset \mathbb{R}^{2n+1}$ be a neighbourhood of the point $(t_2, \widehat{x}(t_2), \dot{\widehat{x}}(t_2))^T$ at which the functions L, L_x, $L_{\dot{x}}$ are continuous. We consider a nondegenerate interval $[\tau_0, \tau_1] \subset [t_0, t_1]$ which contains t_2 and for which the portion $\{(t, \widehat{x}(t), \dot{\widehat{x}}(t))^T \in \mathbb{R}^{2n+1} \mid t \in [\tau_0, \tau_1]\}$ of the extended graph of $\widehat{x}(\cdot)$ lies in U. We have $\langle y, \Lambda v(\cdot)\rangle = \int_{\tau_0}^{\tau_1} (\widehat{L}_{\dot{x}}(t) \cdot \dot{v}(t) + \widehat{L}_x(t) \cdot v(t))dt = 0$ for any $v(\cdot) \in C^2[\tau_0, \tau_1] \cap H$ supported in $[\tau_0, \tau_1]$. If $\tau_0 = t_0$ (or $\tau_1 = t_1$), we assume in addition that $v(t_0) = \dot{v}(t_0) = 0$ (or, respectively, $v(t_1) = \dot{v}(t_1) = 0$). Integrating by parts, gives

$$0 = \int_{\tau_0}^{\tau_1} \Big(\widehat{L}_{\dot{x}}(t) + \int_t^{\tau_1} \widehat{L}_x(\tau)d\tau - c_0\Big) \cdot \dot{v}(t)\,dt$$

$$= \int_{\tau_0}^{\tau_1} \Big(\int_t^{\tau_1} \big(\widehat{L}_{\dot{x}}(\tau) + \widehat{L}_x(\tau)(\tau - t) - c_0\big)\,d\tau - c_1\Big) \cdot \ddot{v}(t)\,dt$$

for any $c_0, c_1 \in \mathbb{R}^n$. We choose c_0, c_1 and a function $v(\cdot)$ so as to have

$$\ddot{v}(t) = \int_t^{\tau_1} \big(\widehat{L}_{\dot{x}}(\tau) + \widehat{L}_x(\tau)(\tau - t) - c_0\big)\,d\tau - c_1\,, \quad v(\tau_0) = \dot{v}(\tau_0) = 0,$$

$$\int_{\tau_0}^{\tau_1} \dot{v}(t)dt = \int_{\tau_0}^{\tau_1} \ddot{v}(t)dt = 0,$$

on $[\tau_0, \tau_1]$ and have $v(t) = 0$ outside the interval $[\tau_0, \tau_1]$. Hence,

$$\int_{\tau_0}^{\tau_1} \Big\| \int_t^{\tau_1} \big(\widehat{L}_{\dot{x}}(\tau) + \widehat{L}_x(\tau)(\tau - t) - c_0\big)\,d\tau - c_1 \Big\|_{\ell_2^n}^2 dt = 0.$$

It follows that $\int_t^{\tau_1} \big(\widehat{L}_{\dot{x}}(\tau) + \widehat{L}_x(\tau)(\tau - t) - c_0\big)\,d\tau - c_1 \equiv 0$ on $[\tau_0, \tau_1]$ and that $\widehat{L}_{\dot{x}}(t)$ is continuously differentiable. Differentiating, we arrive at the Euler equation on the interval $[\tau_0, \tau_1]$, and hence, at all points of the set $Q(\widehat{x})$.

Since $t_0 \in Q(\widehat{x})$, at this point we have, in addition,

$$\int_{t_0}^{\tau_1} (\widehat{L}_{\dot{x}}(t) \cdot \dot{v}(t) + \widehat{L}_x(t) \cdot v(t))dt + \ell_{x(t_0)}(\widehat{x}(t_0), \widehat{x}(t_1)) \cdot v(t_0) = 0$$

for any $v(\cdot) \in C^2[\tau_0, \tau_1] \cap H$ supported in $[t_0, \tau_1]$. Integrating by parts and using the Euler equation on $[t_0, \tau_1]$, we arrive at the equality

$$\big(\ell_{x(t_0)}(\widehat{x}(t_0), \widehat{x}(t_1)) - \widehat{L}_{\dot{x}}(t_0)\big) \cdot v(t_0) = 0$$

for any $v(t_0) \in \mathbb{R}^n$. Hence, the transversality condition also holds.

In the second case, there exists a subspace M generated by the functions $\{v_i(\cdot)\}_{i=1}^{m+1} \subset H$ such that $\Lambda(M) = \mathbb{R}^{m+1}$. This means that $\Lambda\big|_M$ is an isomorphism between M and $\mathbb{R}^{m+1}$. Let $S = \cup_{i=1}^{m+1} \mathrm{supp}\,(v_i)$ denote the union of the supports of the functions $v_i(\cdot)$, $i = 1, \dots, m+1$. Note that S is a compact subset of $Q(\widehat{x})$ and $\operatorname{supp}(v) \subset S$ for any $v(\cdot) \in M$. By the definition of $Q(\widehat{x})$ it follows that in some neighbourhood of the set $\{(t, \widehat{x}(t), \dot{\widehat{x}}(t))^T \in \mathbb{R}^{2n+1} \mid t \in S\}$ both the functions $L_i(t, x, \dot{x})$, $i = 0, \dots, m$, and their partial derivatives in x and in $\dot{x}$ are defined and continuous.

Let $\vartheta \in \mathbb{R}^{m+1}$, $\psi(\vartheta) = (\mathcal{J}_0(\widehat{x}(\cdot) + v(\cdot)), \dots, \mathcal{J}_m(\widehat{x}(\cdot) + v(\cdot)))^T$, where $v(\cdot) = (\Lambda\big|_M)^{-1}(\vartheta)$. By the definition $\psi'(0)[\vartheta] = \Lambda v(\cdot) = \vartheta$; that is, $\psi'(0)$ is the identity operator. Next, by the inverse function theorem, applied to $\psi(\cdot)$ at the point $\vartheta = 0$, at any neighbourhood of $\widehat{x}(\cdot)$ one may solve the system of equations $\mathcal{J}_0(x(\cdot)) = \mathcal{J}_0(\widehat{x}(\cdot)) + \varepsilon$, $\mathcal{J}_1(x(\cdot)) = \ldots = \mathcal{J}_m(x(\cdot)) = 0$ for each ε sufficiently close to 0. But this contradicts the fact that $\widehat{x}(\cdot)$ is a local extremum of the problem. $\square$

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