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EMBEDDINGS AND WIDTHS OF WEIGHTED SOBOLEV CLASSES

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Abstract. In this paper, embedding theorems for reduced weighted Sobolev classes $\hat{W}^r_{p,g}(\Omega)$ in Lebesgue spaces $L_{q,v}(\Omega)$ are obtained. Here weight functions have singularity at the origin and $v \notin L_q(\Omega)$. For some special weight functions order estimates for Kolmogorov, Gelfand and linear widths are obtained.

Let $(X, \|\cdot\|_X)$ be a normed space, let X^* be its dual, and let $\mathcal{L}_n(X)$, $n \in \mathbb{Z}_+$, be the family of subspaces of X of dimension at most n. Denote by $L(X, Y)$ the space of continuous linear operators from X to a normed space Y. Also, by $\mathrm{rk}\, A$ denote the dimension of the image of an operator $A \in L(X, Y)$, and by $\|A\|_{X\to Y}$, its norm.

By the Kolmogorov n-width of a set $M \subset X$ in the space X, we mean the quantity

$$d_n(M, X) = \inf_{L\in\mathcal{L}_n(X)} \sup_{x\in M} \inf_{y\in L} \|x - y\|_X,$$

by the linear n-width, the quantity

$$\lambda_n(M, X) = \inf_{A\in L(X, X),\, \mathrm{rk} A\le n} \sup_{x\in M} \|x - Ax\|_X,$$

and by the Gelfand n-width, the quantity

$$d^n(M, X) = \inf_{x_1^*, \dots, x_n^* \in X^*} \sup\{\|x\| :\ x \in M,\ x_j^*(x) = 0,\ 1 \le j \le n\}.$$

The problem of estimating the widths of Sobolev classes in the Lebesgue space and finite-dimensional balls was studied in the 1960s–1980s (see, e.g., [3, 4, 5, 12, 13]). At the same time, the first results on estimates of n-widths of weighted Sobolev classes in weighted Lebesgue spaces were obtained [2, 14]. The extensive study of this problem began in the 1990s (for details, see [17]).

In [17] estimates for n-widths of the weighted Sobolev class $W^r_{p,g}(\Omega)$ in the weighted Lebesgue space $L_{q,v}(\Omega)$ were obtained, where $\Omega \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ was a John domain (see definitions below) and $0 \in \overline{\Omega}$. The weights were defined by

$$g(x) = |x|^{-\beta_g} |\ln|x||^{-\alpha_g} \rho_g(|\ln|x||), \quad v(x) = |x|^{-\beta_v} |\ln|x||^{-\alpha_v} \rho_v(|\ln|x||); \qquad (1)$$

here

$$\beta_g + \beta_v = r + \frac{d}{q} - \frac{d}{p} > 0, \quad \alpha_g + \alpha_v > \left(\frac{1}{q} - \frac{1}{p}\right)_+, \tag{2}$$

$\rho_g, \rho_v : (0, \infty) \to (0, \infty)$ were absolutely continuous functions such that

$$\lim_{y\to\infty} \frac{y\rho_g'(y)}{\rho_g(y)} = \lim_{y\to\infty} \frac{y\rho_v'(y)}{\rho_v(y)} = 0 \tag{3}$$

(the case $p = q$ was considered by Triebel [15] and Mieth [7], [8] as well). In addition, β_v satisfied the condition $\beta_v < \frac{d}{q}$. For $d = 1$, in [16] similar results were obtained for $\beta_v \in \mathbb{R} \backslash \left\{\frac{1}{q}, \frac{1}{q}+1, \ldots, \frac{1}{q}+r-1\right\}$; here estimates for norms of two-weighted Riemann – Liouville operators [11] and their compositions [6] were applied.

In this paper we present estimates of n-widths for $d \ge 2$, $\beta_v \ge \frac{d}{q}$, $\beta_v \notin \left\{\frac{d}{q}, \frac{d}{q}+1, \ldots, \frac{d}{q}+r-1\right\}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (an open connected set), and let $g, v : \Omega \to (0, \infty)$ be measurable functions. For each measurable vector-valued function $\psi : \Omega \to \mathbb{R}^l$, $\psi = (\psi_k)_{1\le k\le l}$, and for each $p \in [1, \infty]$, we put

$$\|\psi\|_{L_p(\Omega)} = \left\| \max_{1\le k\le l} |\psi_k| \right\|_p = \left(\int_\Omega \max_{1\le k\le l} |\psi_k(x)|^p \, dx \right)^{1/p}$$

(modifications in the case $p = \infty$ are clear). Let $\overline{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d$, $|\overline{\beta}| = \beta_1 + \ldots + \beta_d$. For any distribution f defined on Ω we write $\nabla^r f = \left(\partial^r f / \partial x^{\overline{\beta}}\right)_{|\overline{\beta}|=r}$ (here the partial derivatives are taken in the sense of distributions), and denote by $l_{r,d}$ the number of components of the vector-valued distribution $\nabla^r f$. We set

$$W^r_{p,g}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \middle| \, \exists \psi : \Omega \to \mathbb{R}^{l_{r,d}} : \|\psi\|_{L_p(\Omega)} \le 1, \, \nabla^r f = g \cdot \psi \right\}$$

$\Big($we denote the corresponding function ψ by $\dfrac{\nabla^r f}{g}\Big)$,

$$\|f\|_{L_{q,v}(\Omega)} = \|f\|_{q,v} = \|fv\|_{L_q(\Omega)}, \qquad L_{q,v}(\Omega) = \{ f : \Omega \to \mathbb{R} | \ \|f\|_{q,v} < \infty \}.$$

We call the set $W^r_{p,g}(\Omega)$ a weighted Sobolev class.

Denote by $AC[t_0, t_1]$ the space of absolutely continuous functions on an interval $[t_0, t_1]$. For $x \in \mathbb{R}^d$ and $a > 0$ we shall denote by $B_a(x)$ the closed Euclidean ball of radius a in $\mathbb{R}^d$ centered at the point x.

Definition 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $a > 0$. We say that $\Omega \in \mathbf{FC}(a)$ if there exists a point $x_* \in \Omega$ such that, for any $x \in \Omega$, there exist a number $T(x) > 0$ and a curve $\gamma_x : [0, T(x)] \to \Omega$ with the following properties:
1) $\gamma_x \in AC[0, T(x)]$, $\left|\frac{d\gamma_x(t)}{dt}\right| = 1$ a.e.,
2) $\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$,
3) $B_{at}(\gamma_x(t)) \subset \Omega$ for any $t \in [0, T(x)]$.

Definition 2. We say that Ω satisfies the John condition (and call Ω a John domain) if $\Omega \in \mathbf{FC}(a)$ for some $a > 0$.

In [9, 10], Reshetnyak found an integral representation for smooth functions on a John domain Ω in terms of their rth order derivatives. This integral representation implies that for $p > 1$, $1 \le q < \infty$ and $\frac{r}{d}+\frac{1}{q}-\frac{1}{p} \ge 0$ (correspondingly $\frac{r}{d}+\frac{1}{q}-\frac{1}{p} > 0$) the class $W^r_p(\Omega)$ is continuously (correspondingly, compactly) embedded in $L_q(\Omega)$; i.e., the conditions of a continuous and compact embedding are the same as for $\Omega = [0, 1]^d$).

Given nonnegative sequences $u = (u_j)_{j\ge 0}$, $w = (w_j)_{j\ge 0}$, $1 \le p, q \le \infty$, we denote by $\mathfrak{S}^{p,q}_{u,w}$ and $\tilde{\mathfrak{S}}^{p,q}_{u,w}$ the minimal constants C in the inequalities

$$\left(\sum_{j=0}^{\infty} w_j^q \left(\sum_{i=0}^{j} u_i f_i\right)^q\right)^{\frac{1}{q}} \le C \left(\sum_{j=0}^{\infty} |f_j|^p\right)^{\frac{1}{p}}, \quad (f_j)_{j=0}^{\infty} \in l_p, \quad f_j \ge 0,$$

and

$$\left(\sum_{j=0}^{\infty} w_j^q \left(\sum_{i=j}^{\infty} u_i f_i\right)^q\right)^{\frac{1}{q}} \le C \left(\sum_{j=0}^{\infty} |f_j|^p\right)^{\frac{1}{p}}, \quad (f_j)_{j=0}^{\infty} \in l_p, \quad f_j \ge 0,$$

correspondingly. Two-sided sharp estimates for $\mathfrak{S}^{p,q}_{u,w}$ and $\tilde{\mathfrak{S}}^{p,q}_{u,w}$ were obtained by Bennett [1].

We use the following notation for order inequalities. Let X, Y be sets, and let $f_1, f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \underset{y}{\lesssim} f_2(x, y)$ (or $f_2(x, y) \underset{y}{\gtrsim} f_1(x, y)$) if for any $y \in Y$ there exists $c(y) > 0$ such that $f_1(x, y) \le c(y) f_2(x, y)$ for any $x \in X$; $f_1(x, y) \underset{y}{\asymp} f_2(x, y)$ if $f_1(x, y) \underset{y}{\lesssim} f_2(x, y)$ and $f_2(x, y) \underset{y}{\lesssim} f_1(x, y)$.

Theorem A. [1]. *Let $1 < p \le \infty$, $1 \le q < \infty$, and let $u = \{u_n\}_{n\in\mathbb{Z}_+}$, $w = \{w_n\}_{n\in\mathbb{Z}_+}$ be nonnegative sequences. We set*

$$M_{u,w} := \sup_{m\in\mathbb{Z}_+} \left(\sum_{n=m}^{\infty} w_n^q\right)^{\frac{1}{q}} \left(\sum_{n=0}^{m} u_n^{p'}\right)^{\frac{1}{p'}} < \infty \quad \textit{if} \quad 1 < p \le q < \infty,$$

$$M_{u,w} := \left(\sum_{m=0}^{\infty} \left(\left(\sum_{n=m}^{\infty} w_n^q\right)^{\frac{1}{p}} \left(\sum_{n=0}^{m} u_n^{p'}\right)^{\frac{1}{p'}}\right)^{\frac{pq}{p-q}} w_m^q\right)^{\frac{1}{q}-\frac{1}{p}} < \infty$$
$$\textit{if} \quad 1 \le q < p \le \infty,$$

$$\tilde{M}_{u,w} := \sup_{m\in\mathbb{Z}_+} \left(\sum_{n=0}^{m} w_n^q\right)^{\frac{1}{q}} \left(\sum_{n=m}^{\infty} u_n^{p'}\right)^{\frac{1}{p'}} < \infty \quad \textit{if} \quad 1 < p \le q < \infty,$$

$$\tilde{M}_{u,w} := \left(\sum_{m=0}^{\infty} \left(\left(\sum_{n=m}^{\infty} u_n^{p'}\right)^{\frac{1}{q'}} \left(\sum_{n=0}^{m} w_n^q\right)^{\frac{1}{q}}\right)^{\frac{pq}{p-q}} u_m^{p'}\right)^{\frac{1}{q}-\frac{1}{p}} < \infty$$
$$\textit{if} \quad 1 \le q < p \le \infty.$$

Then $\mathfrak{S}^{p,q}_{u,w} \underset{p,q}{\asymp} M_{u,w}$, $\tilde{\mathfrak{S}}^{p,q}_{u,w} \underset{p,q}{\asymp} \tilde{M}_{u,w}$.

Let $a > 0$, $\Omega \in \mathbf{FC}(a)$, $0 \in \overline{\Omega}$. Consider the weight functions

$$g(x) = \varphi_g(|x|), \quad v(x) = \varphi_v(|x|), \tag{4}$$

where φ_g, $\varphi_v : (0, \infty) \to (0, \infty)$. Suppose that there exists $c_0 \geq 1$ such that

$$\frac{\varphi_g(t)}{\varphi_g(s)} \leq c_0, \quad \frac{\varphi_v(t)}{\varphi_v(s)} \leq c_0, \quad t, s \in [2^{-j-1}, 2^{-j+1}], \quad j \in \mathbb{Z}. \tag{5}$$

Let $r \in \mathbb{N}$, $1 < p \leq \infty$, $1 \leq q < \infty$. We set $\overline{u} = (\overline{u}_j)_{j=0}^{\infty}$, $\overline{w} = (\overline{w}_j)_{j=0}^{\infty}$,

$$\overline{u}_j = \varphi_g(2^{-j}) \cdot 2^{-\left(r - \frac{d}{p}\right)j}, \quad \overline{w}_j = \varphi_v(2^{-j}) \cdot 2^{-\frac{dj}{q}}.$$

Denote by $\mathcal{P}_{r-1}(\mathbb{R}^d)$ the space of polynomials on $\mathbb{R}^d$ of degree not exceeding $r-1$. For a measurable set $E \subset \mathbb{R}^d$ we put $\mathcal{P}_{r-1}(E) = \{f|_E : f \in \mathcal{P}_{r-1}(\mathbb{R}^d)\}$.

If $\mathfrak{S}^{p,q}_{\overline{u},\overline{w}} < \infty$, $r + \frac{d}{q} - \frac{d}{p} > 0$, then $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and there exists a linear continuous projection $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for each function $f \in W^r_{p,g}(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega)} \underset{p,q,r,d,a,c_0}{\lesssim} \mathfrak{S}^{p,q}_{\overline{u},\overline{w}} \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)}$$

(see [17, proofs of Lemmas 5, 6]). Observe that by Theorem A it follows that in this case $\sum\limits_{j=0}^{\infty} \overline{w}_j^q < \infty$; i.e., $v \in L_q(\Omega)$ (the last property follows from [17, formula (37), Corollary 1]).

In this paper we consider the case $v \notin L_q(\Omega)$. Since $W^r_{p,g}(\Omega) \supset \mathcal{P}_{r-1}(\Omega)$, the inclusion $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ fails. Therefore, instead of $W^r_{p,g}(\Omega)$ we consider the reduced Sobolev class $\hat{W}^r_{p,g}(\Omega)$.

Definition 3. We denote by $\hat{W}^r_{p,g}(\Omega)$ the completion of the set

$$W^r_{p,g}(\Omega) \cap \{f \in C^\infty(\Omega) : \exists \varepsilon > 0 : f|_{B_\varepsilon}(0) = 0\}$$

under the norm $\|f\|_{W^r_{p,g}(\Omega)} := \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)}$.

Proposition 1. *Let the functions φ_g, φ_v satisfy (5), and let $\sum\limits_{j=0}^{\infty} \varphi_v^q(2^{-j}) \cdot 2^{-jd} = \infty$. Then there exists a John domain Ω such that for any $n \in \mathbb{Z}_+$*

$$d_n(\hat{W}^r_{p,g}(\Omega), L_{q,v}(\Omega)) = \infty.$$

Therefore, there exists a John domain Ω such that for any finite-dimensional subspace $L \subset L_{q,v}(\Omega)$ and for any bounded subset $M \subset L_{q,v}(\Omega)$ we get $\hat{W}^r_{p,g}(\Omega) \not\subset L + M$; i.e., the class $\hat{W}^r_{p,g}(\Omega)$ cannot be imbedded continuously in $L_{q,v}(\Omega)$. For this reason we need to impose some more assumptions on domains Ω.

Definition 4. Let $b > 0$, $c \geq 1$. We say that $\Omega \in \mathbf{FC}'(b, c)$ if for any $x \in \Omega$ there exists a curve $\tilde{\gamma}_x : [0, \tilde{T}(x)] \to \Omega \cup \{0\}$ with the following properties:
1) $\tilde{\gamma}_x$ is absolutely continuous, $\left|\frac{d\tilde{\gamma}_x}{dt}\right| = 1$ a.e.,
2) $\tilde{\gamma}_x(0) = 0$, $\tilde{\gamma}_x(\tilde{T}(x)) = x$,
3) $c^{-1}|x| \leq \tilde{T}(x) \leq c|x|$,
4) $B_{b\cdot\min\{t, \tilde{T}(x)-t\}}(\tilde{\gamma}_x(t)) \subset \Omega$ for any $t \in (0, \tilde{T}(x))$.

Suppose that $\Omega \in \mathbf{FC}(a) \cap \mathbf{FC}'(\{0\}; b, c)$ and the weights $g, v : \Omega \to (0, \infty)$ satisfy (4), (5). Let $1 < p \leq \infty$, $1 \leq q < \infty$, $r \in \mathbb{N}$, $\delta := r + \frac{d}{q} - \frac{d}{p} > 0$.

Denote $\mathfrak{Z} = (p, q, r, d, a, b, c, c_0)$.

For each $0 \leq k \leq r-1$ we set $\overline{u}_{(k)} = \{\overline{u}_{k,j}\}_{j\in\mathbb{Z}_+}$, $\overline{w}_{(k)} = \{\overline{w}_{k,j}\}_{j\in\mathbb{Z}_+}$, where

$$\overline{u}_{k,j} = 2^{-j\left(r-k-\frac{d}{p}\right)}\varphi_g(2^{-j}), \quad \overline{w}_{k,j} = 2^{-j\left(k+\frac{d}{q}\right)}\varphi_v(2^{-j}).$$

Theorem 1. *Let $\tilde{\mathfrak{S}}^{p,q}_{\overline{u}_{(r-1)},\overline{w}_{(r-1)}} < \infty$. Then $\hat{W}^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and for any function $f \in \hat{W}^r_{p,g}(\Omega)$*

$$\|f\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}}{\lesssim} \tilde{\mathfrak{S}}^{p,q}_{\overline{u},\overline{w}} \left\|\frac{\nabla^r f}{g}\right\|_{L_p(\Omega)}.$$

Theorem 2. *Let $0 \leq k \leq r-2$, $\tilde{\mathfrak{S}}^{p,q}_{\overline{u}_{(k)},\overline{w}_{(k)}} < \infty$, $\mathfrak{S}^{p,q}_{\overline{u}_{(k+1)},\overline{w}_{(k+1)}} < \infty$. Then $\hat{W}^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and there exists a linear continuous projection $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any function $f \in \hat{W}^r_{p,g}(\Omega)$*

$$\|f - Pf\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}}{\lesssim} \max\{\tilde{\mathfrak{S}}^{p,q}_{\overline{u}_{(k)},\overline{w}_{(k)}}, \mathfrak{S}^{p,q}_{\overline{u}_{(k+1)},\overline{w}_{(k+1)}}\} \left\|\frac{\nabla^r f}{g}\right\|_{L_p(\Omega)}.$$

Let $\Omega \in \mathbf{FC}(a) \cap \mathbf{FC}'(\{0\}; b, c)$, $\overline{\Omega} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$, $R = \operatorname{diam}\Omega$, and let the weights g and v be defined by (1). We set $\mathfrak{Z}_* = (p, q, r, d, a, b, c, g, v, R)$. Denote $\alpha = \alpha_g + \alpha_v$, $\rho(y) = \rho_g(y)\rho_v(y)$.

In estimating the Kolmogorov, linear, and Gelfand widths we set, respectively, $\vartheta_l(M, X) = d_l(M, X)$ and $\hat{q} = q$, $\vartheta_l(M, X) = \lambda_l(M, X)$ and $\hat{q} = \min\{q, p'\}$, $\vartheta_l(M, X) = d^l(M, X)$ and $\hat{q} = p'$.

Theorem 3. *Let $r \in \mathbb{N}$, $1 < p \leq \infty$, $1 \leq q < \infty$, $\delta := r + \frac{d}{q} - \frac{d}{p} > 0$, $\Omega \in \mathbf{FC}(a) \cap \mathbf{FC}'(\{0\}; b, c)$. Suppose that (1), (2) and (3) hold, $\beta_v \in \mathbb{R}\setminus\left\{\frac{d}{q}, \frac{d}{q}+1, \dots, \frac{d}{q}+r-1\right\}$.*

1. *Let either $p \geq q$ or $p < q$, $\hat{q} \leq 2$. Suppose that $\alpha \neq \frac{\delta}{d}$. We set $\theta_1 = \frac{\delta}{d}$, $\theta_2 = \alpha$, $\sigma_1 = 0$, $\sigma_2 = 1$. Let $j_* \in \{1, 2\}$ be such that $\theta_{j_*} = \min\{\theta_1, \theta_2\}$. Then*

$$\vartheta_n(\hat{W}^r_{p,g}(\Omega), L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{\left(\frac{1}{q}-\frac{1}{p}\right)_+-\theta_{j_*}}\rho(n^{\sigma_{j_*}}).$$

2. *Let $p < q$, $\hat{q} > 2$. We set $\theta_1 = \frac{\delta}{d} + \min\left\{\frac{1}{2} - \frac{1}{\hat{q}}, \frac{1}{p} - \frac{1}{q}\right\}$, $\theta_2 = \frac{\hat{q}\delta}{2d}$, $\theta_3 = \alpha + \min\left\{\frac{1}{2} - \frac{1}{\hat{q}}, \frac{1}{p} - \frac{1}{q}\right\}$, $\theta_4 = \frac{\hat{q}\alpha}{2}$, $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = 1$, $\sigma_4 = \frac{\hat{q}}{2}$. Suppose that there exists $j_* \in \{1, 2, 3, 4\}$ such that $\theta_{j_*} < \min_{j\neq j_*}\theta_j$. Then*

$$\vartheta_n(\hat{W}^r_{p,g}(\Omega), L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\theta_{j_*}}\rho(n^{\sigma_{j_*}}).$$

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