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NURZHAN BOKAYEV

(to the 60th birthday)

On January 5, 2016 was the 60th birthday of Doctor of Physical-Mathematical Sciences (1996), Professor Nurzhan Adilkhanovich Bokayev. Professor Bokayev is the head of the department "Higher Mathematics" of the L.N. Gumilyov Eurasian National University (since 2009), the Vice-President of Mathematical Society of the Turkic World (since 2014), and a member of the Editorial Board of our journal.

N.A. Bokayevwas born in the Urnek village, Karabalyk district, Kostanay region. He graduated from the E.A. Buketov Karaganda State University in 1977 and the M.V. Lomonosov Moscow State University's full-time postgraduate study in 1984.

Scientific works of Professor Bokayev are devoted to studying problems of the theory of functions, in particular of the theory of orthogonal series.

He proved renewal and uniqueness theorems for series with respect to periodic multiplicative systems and Haar-type systems, constructed continual sets of uniqueness (U-sets) and sets of non-uniqueness (M-sets) for multiplicative systems; investigated Besov-type function spaces with respect to the Price bases; studied the Price - and Haar-type p-adic operators; introduced new classes of Faber-Schauder-type systems of functions and spaces of multivariable functions of bounded p-variation and of bounded p-fluctuation, obtained estimates for the best approximation of functions in these spaces by polynomials with respect to the Walsh and Haar systems, established weighted integrability conditions of the sum of multiple trigonometric series and series with respect to multiplicative systems, found the embedding criterion for the Nikol'skii-Besov spaces with respect to multiplicative bases and the coefficient criterion for belonging of functions to such spaces.

His scientific results have made essential contribution to the theory of series with respect to the Walsh and Haar systems and multiplicative systems.

N.A. Bokayev has published more than 150 scientific papers. Under his supervision 8 dissertations have been defended: 4 candidate of sciences dissertations and 4 PhD dissertations.

The Editorial Board of the Eurasian Mathematical Journal congratulates Nurzhan Adilkhanovich Bokayev on the occasion of his 60th birthday and wishes him good health and successful work in mathematics and mathematical education.

The EMJ has been included in the Emerging Sources Citation Index

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On behalf of the Editorial Board of the EMJ
V.I. Burenkov, T.V. Tararykova, A.M. Temirkhanova

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BOUNDEDNESS, COMPACTNESS FOR A CLASS OF FRACTIONAL INTEGRATION OPERATORS OF WEYL TYPE

A.M. Abylayeva

Communicated by E.D. Nursultanov

Key words: fractional integration operator, Weyl operator, Riemann-Liouville operator, Hadamard operator, Erdelyi-Kober operator, boundedness, compactness.

AMS Mathematics Subject Classification: 26A33, 26D10, 47G10.

Abstract. We establish criteria for the boundedness and compactness for a class of operators of fractional integration involving the Weyl operator.

1 Introduction

Let $I = (a, b)$, $0 \leq a < b \leq \infty$, $0 < q, p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let u, v be almost everywhere positive and locally integrable functions on I. By $L_{p,u} \equiv L_p(u, I)$ we denote the set of all measurable functions f on I such that

$$\|f\|_{p,u} = \left(\int_a^b |f(x)|^p u(x) dx \right)^{\frac{1}{p}} < \infty.$$

In the case $u \equiv 1$ we write $L_p \equiv L_p(I)$. Let W be a positive strictly increasing and locally absolutely continuous function on I. Suppose $\frac{dW(x)}{dx} \equiv w(x)$ for almost everywhere $x \in I$.

Let $1 > \alpha > 0$. We consider the operator

$$K_{\alpha,\beta} f(x) = \int_x^b \frac{u(s) W^\beta(s) f(s) w(s) ds}{(W(s) - W(x))^{1-\alpha}}, \quad x \in I. \tag{1.1}$$

In the case $\beta = 0$, $u \equiv 1$ the dual operator to operator (1.1) has the form

$$K^*_{\alpha,\beta} f(x) = \int_a^x \frac{f(s) w(s) ds}{(W(x) - W(s))^{1-\alpha}}, \quad x \in I. \tag{1.2}$$

Operator (1.2) is called [12] the operator of fractional integration of the function f of the function W. Weighted estimates for operator (1.2) were previously considered in [9], [1].

When $W(x) = x$, $u \equiv 1$, $\beta = 0$ operator (1.1) is the Weyl operator

$$I^*_\alpha f(x) = \int\limits_x^b \frac{f(s)ds}{(s-x)^{1-\alpha}}, \quad x \in I, \tag{1.3}$$

which is dual to the Riemann-Liouville operator

$$I_\alpha g(s) = \int\limits_a^s \frac{g(x)dx}{(s-x)^{1-\alpha}}, \quad s \in I. \tag{1.4}$$

Operators (1.3) and (1.4) acting from the weighted space $L_{p,u}$ to the weighted space $L_{q,v}$ are investigated in papers [2], [3], [4], [8], [10], [11] and others, where necessary and sufficient conditions for their boundedness, compactness are obtained for various relations between the parameters α, p, q and under various assumptions regarding the weight functions u and v. Two-sided estimates of their norms are also obtained.

We investigate operator (1.1) acting from the space $L_{p,w}$ to $L_{q,v}$. From the obtained results new assertions follow, in simple terms, for operators (1.3) and (1.4), generalizing the results of [4], [8], [10].

The positivity and monotonicity of W implies the existence of the non-negative limit $\lim\limits_{x \to a^+} W(x) \equiv W(a)$. Futher, we assume $W(a) = 0$ and otherwise, we consider the operator $K_{\alpha,\beta}$ in the form, where function $W(x)$ is replaced by the function $W_0(x) = W(x) - W(a), x \in I$.

Further, the norm of the linear operator T from a normed space to another one is denoted briefly by $\|T\|$. Which spaces are meant will be clear from the context.

Throughout the paper the products of the form $0 \cdot \infty$ are supposed be equal to zero. Relations $A \ll B$, $A \gg B$ mean $A \le cB$ with a constant c depending only on p, q, α which can be different in different places. If $A \ll B$ and $A \gg B$ then we write $A \approx B$. By $\mathbb{Z}$ we denote the set of all integer numbers, χ_E denotes the characteristic function of the set E.

2 Auxiliary assertions

To prove the main results we need some well-known assertions.

Along with operator (1.1) we consider the Hardy operator

$$H_{\alpha,\beta} f(x) = \int\limits_x^b u(s) W^{\beta+\alpha-1}(s) f(s) w(s) ds. \tag{2.1}$$

It is easy to see that for $f \ge 0$

$$K_{\alpha,\beta} f(x) \ge H_{\alpha,\beta} f(x), \;\; \forall x \in I. \tag{2.2}$$

Issues of boundedness and compactness of operator (2.1) in weighted Lebesgue spaces were studied quite completely. A summary of the results can be found in [7]. The following Theorem A and Theorem B are corollaries of Theorem 5 and Theorem 6 in [7].

Theorem A. *Let $1 < p \le q < \infty$. The operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$A_{\alpha,\beta} = \sup_{z\in I} \left(\int_a^z v(x)dx \right)^{\frac{1}{q}} \left(\int_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds \right)^{\frac{1}{p'}} < \infty.$$

Moreover, $\|H_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.

Theorem B. *Let $0 < q < p < \infty$, $p > 1$. The operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$B_{\alpha,\beta} = \left(\int_a^b \left(\int_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}w(s)ds \right)^{\frac{q(p-1)}{p-q}} \times \left(\int_a^z v(x)dx \right)^{\frac{q}{p-q}} v(z)dz \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.

Remark 1. *In the case $1 < q < p < \infty$, $p > 1$ the value $B_{\alpha,\beta}$ is equivalent to the value*

$$\widetilde{B}_{\alpha,\beta}(a,b) = \left(\int_a^b \left(\int_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds \right)^{\frac{p(q-1)}{p-q}} \times \left(\int_a^z v(x)dx \right)^{\frac{p}{p-q}} u^{p'}(z)W^{p'(\alpha+\beta-1)}(z)w(z)dz \right)^{\frac{p-q}{pq}}.$$

Remark 2. *Note that a function u non-decreasing on I and such that $uW^{\beta+\alpha-1} \in L_{p',w}(z,b)$, for all $z \in I$, exists if and only if $W^{\beta+\alpha-1} \in L_{p',w}(z,b)$ for all $z \in I$.*

3 Boundedness of the operator $K_{\alpha,\beta}$

Theorem 3.1. *Let $0 < \alpha < 1$, $\frac{1}{\alpha} < p \le q < \infty$ and $\beta \le 0$ ($\beta < \frac{1}{p} - \alpha$ when $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $A_{\alpha,\beta} < \infty$. Moreover, $\|K_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.*

Proof. Necessity. Let the operator $K_{\alpha,\beta}$ be bounded from $L_{p,w}$ to $L_{q,v}$. Then, in view of (2.2), the operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ and $\|K_{\alpha,\beta}\| \ge \|H_{\alpha,\beta}\|$, therefore by Theorem A the value $A_{\alpha,\beta} < \infty$ and

$$\|K_{\alpha,\beta}\| \gg A_{\alpha,\beta}. \tag{3.1}$$

Sufficiency. Since the function W is continuous and strictly increasing on I and $W(a) = 0$, then for any $k \in \mathbb{Z}$ the set $\left\{x : W(x) \leq 2^k, x \in I\right\}$ is non-empty. Denoting $x_k = \sup\left\{x : W(x) \leq 2^k, x \in I\right\}$ we obtain a sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ such that $0 < x_k \leq x_{k+1}$, $\forall k \in \mathbb{Z}$, and if $x_k < b$, then $W(x_k) = 2^k$, $2^k \leq W(x) \leq 2^{k+1}$ for $x_k \leq x \leq x_{k+1}$, $\int\limits_{x_{k-1}}^{x_k} w(s)ds = 2^{k-1}$, and if $x_{k+1} = b$, then $\int\limits_{x_k}^{x_{k+1}} w(s)ds \leq 2^k$. These facts will be used below without reminders. We assume that $I_k = [x_k, x_{k+1})$, $k \in \mathbb{Z}$, $\mathbb{Z}_0 = \{k : k \in \mathbb{Z},\ I_k \neq \emptyset\}$. Then $\mathbb{Z}_0 \subseteq \mathbb{Z}$ and $I = \bigcup\limits_{k\in\mathbb{Z}} I_k = \bigcup\limits_{k\in\mathbb{Z}_0} I_k$. Since $I_k = \emptyset$, $\forall k \in \mathbb{Z} \setminus \mathbb{Z}_0$, and integrals over these intervals are equal to zero, then in the sequel, without loss of generality, we suppose that $\mathbb{Z} = \mathbb{Z}_0$.

Let $A_{\alpha,\beta} < \infty$. We need to prove that the inequality

$$\|T_{\alpha,\beta} f\|_{q,v} \ll A_{\alpha,\beta}\|f\|_{p,w}, \quad f \in L_{p,w}, \tag{3.2}$$

holds, which means $\|T_{\alpha,\beta}\| \ll A_{\alpha,\beta}$ and, together with (3.1), gives

$$\|T_{\alpha,\beta}\| \approx A_{\alpha,\beta}.$$

It suffices to prove inequality (3.2) for $f \geq 0$. So let $f \geq 0$. Using the relation $I = \bigcup\limits_k I_k$, we have

$$\|K_{\alpha,\beta} f\|_{q,v}^q = \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left(\int\limits_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx$$

$$= \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left[\left(\int\limits_x^{x_{k+1}} + \int\limits_{x_{k+1}}^b \right) \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right]^q dx$$

$$\ll \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left(\int\limits_x^{x_{k+1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx$$

$$+ \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left(\int\limits_{x_{k+1}}^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx = J_1 + J_2. \tag{3.3}$$

We estimate the values J_1 and J_2 separately. Using Hölder's inequality, nondecreasing of the function u and $\beta \leq 0$ and in view of change of variables $W(s) = W(x)t$ we have

$$J_1 = \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left(\int\limits_x^{x_{k+1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx$$

$$\leq \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left(\int\limits_x^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \left(\int\limits_x^{x_{k+1}} \frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{(1-\alpha)p'}} \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} u^q(x_{k+1}) \int\limits_{x_{k-1}}^{x_k} v(x) W^{\frac{q}{p'}(p'\beta+p'(\alpha-1))}(x)$$

$$\times W^{\frac{q}{p'}}(x) \left(\int\limits_1^{\frac{W(x_{k+1})}{W(x_{k-1})}} t^{p'\beta}(t-1)^{p'(\alpha-1)} dt \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} u^q(x_{k+1})$$

$$\times 2^{\frac{q}{p'}(p'(\beta+\alpha-1))(k-1)} 2^{\frac{q}{p'}k} \left(\int\limits_1^4 t^{p'\beta}(t-1)^{p'(\alpha-1)} dt \right)^{\frac{q}{p'}} \int\limits_{x_{k-1}}^{x_k} v(x)dx. \quad (3.4)$$

By the assumptions of the theorem $\alpha > \frac{1}{p}$, therefore $\int\limits_1^4 t^{p'\beta}(t-1)^{p'(\alpha-1)}dt < \infty$.

The expression $F = u^q(x_{k+1}) 2^{q(\beta+\alpha-1)(k-1)} 2^{\frac{q}{p'}k}$ is estimated as follows. Since $\beta + \alpha - 1 < 0$ then

$$F = u^q(x_{k+1}) 2^{3q|\beta+\alpha-1|} 2^{q(\beta+\alpha-1)(k+2)} 2^{-\frac{q}{p'}} 2^{\frac{q}{p'}(k+1)}$$

$$= 2^{3q|\beta+\alpha-1|-\frac{q}{p'}} u^q(x_{k+1}) 2^{q(\beta+\alpha-1)(k+2)} \left(\int\limits_{x_{k+1}}^{x_{k+2}} w(s)ds \right)^{\frac{q}{p'}}$$

$$\leq 2^{3q|\beta+\alpha-1|-\frac{q}{p'}} \left(\int\limits_{x_{k+1}}^{x_{k+2}} W^{p'(\beta+\alpha-1)}(s) u^{p'}(s) w(s) ds \right)^{\frac{q}{p'}}.$$

Substituting this estimate in (3.4) we obtain

$$J_1 \ll \sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \int\limits_{x_{k-1}}^{x_k} v(x)dx$$

$$\times \left(\int\limits_{x_{k+1}}^{x_{k+2}} u^{p'}(s) W^{p'(\beta+\alpha-1)} w(s)ds \right)^{\frac{q}{p'}}$$

$$\leq A^q_{\alpha,\beta} \sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \leq A^q_{\alpha,\beta} \left(\sum_k \int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}}$$

$$\ll A^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{3.5}$$

Now, we estimate J_2.

$$J_2=\sum_k\int\limits_{x_{k-1}}^{x_k}v(x)\left(\int\limits_{x_{k+1}}^{b}\frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\le\sum_k\int\limits_{x_{k-1}}^{x_k}v(x)\left(\int\limits_{x_{k+1}}^{b}\frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x_k))^{1-\alpha}}\right)^q dx$$

$$\le\sum_k\int\limits_{x_{k-1}}^{x_k}v(x)\left(\int\limits_{x_{k+1}}^{b}\frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-\frac{1}{2}W(x_{k+1}))^{1-\alpha}}\right)^q dx$$

$$\le 2^{q(1-\alpha)}\sum_k\int\limits_{x_{k-1}}^{x_k}v(x)\left(\int\limits_{x_{k+1}}^{b}\frac{u(s)W^\beta(s)f(s)w(s)}{(W(s))^{1-\alpha}}\right)^q dx$$

$$\ll\int\limits_a^b v(x)\left(\int\limits_x^b u(s)W^{\beta+\alpha-1}(s)f(s)w(s)ds\right)^q dx=\|H_{\alpha,\beta}f\|^q_{q,v}. \tag{3.6}$$

Then, by Theorem A

$$J_2\ll A^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{3.7}$$

Inequalities (3.3), (3.5) and (3.7) imply inequality (3.2). □

Theorem 3.2. *Let $0<\alpha<1$, $0<q<p<\infty$, $p>\frac{1}{\alpha}$ and $\beta\le 0$ ($\beta<\frac{1}{p}-\alpha$ in the case $W(b)=\infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $B_{\alpha,\beta}<\infty$. Moreover, $\|K_{\alpha,\beta}\|\approx B_{\alpha,\beta}$.*

Proof. Necessity and the estimate

$$\|K_{\alpha,\beta}\|\gg B_{\alpha,\beta} \tag{3.8}$$

follows by relation (2.2) and Theorem B.
Sufficiency. Let $B_{\alpha,\beta}<\infty$. If the inequality

$$\|K_{\alpha,\beta}f\|_{q,v}\ll B_{\alpha,\beta}\|f\|_{p,w}, \tag{3.9}$$

holds then by (3.8) and (3.9) we obtain $\|K_{\alpha,\beta}\|\approx B_{\alpha,\beta}$.

To prove (3.9) we use relation (3.3) of Theorem 3.1. Estimate for J_2 directly follows by (3.6) and Theorem B:

$$J_2\ll B^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{3.10}$$

By (3.5) we have

$$J_1 \ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \int_{x_{k-1}}^{x_k} v(x) dx \times \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q}{p'}}$$

$\Big($applying the Hölder inequality with the exponents $\frac{p}{q}$, $\frac{p}{p-q}\Big)$

$$\leq \left(\sum_k \left(\int_{x_{k-1}}^{x_k} v(x) dx \right)^{\frac{p}{p-q}} \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}} \times \left(\sum_k \int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \leq G \|f\|_{p,w}^q, \quad (3.11)$$

where

$$G = \left(\sum_k \left(\int_{x_{k-1}}^{x_k} v(x) dx \right)^{\frac{p}{p-q}} \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}.$$

Using the relation

$$\left(\int_{x_{k-1}}^{x_k} v(x) dx \right)^{\frac{p}{p-q}} = \frac{p}{p-q} \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k-1}}^{x} v(t) dt \right)^{\frac{q}{p-q}} dx$$

we estimate G:

$$G \ll \left(\sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k-1}}^{x} v(t) dt \right)^{\frac{q}{p-q}} dx \times \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\leq \left(\sum_k \int\limits_{x_{k-1}}^{x_k} \left(\int\limits_a^x v(t)dt \right)^{\frac{q}{p-q}} \right.$$

$$\left. \times \left(\int\limits_x^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} v(x)dx \right)^{\frac{p-q}{p}}$$

$$\leq B^q_{\alpha,\beta}. \tag{3.12}$$

By (3.11) and (3.12) it follows that

$$J_1 \ll B^q_{\alpha,\beta} \|f\|^q_{p,w}. \tag{3.13}$$

Therefore, by (3.3), (3.10) and (3.13) it follows that inequality (3.9) holds. □

4 The compactness of the operator $K_{\alpha,\beta}$

Theorem 4.1. *Let $0 < \alpha < 1$, $\frac{1}{\alpha} < p \leq q < \infty$ and $\beta \leq 0$ ($\beta < \frac{1}{p} - \alpha$ if $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $A_{\alpha,\beta} < \infty$ and*

$$\lim_{z \to a^+} A_{\alpha,\beta}(z) = \lim_{z \to b^-} A_{\alpha,\beta}(z) = 0,$$

where

$$A_{\alpha,\beta}(z) = \left(\int\limits_a^z v(x)dx \right)^{\frac{1}{q}} \left(\int\limits_z^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{1}{p'}}.$$

Proof. Necessity. Let the operator $K_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded and therefore, by Theorem 3.1, $A_{\alpha,\beta} < \infty$. First, we prove that $\lim\limits_{z \to b^-} A_{\alpha,\beta}(z) = 0$.

Let $F(t) = \int\limits_t^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds$. Since $A_{\alpha,\beta} < \infty$ and function u non-decreasing then $0 < F(t) < \infty$ for $t \in I$. Consider the family of functions $\{f_t\}_{t \in I}$, where

$$f_t(x) = \chi_{(t,b)}(x) u^{p'-1}(x) W^{(p'-1)(\beta+\alpha-1)}(x) (F(t))^{-\frac{1}{p}}. \tag{4.1}$$

Then

$$\int\limits_a^b |f_t(x)|^p w(x)dx = (F(t))^{-1} \int\limits_t^b u^{p'}(x) W^{p'(\beta+\alpha-1)}(x) w(x)dx \equiv 1. \tag{4.2}$$

We show that the family of functions $\{f_t\}$ weakly converges to zero in $L_{p,w}$. Let $g \in L_{p',w^{1-p'}} = (L_{p,w})^*$.

Applying the Holder inequality and using (4.2) we have

$$\int_a^b f_t(x)g(x)dx \le \left(\int_t^b |f_t(x)|^p w(x)dx\right)^{\frac{1}{p}} \left(\int_t^b |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_t^b |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}}.$$

Since $g \in L_{p',w^{1-p'}}$ then the last integral converges to zero as $t \to b$, which means the weak convergence to zero the family of function $\{f_t\}$. Then, by the compactness of the operator $K_{\alpha,\beta}$ from $L_{p,w}$ to $L_{q,v}$

$$\lim_{z\to b^-} \|K_{\alpha,\beta} f_t\|_{q,v} = 0. \tag{4.3}$$

We have

$$\|K_{\alpha,\beta} f_t\|_{q,v}^q = \int_a^b v(x) \left(\int_x^b \frac{u(s)W^\beta(s) f_t(s) w(s) ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\ge \int_a^t v(x) \left(\int_t^b \frac{u(s)W^\beta(s) f_t(s) w(s) ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\ge \int_a^t v(x)dx \left(\int_t^b u(s)W^{\beta+\alpha-1}(s) f_t(s) w(s) ds\right)^q$$

$$= (F(t))^{-\frac{q}{p}} \left(\int_t^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\right)^q \int_a^t v(x)dx = (A_{\alpha,\beta}(t))^q. \tag{4.4}$$

By (4.3) and (4.4) we obtain that $\lim_{t\to b^-} A_{\alpha,\beta}(t) = 0$.

Now, we show $\lim_{t\to a^+} A_{\alpha,\beta}(t) = 0$.

The compactness of the operator $K_{\alpha,\beta} : L_{p,w} \longrightarrow L_{q,v}$ implies the compactness of the adjoint operator

$$K^*_{\alpha,\beta} g(x) = u(s)W^\beta(s)w(s) \int_a^s \frac{g(x)dx}{(W(s)-W(x))^{1-\alpha}}$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$.

We introduce the family of functions $\{g_t\}_{t\in I}$, where

$$g_t(x) = \chi_{(a,t)}(x) \left(\int_a^t v(x)dx\right)^{-\frac{1}{q'}} v(x).$$

Since almost everywhere $v > 0$ and $A_{\alpha,\beta} < \infty$ then the function g_t is well defined.
In view of the equality

$$\int_a^b |g_t(x)|^{q'} v^{1-q'}(x)dx = \left(\int_a^t v(x)dx\right)^{-1} \left(\int_a^t v(x)dx\right) = 1$$

for $f \in L_{q,v} = (L_{q',v^{1-q'}})^*$ we have

$$\int_a^b f(x)g_t(x)dx \le \left(\int_a^t |f(x)|^q v(x)dx\right)^{\frac{1}{q}} \left(\int_a^t |g_t(x)|^{q'} v^{1-q'}(x)dx\right)^{\frac{1}{q'}}$$

$$= \left(\int_a^t |f(x)|^q v(x)dx\right)^{\frac{1}{q}}.$$

Consequently $\lim\limits_{t\to a^+} \int_a^b f(x)g_t(x)dx = 0$ for any $f \in L_{q,v}$, which means the weak convergence to zero the family of functions g_t. Then by the compactness of the operator $K^*_{\alpha,\beta}$ from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$

$$\lim_{t\to a^+} \|K^*_{\alpha,\beta} g_t\|_{p',w^{1-p'}} = 0. \tag{4.5}$$

We have

$$\|K^*_{\alpha,\beta} g_t\|^{p'}_{p',w^{1-p'}} \ge \int_t^b |u(s)W^\beta(s)w(s)|^{p'} \left(\int_a^t \frac{g_t(x)dx}{(W(s)-W(x))^{1-\alpha}}\right)^{p'} w^{1-p'}(s)ds$$

$$\ge \int_t^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds \left(\int_a^t v(x)dx\right)^{-\frac{p'}{q'}} \left(\int_a^t v(x)dx\right)^{p'} = (A_{\alpha,\beta}(t))^{p'}. \tag{4.6}$$

By (4.5) and (4.6) it follows that $\lim\limits_{t\to a^+} A_{\alpha,\beta}(t) = 0$. The necessity is proved.

Sufficiency. Let $A_{\alpha,\beta} < \infty$ and $\lim\limits_{z\to a^+} A_{\alpha,\beta}(z) = \lim\limits_{z\to b^-} A_{\alpha,\beta}(z) = 0$.

Yet for $a < c < d < b$

$$P_c f = \chi_{(a,c]} f, \quad P_{cd} f = \chi_{(c,d]} f, \quad Q_d f = \chi_{(d,b)} f.$$

Then $f = P_c f + P_{cd} f + Q_d f$ and by the equalities $P_{cd}K_{\alpha,\beta}Q_d \equiv 0$, $P_{cd}K_{\alpha,\beta}P_c \equiv 0$, $Q_d K_{\alpha,\beta} P_c \equiv 0$, we have

$$K_{\alpha,\beta} f = P_{cd}K_{\alpha,\beta}P_{cd} f + Q_d K_{\alpha,\beta} Q_d f + P_{cd} K_{\alpha,\beta} Q_d f + P_c K_{\alpha,\beta} f. \tag{4.7}$$

We show that the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}$ to $L_{q,v}$. Since $P_{cd}K_{\alpha,\beta}P_{cd} f(x) = 0$ when $x \in I\backslash(c,d]$ then it suffices to show that the operator

$P_{cd}K_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$ and this is equivalent to the compactness of the operator $Kf(x)=\int\limits_c^d K(x,s)f(s)ds$ with the kernel

$$K(x,s)=\frac{u(s)W^{\beta}(s)v^{\frac{1}{q}}(x)\chi_{(c,d)}(s-x)w^{\frac{1}{p'}}(s)}{(W(s)-W(x))^{1-\alpha}}$$

from L_p to L_q.

Let $\{x_k\}_{k\in\mathbb{Z}}$ be a sequence of points introduces in the proof of Theorem 3.1. There are the points x_{i-1}, x_n, $x_{i-1}<x_n$ such that $x_{i-1}\le c<x_i$, $x_{n-1}<d\le x_n$. We assume that the number c,d are chosen so that $x_i<x_{n-1}$. Similarly to obtaining estimates of J_1, J_2 in Theorem 3.1, we have

$$\int\limits_c^d\left(\int\limits_c^d|K(x,s)|^{p'}ds\right)^{\frac{q}{p'}}dx=\int\limits_c^d v(x)\left(\int\limits_x^d\frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}}dx$$

$$\le\sum_{k=i}^{n}\int\limits_{x_{k-1}}^{x_k}v(x)\left[\left(\int\limits_{x_{k+1}}^{b}+\int\limits_x^{x_{k+1}}\right)\frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{p'(1-\alpha)}}\right]^{\frac{q}{p'}}dx$$

$$\le\mu(n-i+1)A^q_{\alpha,\beta}<\infty,$$

where the constant μ does not depend on i,n. Therefore, on the basis of the theorem in Kantorovich and Akilov [5] (page 420), the operator K is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ from $L_{p,w}$ to $L_{q,v}$.

By (4.7) we have

$$\|K_{\alpha,\beta}-P_{cd}K_{\alpha,\beta}P_{cd}\|\le\|Q_dK_{\alpha,\beta}Q_d\|+\|P_{cd}K_{\alpha,\beta}Q_d\|+\|P_cK_{\alpha,\beta}\|. \tag{4.8}$$

We shall show that the right-hand side of (4.8) tends to zero as $c\to a^+$, $d\to b^-$. This will imply that the operator $K_{\alpha,\beta}$ being a uniform limit of compact operators, is compact from $L_{p,w}$ to $L_{q,v}$.

On the basis of Theorem 3.1, we have:

$$\|Q_dK_{\alpha,\beta}Q_df\|_{q,v}=\left(\int\limits_d^b v(x)\left(\int\limits_x^b\frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\ll\sup_{d<z<b}\left(\int\limits_d^z v(x)dx\right)^{\frac{1}{q}}\left(\int\limits_z^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\right)^{\frac{1}{p'}}\|f\|_{p,w}$$

$$\le\sup_{d<z<b}A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Hence

$$\lim_{d\to b^-}\|Q_dK_{\alpha,\beta}Q_df\|\ll\lim_{d\to b^-}\sup_{d<z<b}A_{\alpha,\beta}(z)=\lim_{z\to b^-}A_{\alpha,\beta}(z)=0; \tag{4.9}$$

Let $1 > \varepsilon > 0$. To estimate $\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v}$ we introduce the functions v_ε, u_ε defined by $v_\varepsilon(x) = v(x)$ for $x \in (a,d]$ and $v_\varepsilon(x) = \varepsilon^q v(x)$ for $x \in I \setminus (a,d]$, $u_\varepsilon(s) = u(s)$ for $s \in (d,b)$ and $u_\varepsilon(s) = \varepsilon u(s)$ for $s \in I \setminus (d,b)$. Obviously, the function u_ε is non-decreasing on I. Then by Theorem 3.1

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v} = \left(\int_c^d v(x)\left(\int_d^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b v_\varepsilon(x)\left(\int_x^b \frac{u_\varepsilon(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}} \ll A^\varepsilon_{\alpha,\beta}\|f\|_{p,w}, \tag{4.10}$$

where

$$A^\varepsilon_{\alpha,\beta} = \sup_{z\in I}\left(\int_a^z v_\varepsilon(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u_\varepsilon^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}.$$

We estimate $A^\varepsilon_{\alpha,\beta}$.

$$A^\varepsilon_{\alpha,\beta} \leq \sup_{a<z<d}\left(\int_a^z v(x)dx\right)^{\frac{1}{q}}\left(\varepsilon^{p'}\int_z^d u^{p'}(s)W^{p'\beta}(s)w(s)ds + \int_d^b u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}$$

$$+ \sup_{d<z<b}\left(\int_a^d v(x)dx + \varepsilon^q\int_d^z v(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}$$
$$\ll 2(\varepsilon A_{\alpha,\beta} + A_{\alpha,\beta}(d)).$$

Hence, by (4.10) we have

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v} \ll (\varepsilon A_{\alpha,\beta} + A_{\alpha,\beta}(d))\|f\|_{p,w}. \tag{4.11}$$

Where, due to the independence of the left-hand side of (28) of $\varepsilon > 0$, by letting $\varepsilon \to 0^+$, we obtain

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v} \ll A_{\alpha,\beta}(d)\|f\|_{p,\omega}.$$

Then

$$\lim_{d\to b^-}\|P_{cd}K_{\alpha,\beta}Q_d\| \ll \lim_{d\to b^-} A_{\alpha,\beta}(d) = 0. \tag{4.12}$$

Similarly, we obtain

$$\|P_c K_{\alpha,\beta}\|_{q,v} = \left(\int_a^c v(x)\left(\int_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{a<z<c} A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Therefore

$$\lim_{c\to a^+} \|P_c K_{\alpha,\beta} f\| \ll \lim_{c\to a^+} \sup_{a<z<c} A_{\alpha,\beta}(z) = \lim_{z\to a^+} A_{\alpha,\beta}(z) = 0. \quad (4.13)$$

By (4.8), (4.9), (4.12) and (4.13) it follows that $\lim\limits_{c\to a^+, d\to b^-} \|K_{\alpha,\beta} - P_{cd}K_{\alpha,\beta}P_{cd}\| = 0.$ □

Theorem 4.2. *Let $0<\alpha<1$, $p>\frac{1}{\alpha}$ and $\beta\le 0$ ($\beta<\frac{1}{p}-\alpha$ in the case $W(b)=\infty$). Let u be a non-decreasing function on I. If $b<\infty$ and $0<q<p<\infty$ or $a=0, b=\infty$ and $1<q<p<\infty$, then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $B_{\alpha,\beta}<\infty$.*

Proof. In the case $b<\infty$ and $0<q<p<\infty$ the statement of Theorem 4.2 follows by Ando Theorem and its generalizations [6]. Therefore, we prove Theorem 4.2 in the case $a=0, b=\infty$ and $1<q<p<\infty$.
Necessity. Let the operator $K_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded. Hence, by Theorem 3.2 $B_{\alpha,\beta}<\infty$.
Sufficiency. Let $B_{\alpha,\beta}<\infty$. Here $K_{\alpha,\beta}f = P_dK_{\alpha,\beta}P_df + P_dK_{\alpha,\beta}Q_dF + Q_dK_{\alpha,\beta}f$. Therefore

$$\|K_{\alpha,\beta} - P_dK_{\alpha,\beta}P_d\| \le \|P_dK_{\alpha,\beta}Q_d\| + \|Q_dK_{\alpha,\beta}\|. \quad (4.14)$$

Since $d<\infty$ then the operator $P_dK_{\alpha,\beta}P_d$ is compact from $L_{p,w}(0,d)$ to $L_{q,v}(0,d)$, which is equivalent to its compactness from $L_{p,w}$ to $L_{q,v}$. We show that the right-hand side of (4.14) tends to zero as $d\to\infty$. Then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ as the uniform limit of compact operators. On the basis of Theorem 3.2

$$\|Q_dK_{\alpha,\beta}\| \le \left(\int\limits_d^\infty \left(\int\limits_z^\infty u^{p'}(s)W^{p'(\alpha+\beta-1)}w(s)ds\right)^{\frac{q(p-1)}{p-q}} \times \left(\int_d^z v(x)dx\right)^{\frac{q}{p-q}} v(z)dz\right)^{\frac{(p-q)}{pq}}.$$

Hence, since $B_{\alpha,\beta}<\infty$, it follows that

$$\lim_{d\to\infty} \|Q_dK_{\alpha,\beta}\| = 0. \quad (4.15)$$

Let $1>\varepsilon>0$. To estimate $\|P_dK_{\alpha,\beta}Q_df\|$ we suppose as above, that $v_\varepsilon(x)=v(x)$ for $x\in(0,d]$ and $v_\varepsilon(x)=\varepsilon^q v(x)$ for $x\in(d,\infty)$, $u_\varepsilon(s)=u(s)$ for $s\in(d,\infty)$ and $u_\varepsilon(s)=\varepsilon u(s)$ for $s\in(0,d]$. Obviously, the function u_ε is non-decreasing on $I=(0,\infty)$. Now, by Theorem 3.2, estimating the norm $\|P_dK_{\alpha,\beta}Q_d\|$ as in (4.10), and then passing to the limit as $\varepsilon\to 0^+$, we obtain

$$\|P_dK_{\alpha,\beta}Q_d\| \ll \left(\int\limits_0^d v(x)dx\right)^{\frac{1}{q}} \left(\int\limits_d^\infty u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds\right)^{\frac{1}{p'}} = A_{\alpha,\beta}(d). \quad (4.16)$$

By Remark 1 $B_{\alpha,\beta} \approx \widetilde{B}_{\alpha,\beta}(0,\infty)$. Since $A_{\alpha,\beta}(d) \ll \widetilde{B}_{\alpha,\beta}(d,\infty)$ then by (4.16) it follows that $\lim\limits_{d\to\infty} \|P_d K_{\alpha,\beta} Q_d\| = 0$. Hence by (4.15) it follows that the right-hand side of (4.14) tends to zero as $d \to \infty$. □

5 Dual case

We consider the operator

$$T_{\alpha,\beta} f(x) = u(x) W^{\beta}(x) \int\limits_a^x \frac{v(s) f(s) ds}{(W(x) - W(s))^{1-\alpha}}$$

acting from $L_{p,v}$ to $L_{q,w}$.

Assume that

$$A^*_{\alpha,\beta}(z) = \left(\int\limits_a^z v(x) dx \right)^{\frac{1}{p'}} \left(\int\limits_z^b u^q(x) W^{q(\beta+\alpha-1)}(x) w(x) dx \right)^{\frac{1}{q}},$$

$$A^*_{\alpha,\beta} = \sup_{z\in I} A^*_{\alpha,\beta}(z).$$

Theorem 5.1. *Let $0 < \alpha < 1$, $1 < p \le q < \frac{1}{1-\alpha}$ and $\beta \le 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ in the case $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $T_{\alpha,\beta}$*

i) *is bounded from $L_{p,v}$ to $L_{q,w}$ if and only if $A^*_{\alpha,\beta}(z) < \infty$, moreover, $\|T_{\alpha,\beta}\| \approx A^*_{\alpha,\beta}$,*
ii) *is compact from $L_{p,v}$ to $L_{q,w}$ if and only if $A^*_{\alpha,\beta}(z) < \infty$ and*

$$\lim_{z\to a} A^*_{\alpha,\beta}(z) = \lim_{z\to b} A^*_{\alpha,\beta}(z) = 0.$$

Proof. The operator $T_{\alpha,\beta}$ acting from $L_{p,v}$ to $L_{q,w}$ is adjoint to the operator

$$\widetilde{K}_{\alpha,\beta} f(x) = v(x) \int\limits_x^b \frac{u(s) W^{\beta}(s) f(s) ds}{(W(x) - W(s))^{1-\alpha}}$$

acting from $L_{q',w^{1-q'}}$ to $L_{p',v^{1-p'}}$, which is equivalent to the action of the operator $K_{\alpha,\beta}$ from $L_{q',\omega}$ to $L_{p',v}$. Consequently, the operator $T_{\alpha,\beta}$ is bounded and compact from $L_{p,v}$ to $L_{q,\omega}$ if and only if the operator $K_{\alpha,\beta}$ is bounded and compact from $L_{q',\omega}$ to $L_{p',v}$ respectively. Since by the assumptions of Theorem 5.1 it follows that $\frac{1}{\alpha} < q' \le p' < \infty$ then on the basis of Theorems 3.1 and 4.1 the validity of the Statements i) and ii) of Theorem 5.1 follows. □

Similarly, on the basis of Theorem 4.2, we have

Theorem 5.2. *Yet $0 < \alpha < 1$, $1 < q < \min\{p, \frac{1}{1-\alpha}\}$, $p > 1$ and $\beta \leq 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ in the case $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $T_{\alpha,\beta}$ is bounded and compact from $L_{p,v}$ to $L_{q,w}$ if and only if $B^*_{\alpha,\beta}(z) < \infty$, where*

$$B^*_{\alpha,\beta} = \left(\int_a^b \left(\int_a^x v(x)dx \right)^{\frac{p(q-1)}{p-q}} \left(\int_x^b u^q(s) W^{q(\beta+\alpha-1)} w(s)ds \right)^{\frac{p}{p-q}} v(x)dx \right)^{\frac{p-q}{p}}.$$

6 Applications

We consider the weighted Weyl operator

$$\widetilde{I}^*_\alpha g(s) = \omega(s) \int_s^\infty \frac{\rho(x)g(x)dx}{(x-s)^{1-\alpha}}, \quad s > 0$$

and the weighted Riemann-Liouville operator

$$\widetilde{I}_\alpha f(x) = \rho(x) \int_0^x \frac{\omega(s)f(s)ds}{(x-s)^{1-\alpha}}, \quad x > 0$$

acting from L_p to L_q, where the weight functions ρ and ω are almost everywhere positive and locally integrable on $I = (0, \infty)$. The actions of the operator $K_{\alpha,\beta}$ from $L_{p,\omega}$ to $L_{q,v}$ and the operator $T_{\alpha,\beta}$ from $L_{p,v}$ to $L_{q,\omega}$ are equivalent to the actions of the operators

$$\widetilde{K}_{\alpha,\beta} g(s) = v^{\frac{1}{q}}(s) \int_s^b \frac{u(x)W^\beta(x) w^{\frac{1}{p'}}(x) g(x)dx}{(W(x) - W(s))^{1-\alpha}}, \tag{6.1}$$

$$\widetilde{T}_{\alpha,\beta} f(x) = w^{\frac{1}{q}}(x) u(x) W^\beta(x) \int_a^x \frac{v^{\frac{1}{p'}}(s) f(s)ds}{(W(x) - W(s))^{1-\alpha}}, \tag{6.2}$$

from L_p to L_q, respectively.

Let $\omega(s) = v^{\frac{1}{q}}(s)$ in (6.1) and $\omega(s) = v^{\frac{1}{p'}}(s)$ in (6.2). If $W(x) = x$, $a = 0, b = \infty$ and $\rho(x) = u(x)x^\beta$ then the operators (6.1) and (6.2) coincide with the operators I^*_α and I_α, respectively. Therefore, by Theorems 3.1- 4.2 we have

Corollary 6.1. *Let $0 < \alpha < 1$, $\beta < \frac{1}{p} - \alpha$ and $\rho(x) = u(x)x^\beta$, where u is a non-decreasing function on $I = (0, \infty)$. Then the operator $\widetilde{I}^*_\alpha$*

i) for $\frac{1}{\alpha} < p \leq q < \infty$ is bounded from L_p to L_q if and only if $\widetilde{A}_\alpha < \infty$, moreover, $\|\widetilde{I}^*_\alpha\| \approx \widetilde{A}_\alpha$, and is compact from L_p to L_q if and only if $\widetilde{A}_\alpha < \infty$ and $\lim\limits_{z\to 0^+} \widetilde{A}_\alpha(z) = \lim\limits_{z\to\infty} \widetilde{A}_\alpha(z) = 0$, where

$$\widetilde{A}_\alpha(z) = \left(\int_z^\infty \rho^{p'}(x) x^{p'(\alpha-1)} dx \right)^{\frac{1}{p'}} \left(\int_0^z \omega^q(s)ds \right)^{\frac{1}{q}}, \quad \widetilde{A}_\alpha = \sup_{z\in I} \widetilde{A}_\alpha(z);$$

Corollary 6.3. *Let $0 < \alpha < 1$, $\beta < 1 - \frac{1}{q} - \alpha$ and $\rho(x) = u(x)x^{\sigma\beta + \frac{\sigma-1}{q}}$, where u is a non-decreasing function on $I = (0, \infty)$. Then the operator $E_{\alpha,\gamma}$*

i) for $1 < p \le q < \frac{1}{1-\alpha}$ is bounded from L_p to L_q if and only if $A^{\circ}_{\alpha,\gamma} < \infty$, moreover, $\|E_{\alpha,\gamma}\| \approx A^{\circ}_{\alpha,\gamma}$ and compact from L_p to L_q if and only if $A^{\circ}_{\alpha,\gamma} < \infty$ and $\lim\limits_{z\to 0^+} A^{\circ}_{\alpha,\gamma}(z) = \lim\limits_{z\to\infty} A^{\circ}_{\alpha,\gamma}(z) = 0$, where $A^{\circ}_{\alpha,\gamma} = \sup\limits_{z\in I} A^{\circ}_{\alpha,\gamma}(z)$,

$$A^{\circ}_{\alpha,\gamma}(z) = \left(\int_z^{\infty} \rho^q(x) x^{q\sigma(\alpha-1)} dx \right)^{\frac{1}{q}} \left(\int_0^{z} \omega^{p'}(s) s^{p'(\sigma\gamma+\sigma-1)} ds \right)^{\frac{1}{p'}} .$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ is bounded (compact) from L_p to L_q if and only if $B^{\circ}_{\alpha,\gamma} < \infty$, moreover, $\|E_{\alpha,\gamma}\| \approx B^{\circ}_{\alpha,\gamma}$, where

$$B^{\circ}_{\alpha,\gamma} = \left(\int_0^{\infty} \left(\int_z^{\infty} \rho^q(x) x^{q\sigma(\alpha-1)} dx \right)^{\frac{p}{p-q}} \left(\int_0^{z} \omega^{p'}(s) s^{p'(\sigma\gamma+\sigma-1)} ds \right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z) z^{p'(\sigma\gamma+\sigma-1)} dz \right)^{\frac{p-q}{pq}} .$$

Corollary 6.4. *Let $a > 0, 0 < \alpha < 1$, $\beta \le 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ in the case $b = \infty$) and $\rho(x) = u(x)x^{-\frac{1}{q}}(ln\frac{x}{a})^{\beta}$, where u is non-decreasing function on $I = (a, b)$. Then the operator $\mathcal{H}_\alpha$*

i) for $1 < p \le q < \frac{1}{1-\alpha}$ is bounded from L_p to L_q if and only if $A^1_\alpha < \infty$, moreover, $\|H_\alpha\| \approx A^1_\alpha$, and compact from L_p to L_q if and only if $A^1_\alpha < \infty$ and $\lim\limits_{z\to a^+} A^1_\alpha(z) = \lim\limits_{z\to b^-} A^1_\alpha(z) = 0$, where $A^1_\alpha = \sup\limits_{z\in I} A^1_\alpha(z)$,

$$A^1_\alpha(z) = \left(\int_z^{b} \rho^q(x) \left(\ln\frac{x}{a} \right)^{q(\alpha-1)} dx \right)^{\frac{1}{q}} \left(\int_a^{z} \omega^{p'}(s) s^{-p'} ds \right)^{\frac{1}{p'}} ;$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ is bounded (compact) from L_p to L_q if and only if $B^1_\alpha < \infty$, moreover, $\|H_\alpha\| \approx B^1_\alpha$, where

$$B^1_\alpha = \left(\int_a^{b} \left(\int_z^{b} \rho^q(x) \left(ln\frac{x}{a} \right)^{q(\alpha-1)} dx \right)^{\frac{p}{p-q}} \left(\int_a^{z} \omega^{p'}(s) s^{-p'} ds \right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z) z^{-p'} dz \right)^{\frac{p-q}{pq}} .$$

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References

[1] A.M. Abylaeva, D. Kaskirbaeva, *Boundedness and compactness of fractional integration operator of Holmgren type in weighted Lebesgue spaces,* Evraziiskii Matematicheskii Zhurnal, 2 (2007), 75-86. (in Russian).

[2] K.F. Andersen, E.T. Sawyer, *Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators,* Trans. Amer. Math. Soc. 308 (1988), 547-558.

[3] D.E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and compact integral operators,* Kluwer Academic Publishers, Dordrecht 2002.

[4] S.M. Farsani, *On the boundedness and compactness of the fractional Riemann-Liouville operators,* Sibirsk. Mat. J. 54 (2013), no. 2, 468-479. (in Russian).

[5] L.V. Kantarovich, G.R. Akilov, *Functional analysis,* M .: Nauka 1977. (in Russian).

[6] M.A. Krasnosel'skii, P.P. Zabreiko, E.N. Pustilnik, P.E. Sobolevski, *Integral operators in spaces of summable functions,* Moscow: Nauka, 1966. (in Russian).

[7] A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy inequality - about its history and some related results,* University of West Bohemia, Plzen, 2007, 152 pp.

[8] A. Meskhi, *Solution of some weight problems for the Riemann-Liouville and Weyl operators,* Georgian Math. J. 106 (1989), 727-733.

[9] R. Oinarov, A.M. Abylaeva, *Criteria for the boundedness of a class of fractional integration,* Mathematical Journal. 4 (2004), no. 2(12), 5-14. (in Russian).

[10] D.V. Prokhorov, *On the boundedness and compactness of a class of integral operators,* J. London Math. Soc. 61 (2000), no. 2, 617-628.

[11] D.V. Prokhorov, V.D. Stepanov, *Weighted estimates for the Riemann-Liouville operators and applications,* Proc. Steklov Math. 248 (2003), 289-312 (in Russian).

[12] S.G. Samko, A.A. Kilbac, O.I. Marichev, *Integrals and derivatives of fractional order and some of their applications,* Minsk: Science and Technology, 1987, 688 pp. (in Russian).

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