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MULTIDIMENSIONAL VARIATIONAL FUNCTIONALS WITH SUBSMOOTH INTEGRANDS

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Abstract. In the present paper, we establish a base of investigation of multidimensional variational functionals having C^1-subsmooth or C^2-subsmooth integrands. First, an estimate of the first K-variation for the multidimensional variational functional having a C^1-subsmooth integrand is obtained and numerous partial cases are studied. Secondly, we have obtained C^1-subsmooth generalizations of the basic variational lemma and Euler–Ostrogradskii equation. Finally, for the C^2-subsmooth case, an estimate of the second K-variational is obtained and a series of the partial cases is studied as well.

1 Introduction

The subdifferentials are one of the main tools of modern nonsmooth and convex analysis. Starting from the classical Rockafellar subdifferential [19], many types of the subdifferentials were introduced aimed to research different extremal and other problems of analysis (see, e.g.,[1]-[3], [5], [16]-[18]). Last decade, by the first of the authors jointly with F.S. Stonyakin and Z.I. Khalilova, the so-called *compact subdifferentials* (or, *K-subdifferentials*) were introduced and studied in detail (see [8], [12]-[15], [20]). In particular, applications to the extremal one-dimensional variational problems, for the case so called *subsmooth integrand*, were investigated explicitly (see [8], [12], [13]).

So, the natural and opportune problem is the investigation of multidimensional extremal variational problems with subsmooth integrands. In the present paper, we establish a base of such investigation. First, an estimate of the first K-variation for the multidimensional variational functional having C^1-subsmooth integrand is obtained and numerous partial cases are studied. Secondly, we have obtained the C^1-subsmooth generalizations of the basic variational lemma and Euler–Ostrogradskii equation. Finally, for the C^2-subsmooth case, an estimate of the second K-variational is obtained, and a series of the partial cases is studied as well. Some of these problems in the smooth case were investigated in [11].

Let us recall the necessary definitions and facts (see [10]). In what follows, E, F are real Banach spaces, $U(y)$ is a neighborhood of a point $y \in E$, $\Phi : E \supset U(y) \to F$, $\{B_\delta\}_{\delta>0}$ is a decreasing as $\delta \searrow_{+0}$ system of closed convex subsets of F, $B = \bigcap\limits_{\delta>0} B_\delta$,

$\overline{co}$ is the closed convex hull of a set in F, F_K is the normalized cone of a non-empty convex compact $K \subset F$.

Definition 1. A set B is called the *K-limit* of the system $\{B_\delta\}_{\delta>0}$, briefly $B = K\text{-}\lim\limits_{\delta\to+0} B_\delta$, if
(i) $\forall\varepsilon > 0\ \exists\delta(\varepsilon) > 0 : (0 < \delta < \delta(\varepsilon)) \Rightarrow (B \subset B_\delta \subset B + U_\varepsilon(0))$;
and (ii) B is a nonempty compact set.

Thus, the K-limit is characterized by a uniform topological contraction of the system $\{B_\delta\}_{\delta>0}$ to its non-empty compact intersection. The concept of the K-limit is used in the following basic definition.

Definition 2. The *K-subdifferential* of a mapping Φ at a point $y \in E$ in the direction $h \in E$ is the following K-limit (if it exists):

$$\partial_K\Phi(y,h) = K - \lim_{\delta\to+0} \overline{co} \left\{ \frac{\Phi(y+th) - \Phi(y)}{t} \mid 0 < t < \delta \right\}. \tag{1.1}$$

We say that Φ *is weakly K–subdifferentiable* at y, if $\partial_K\Phi(y,h)$ exists for any $h \in E$ and it is a sublinear operator $\partial_K\Phi(y) : E \to F_K$. If, in addition, the operator $\partial_K\Phi(y)$ is bounded, then we say that Φ is *K-subdifferentiable by Gato* at the point y. Finally, if, in addition, the convergence in K-limit (1.1) is uniform with respect to all directions h satisfying $||h|| \le 1$, then we say that Φ is *K-subdifferentiable by Frechet* (or, *strongly K–subdifferentiable*) at the point y.

Definition 2 is easily generalized to a case of the normalized cone F. Properties of strongly K-subdifferentiable mappings are described in detail in [13]. Here we recall only a simple sufficient condition of K-subdifferentiability, which is related to the concept of *subsmoothness*.

Definition 3. A mapping $\Lambda : E \supset \dot{U}(y) \to F_K$ is called *subcontinuous* at a point $y \in E$ ($\Lambda \in C_{sub}(y)$), if

$$\exists\Lambda_y \in F_K\ \forall\varepsilon > 0\ \exists\delta > 0\ (0 < ||h|| < \delta) \Rightarrow (\Lambda(y+h) \subset \Lambda_y + U_\varepsilon(0)).$$

A mapping $\Phi : E \supset U(y) \to F$ is called *subsmooth* at a point $y \in E$ ($\Phi \in C^1_{sub}(y)$), if $\partial_K\Phi \in C_{sub}(y)$.

Theorem 1.1. *If $\Phi \in C^1_{sub}(y)$, then Φ is strongly K–subdifferentiable at the point y.*

Note also that in the case of functionals $f : \mathbb{R}^n \to \mathbb{R}$ the subsmoothness is equivalent to the following simple condition: *all upper derivatives $(\overline{\partial f/\partial y_i})$ are upper semicontinuous at the point y and all lower derivaties $(\underline{\partial f/\partial y_i})$ are lower semicontinuous at the point y.*

The concept of repeated K - subdifferentiability is introduced in the usual inductive way.

Definition 4. Let a mapping $\Phi : E \supset U(y) \to F$ be strongly K-subdifferentiable in $U(y)$. If the mapping $\partial_K\Phi : E \supset U(y) \to F_K$ is K-subdifferentiable at the point y, then set

$$\partial^2_K\Phi(y) := \partial_K(\partial_K\Phi)(y).$$

For the repeated K-subdifferentiability there is also a simple sufficient condition associated with the concept of C^2subsmoothness.

Definition 5. A mapping $\Phi : E \supset U(y) \to F$ is called C^2-subsmooth at a point y ($\Phi \in C^2_{sub}(y)$) if $\partial_K \Phi \in C^1_{sub}(y)$.

Theorem 1.2. *If $\Phi \in C^2_{sub}(y)$ then Φ is twice (strongly) K–subdifferentiable at the point y.*

Here also, in the case of functionals $f : \mathbb{R}^n \to \mathbb{R}$, the C^2-subsmoothness is equivalent to the simple condition: *all upper derivatives $\overline{\frac{\partial}{\partial y_i}}(\partial f/\partial y_i)$ are upper semicontinuous at the point y and all lower derivatives $\underline{\frac{\partial}{\partial y_i}}(\partial f/\partial y_i)$ are lower semicontinuous at the point y.*

2 K-subdifferential of the basic variational functional

Recently, in the works [7], [10] the following estimate of the K - subdifferential of a one-dimensional variational functional with subsmooth integrand

$$\Phi(y) = \int_a^b f(x, y, y')dx \quad (f \in C^1_{sub}([a; b] \times \mathbb{R}^2), y \in C^1[a; b])$$

was obtained:

$$\partial_K \Phi(y)h \subset \left[\int_a^b \left(\underline{\frac{\partial f}{\partial y}}(x, y, y')h + \underline{\frac{\partial f}{\partial z}}(x, y, y')h' \right) dx; \right.$$

$$\left. \int_a^b \left(\overline{\frac{\partial f}{\partial y}}(x, y, y')h + \overline{\frac{\partial f}{\partial z}}(x, y, y')h' \right) dx \right]. \tag{2.1}$$

In this section, we generalize estimate (2.1) to the case of a multidimensional variational functional with a subsmooth integrand:

$$\Phi(y) = \int_D f(x, y, \nabla y)dx \quad (f \in C^1_{sub}(D \times \mathbb{R}^{n+1}), y \in C^1(D)), \tag{2.2}$$

where D is a compact domain in $\mathbb{R}^n$ with Lipschitz boundary. Recall that in the case of a smooth integrand $f(x, y, z)$ the classical formula for the first variation of functional (2.2) has the form (see [6]):

$$\partial \Phi(y)h = \int_D \left(\frac{\partial f}{\partial y}(x, y, \nabla y)h + (\nabla_z f(x, y, \nabla y), \nabla h) \right) dx. \tag{2.3}$$

In the subsmooth case the estimate of the K-variation of functional (2.2) takes the following form.

Theorem 2.1. *Let the integrand f of variational functional* (2.2) *be* C^1–subsmooth*:* $f \in C^1_{sub}(D \times \mathbb{R}^{n+1})$ *(see Definition 1.3). Then Φ is strongly K–subdifferentiable everywhere in $C^1(D)$, and the following estimate holds:*

$$\partial_K \Phi(y)h \subset \left[\int\limits_D \left(\underline{\frac{\partial f}{\partial y}}(x, y, \nabla y)h + (\underline{\nabla_z f}(x, y, \nabla y), \nabla h) \right) dx; \right.$$

$$\left. \int\limits_D \left(\overline{\frac{\partial f}{\partial y}}(x, y, \nabla y)h + (\overline{\nabla_z f}(x, y, \nabla y), \nabla h) \right) dx \right]$$

$$(\forall h \in C^1(D)). \tag{2.4}$$

Proof. First introduce the auxiliary linear operator

$$(Ay)(x) = (x, y(x), \nabla y(x)), \;\; A : \; C^1(D) \longrightarrow \mathbb{R}^n \times C^1(D) \times C(D, \mathbb{R}^n).$$

Obviously, the operator A is continuous. Let us introduce also two auxiliary mappings, namely, the nonlinear composition operator

$$B_f(\widetilde{A})(y) = f(\widetilde{A}(y)), \;\; \widetilde{A} \in L(C^1(D); \mathbb{R}^n_x \times C^1(D) \times C(D, \mathbb{R}^n)),$$

$$B_f : L(C^1(D); \mathbb{R}^n_x \times C^1(D) \times C(D, \mathbb{R}^n)) \longrightarrow C(D),$$

and the linear integral functional

$$G(v) = \int\limits_D v(x) dx, \qquad G : C(D) \longrightarrow \mathbb{R}.$$

Then the variational functional Φ can be written as a composition

$$\Phi(y) = (G \circ B_f \circ A)(y). \tag{2.5}$$

Applying to composition (2.5) the theorem on K-subdifferentiation of compositions (see [10] Theorem 3.13, p. 91), we obtain

$$\partial_K \Phi(y, h) = \partial_K (G \circ B_f \circ A)(y)h \subset [\partial_K G(B_f \circ A(y)) \cdot [\partial^{yz}_K B_f(A(y)) \cdot \partial_K (A(y)]]h. \tag{2.6}$$

Now consider separately all the components in right-hand side of (2.6).

1) Since A is a linear continuous operator, then it is Frechet differentiable, and $A'(y) \equiv A$. Therefore,

$$\partial_K (Ay)(x) = (x, y(x), \nabla y(x)).$$

2) For the operator $B_f(u) = B_f((u_1, u_2, u_3))$ we calculate K–subdifferential in u_2, u_3. Applying Theorem 3.13 in [10], p. 91, and Corollary 3.3 in [10], p. 89, we obtain:

$$\partial^{yz}_K B_f(A(y))h \subset$$

$$\subset \left[\underline{\frac{\partial f}{\partial y}}(x, y, \nabla y)h + (\underline{\nabla_z f}(x, y, \nabla y), \nabla h); \overline{\frac{\partial f}{\partial y}}(x, y, \nabla y)h + (\overline{\nabla_z f}(x, y, \nabla y), \nabla h) \right].$$

3) Since G is a continuous linear functional, then it is Frechet differentiable, and $G'(v) \equiv G$.

Hence it follows that

$$\partial_K \Phi(y)h \subset \int_D \left[\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y)h + (\underline{\nabla_z f}(x,y,\nabla y), \nabla h); \right.$$

$$\left. \overline{\frac{\partial f}{\partial y}}(x,y,\nabla y)h + (\overline{\nabla_z f}(x,y,\nabla y), \nabla h) \right] dx$$

$$= \left[\int_D \left(\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y)h + (\underline{\nabla_z f}(x,y,\nabla y), \nabla h) \right) dx; \right.$$

$$\left. \int_D \left(\overline{\frac{\partial f}{\partial y}}(x,y,\nabla y)h + (\overline{\nabla_z f}(x,y,\nabla y), \nabla h) \right) dx \right]. \tag{2.7}$$

□

Let us consider a special case of the obtained estimate. First of all, select an obvious case when estimate (2.4) reduces to the classical equality (2.3).

Corollary 2.1. *If, under the conditions of Theorem 2.1,* $\frac{\partial f}{\partial y}(x,y,z)$ *and* $\nabla_z f(x,y,z)$ *exist for a. e.* $x \in D$ *and for all* y *and* z*, then there exists a classical first variation of the functional* Φ*, which is calculated by formula* (2.3).

Now note, a special case of the estimate (2.4), when the integrand f is formed by an external composition of a sub-smooth function with a smooth one.

Theorem 2.2. *Let*

$$\Phi(y) = \int_D \varphi\left[f(x,y,\nabla y)\right] dx \quad (\varphi \in C^1_{sub}(\mathbb{R}), f \in C^1(D \times \mathbb{R}^{n+1}), y \in C^1(D)). \tag{2.8}$$

Then the following estimate holds:

$$\partial_K \Phi(y)h \subset \left[\int_D \underline{\varphi'}\left(f(x,y,\nabla y)\right) \left(\frac{\partial f}{\partial y}(x,y,\nabla y)h + (\nabla_z f(x,y,\nabla y), \nabla h) \right) dx \right. ;$$

$$\left. \int_a^b \overline{\varphi'}\left(f(x,y,\nabla y)\right) \left(\frac{\partial f}{\partial y}(x,y,\nabla y)h + (\nabla_z f(x,y,\nabla y), \nabla h) \right) dx \right] \quad (\forall h \in C^1(D)). \tag{2.9}$$

Proof. According to formula (2.4) we have:

$$\partial_K\Phi(y)h \subset \Big[\int\limits_D \left(\underline{\frac{\partial}{\partial y}}\varphi(f(x,y,\nabla y))h + (\underline{\nabla z}\varphi(f(x,y,\nabla y)), \nabla h)\right)dx;$$

$$\int\limits_D \left(\frac{\overline{\partial}}{\partial y}\varphi(f(x,y,\nabla y))h + (\overline{\nabla z}\varphi(f(x,y,\nabla y)), \nabla h)\right)dx\Big]. \tag{2.10}$$

Here, using smoothness of f, we have:

$$\underline{\frac{\partial}{\partial y}}\varphi(f(x,y,\nabla y)) = \underline{\varphi}'(f(x,y,\nabla y)) \cdot \frac{\partial f}{\partial y}(x,y,\nabla y);$$

$$\underline{\frac{\partial}{\partial z}}\varphi(f(x,y,\nabla y)) = \underline{\varphi}'(f(x,y,\nabla y)) \cdot \nabla_z f(x,y,\nabla y);$$

$$\frac{\overline{\partial}}{\partial y}\varphi(f(x,y,\nabla y)) = \overline{\varphi}'(f(x,y,\nabla y)) \cdot \frac{\partial f}{\partial y}(x,y,\nabla y);$$

$$\frac{\overline{\partial}}{\partial z}\varphi(f(x,y,\nabla y)) = \overline{\varphi}'(f(x,y,\nabla y)) \cdot \nabla_z f(x,y,\nabla y). \tag{2.11}$$

Substituting (2.11) in (2.10), we arrive at estimate (2.9). □

Let us choose a special case when the variational functional (2.8) with subsmooth integrand possesses a classical variation.

Corollary 2.2. *If, under the conditions of Theorem 2.2, the function φ is differentiable a. e. on the set $\{f(x,y,\nabla y)|x \in D\}$, then the estimate (2.9) turns to the exact equality:*

$$\partial\Phi(y)h = \int\limits_D \varphi'\left(f(x,y,\nabla y)\right)\left(\frac{\partial f}{\partial y}(x,y,\nabla y)h + (\nabla_z f(x,y,\nabla y), \nabla h)\right)dx \quad (\forall h \in C^1(D)).$$

Note, as a special case, the case of an integrand formed by a composition of a smooth function and module.

Example 1. Let

$$\Phi(y) = \int\limits_D |f(x,y,\nabla y)|dx \quad (y \in C^1(D), f \in C^1(D \times \mathbb{R}^{n+1})). \tag{2.12}$$

Here, under the notation of Theorem 2.2, $\varphi(t) = |t|$, whence it follows

$$[\underline{\varphi}(t); \overline{\varphi}(t)] = \begin{cases} sign\ t,\ t \neq 0; \\ [-1;1],\ t = 0. \end{cases} \tag{2.13}$$

Substituting (2.13) in (2.9), after transformations, leads to the estimate

$$\partial_K\Phi(y)h \subset \int\limits_{(f\neq 0)} sign\, f \cdot \left(\frac{\partial f}{\partial y}(x,y,\nabla y)h + (\nabla_z f(x,y,\nabla y), \nabla h)\right)dx$$

$$+[-1;1]\cdot\int\limits_{(f=0)}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)h+(\nabla_z f(x,y,\nabla y),\nabla h)\right)dx\,. \qquad (2.14)$$

In particular, if $mes\,(f=0)=0$, then estimate (2.14) reduces to the exact equality:

$$\partial_K\Phi(y)h=\Phi'(y)h=\int\limits_D sign\,f\cdot\left(\frac{\partial f}{\partial y}(x,y,\nabla y)h+(\nabla_z f(x,y,\nabla y),\nabla h)\right)dx.$$

Next, we generalize the result of Theorem 2.2.

Theorem 2.3. *Let*

$$\Phi(y)=\int\limits_D f(x,y,\nabla y)\cdot\varphi(g(x,y,\nabla y))dx \quad (\varphi\in C^1_{sub}(\mathbb{R});f,g\in C^1(D\times\mathbb{R}^{n+1}),y\in C^1(D)). \qquad (2.15)$$

Then the following estimate is valid:

$$\partial_K\Phi(y)h\subset\int\limits_D\left(\frac{\partial f}{\partial y}\varphi(g(x,y,\nabla y))h+(\nabla_z f\cdot\varphi(g(x,y,\nabla y)),\nabla h)\right)dx$$

$$+\left[\int\limits_D f\cdot\underline{\varphi'}(g(x,y,\nabla y))\left(\frac{\partial g}{\partial y}h+(\nabla_z g,\nabla h)\right)dx;\right.$$

$$\left.\int\limits_D f\cdot\overline{\varphi'}(g(x,y,\nabla y))\left(\frac{\partial g}{\partial y}h+(\nabla_z g,\nabla h)\right)dx\right]. \qquad (2.16)$$

$$(\forall h\in C^1(D)).$$

Proof. According to formula (2.7) we obtain:

$$\partial_K\Phi(y)h\subset\left[\left(\int\limits_D\underline{\frac{\partial}{\partial y}}(f\cdot\varphi(g(x,y,\nabla y)))h+(\underline{\nabla_z}(f\cdot\varphi(g(x,y,\nabla y))),\nabla h)\right)dx;\right.$$

$$\left.\left(\int\limits_D\frac{\overline{\partial}}{\partial y}(f\cdot\varphi(g(x,y,\nabla y)))h+(\overline{\nabla_z}(f\cdot\varphi(g(x,y,\nabla y))),\nabla h)\right)dx\right]. \qquad (2.17)$$

In this case, using the smoothness of f,g, and sub-smoothness of φ we get the following equalities:

$$\underline{\frac{\partial}{\partial y}}(f(x,y,\nabla y)cdot\varphi(g(x,y,\nabla y)))=\frac{\partial f}{\partial y}(x,y,\nabla y)\cdot\varphi(g(x,y,\nabla y))$$

$$+f(x,y,\nabla y)\cdot\underline{\varphi'}(g(x,y,\nabla y))\frac{\partial g}{\partial y};$$

$$\underline{\nabla_z}(f\cdot\varphi(g(x,y,\nabla y)))=\nabla_z f\cdot\varphi(g(x,y,\nabla y))+f\cdot\underline{\varphi'}(g(x,y,\nabla y))\cdot\nabla_z g(x,y,\nabla y);$$

$$\frac{\overline{\partial}}{\partial y}(f \cdot \varphi(g(x,y,\nabla y))) = \frac{\partial f}{\partial y} \cdot \varphi(g(x,y,\nabla y)) + f \cdot \overline{\varphi'}(g(x,y,\nabla y))\frac{\partial g}{\partial y};$$

$$\overline{\nabla_z}(f \cdot \varphi(g(x,y,\nabla y))) = \nabla_z f \cdot \varphi(g(x,y,\nabla y)) + f \cdot \overline{\varphi'}(g(x,y,\nabla y)) \cdot \nabla_z g(x,y,\nabla y). \quad (2.18)$$

Substituting (2.18) in (2.17) leads to estimate (2.16). □

Note a special case when the variational functional (2.15) has a classical variation.

Corollary 2.3. *If the function φ is differentiable a.e. on the set $\{g(x,y,\nabla y)|x \in D\}$, then estimate (2.16) in Theorem 2.3 reduces to the exact equality:*

$$\partial\Phi(y)h = \int\limits_D \left(\frac{\partial f}{\partial y}\varphi(g(x,y,\nabla y))h + (\nabla_z f \cdot \varphi(g(x,y,\nabla y)), \nabla h)\right) dx$$

$$+ \int\limits_D f \cdot \varphi'(g(x,y,\nabla y)) \left(\frac{\partial g}{\partial y}h + (\nabla_z g, \nabla h)\right) dx \quad (\forall h \in C^1(D)).$$

Corollary 2.4. *Let*

$$\Phi(y) = \int\limits_D f \cdot |g(x,y,\nabla y)| dx \quad (f, g \in C^1(D \times \mathbb{R}^{n+1}), y \in C^1(D)).$$

Then the following estimate takes place:

$$\partial_K\Phi(y)h \subset \int\limits_D \left(\frac{\partial f}{\partial y}|g(x,y,\nabla y)|h + (\nabla_z f \cdot |g(x,y,\nabla y)|, \nabla h)\right) dx$$

$$+ \int\limits_{(g\neq 0)} f \cdot sign\, g(x,y,\nabla y)) \left(\frac{\partial g}{\partial y}h + (\nabla_z g, \nabla h)\right) dx$$

$$+ [-1;1] \cdot \int\limits_{(g=0)} f \cdot \left(\frac{\partial g}{\partial y}h + (\nabla_z g, \nabla h)\right) dx \quad (2.19)$$

$$(\forall h \in C^1(D)).$$

In particular, if $mes\,(g=0)=0$, then estimate (2.19) reduces to the exact equality:

$$\partial\Phi(y)h = \int\limits_D \left(\frac{\partial f}{\partial y}|g(x,y,\nabla y)|h + (\nabla_z f \cdot |g(x,y,\nabla y)|, \nabla h)\right) dx$$

$$+ \int\limits_D f \cdot sign\, g(x,y,\nabla y)) \left(\frac{\partial g}{\partial y}h + (\nabla_z g, \nabla h)\right) dx.$$

Corollary 2.5. *Let*

$$\Phi(y) = \int\limits_D f \cdot |f(x,y,\nabla y)| dx \quad (f \in C^1(D \times \mathbb{R}^{n+1}), y \in C^1(D)).$$

Then the following estimate takes place:

$$\begin{aligned} \partial_K \Phi(y)h \subset & \int\limits_D \left(\frac{\partial f}{\partial y} |f(x,y,\nabla y)| h + (\nabla_z f \cdot |f(x,y,\nabla y)|, \nabla h) \right) dx \\ & + \int\limits_{(f \neq 0)} |f(x,y,\nabla y)| \left(\frac{\partial f}{\partial y} h + (\nabla_z f, \nabla h) \right) dx \\ & + [-1;1] \cdot \int\limits_{(f=0)} f \cdot \left(\frac{\partial f}{\partial y} h + (\nabla_z f, \nabla h) \right) dx \qquad (2.20) \\ & (\forall h \in C^1(D)). \end{aligned}$$

In particular, if mes $(f=0)=0$, *then estimate (2.20) reduces to the exact equality:*

$$\begin{aligned} \partial \Phi(y)h = & \int\limits_D \left(\frac{\partial f}{\partial y} |f(x,y,\nabla y)| h + (\nabla_z f \cdot |f(x,y,\nabla y)|, \nabla h) \right) dx \\ & + \int\limits_D |f(x,y,\nabla y)| \left(\frac{\partial f}{\partial y} h + (\nabla_z f, \nabla h) \right) dx. \end{aligned}$$

Next, consider the case of an internal composition with a subsmooth function.

Theorem 2.4. *Let*

$$\Phi(y) = \int\limits_D f(\varphi(x,y,\nabla y)) dx \quad (y \in C^1(D), f \in C^1(D \times \mathbb{R}^{n+1}), \varphi \in C^1_{sub}(\mathbb{R})). \qquad (2.21)$$

Then the following estimate is valid:

$$\begin{aligned} \partial_K \Phi(y)h \subset & \left[\int\limits_D f'(\varphi(x,y,\nabla y)) \left(\underline{\frac{\partial \varphi}{\partial y}}(x,y,\nabla y)h + (\underline{\nabla_z} \varphi(x,y,\nabla y), \nabla h) \right) dx; \right. \\ & \left. \int\limits_D f'(\varphi(x,y,\nabla y)) \left(\overline{\frac{\partial \varphi}{\partial y}}(x,y,\nabla y)h + (\overline{\nabla_z} \varphi(x,y,\nabla y), \nabla h) \right) dx \right] \quad (h \in C^1(D)). \end{aligned} \qquad (2.22)$$

Proof. Using the smoothness of f and subsmoothness of φ we obtain:

$$\underline{\frac{\partial}{\partial y}} f(\varphi(x,y,\nabla y)) = f'(\varphi(x,y,\nabla y)) \underline{\frac{\partial \varphi}{\partial y}}(x,y,\nabla y);$$
$$\underline{\nabla z} f(\varphi(x,y,\nabla y)) = f'(\varphi(x,y,\nabla y)) \cdot \underline{\nabla z}\varphi(x,y,\nabla y);$$
$$\overline{\frac{\partial}{\partial y}} f(\varphi(x,y,\nabla y)) = f'(\varphi(x,y,\nabla y)) \overline{\frac{\partial \varphi}{\partial y}}(x,y,\nabla y);$$
$$\overline{\nabla z} f(\varphi(x,y,\nabla y)) = f'(\varphi(x,y,\nabla y)) \cdot \overline{\nabla z}\varphi(x,y,\nabla y). \tag{2.23}$$

Substituting (2.23) in (2.7) leads to estimate (2.22). □

Note a special cases when a variational functional (2.21) possesses a classical variation.

Corollary 2.6. *If the function φ is differentiable a.e. on the set $\{\varphi(x,y,\nabla y)|x \in D\}$, then estimate (2.22) in Theorem 2.4 reduces to the exact equality:*

$$\partial\Phi(y)h = \int\limits_D f'(\varphi(x,y,\nabla y)) \left(\frac{\partial \varphi}{\partial y}(x,y,\nabla y)h + (\nabla_z \varphi(x,y,\nabla y), \nabla h) \right) dx \quad (\forall h \in C^1(D)).$$

Example 2. Let

$$\Phi(y) = \int\limits_D f(x,y,||\nabla y||)dx \quad (y \in C^1(D), f \in C^1(D \times \mathbb{R}^{n+1})). \tag{2.24}$$

Here, under the notation of Theorem 2.4, $\varphi(x,y,z) = ||z||$, whence it follows

$$\nabla\varphi(z) = \frac{\overline{r}}{r}\ z \neq 0; \quad \underline{\nabla\varphi}(z) = -(1,\dots,1)\ z = 0; \quad \overline{\nabla\varphi}(z) = +(1,\dots,1)\ z = 0; \tag{2.25}$$

here $\overline{r}$ is the radius vector, $r = ||\overline{r}||$. Substituting (2.25) in (2.22), after transformations, leads to the estimate

$$\partial_K\Phi(y)h \subset \left(\int\limits_D \frac{\partial f}{\partial y}(x,y,||\nabla y||)hdx + \int\limits_{(\nabla y \neq 0)} \left(\frac{\overline{r}}{r} \cdot \frac{\partial f}{\partial z}(x,y,||\nabla y||), \nabla h \right) dx \right)$$
$$+[-1;1] \cdot \int\limits_{(\nabla y = 0)} \left(\frac{\partial f}{\partial z}(x,y,0), \nabla h \right) dx \quad (h \in C^1(D)). \tag{2.26}$$

In particular, if $mes\,(\nabla y = 0) = 0$, then estimate (2.26) reduces to the exact equality:

$$\partial_K\Phi(y)h = \Phi'(y)h = \int\limits_D \left[\frac{\partial f}{\partial y}(x,y,||\nabla y||)h + \left(\frac{\overline{r}}{r} \cdot \frac{\partial f}{\partial z}(x,y,||\nabla y||), \nabla h \right) \right] dx.$$

Corollary 2.7. *If the functions φ_0 and φ_1 are differentiable almost everywhere on $\nabla y(D)$, $\varphi(x,y,z) = (x, \varphi_0(y), \varphi_1(z))$, then*

$$\partial\Phi(y)h = \int\limits_D f'(x,\varphi_0(y),\varphi_1(\nabla y))\,(\varphi_0'(y)h; (\nabla_z \varphi_1(\nabla y), \nabla h))\,dx \quad (\forall h \in C^1(D)).$$

3 K–analogues of the basic variational lemma and Euler–Ostrogradskii equation in a domain

With a view to obtain K-analogue of the Euler–Lagrange equation in the one-dimensional subsmooth case, the following K-analogue of the basic variational lemma was stated in [10] (Theorem 5.4, p. 116).

Theorem 3.1. *Let $\varphi_1, \varphi_2 \in L_2[a;b]$. If the inclusion:*

$$0 \in \left[\int_a^b \varphi_1(x)h(x)dx; \int_a^b \varphi_2(x)h(x)dx\right],$$

holds for any $h \in C^1[a;b]$, then $0 \in [\varphi_1;\varphi_2] \subset L_2[a;b]$.

Here we generalize this result to the multidimensional case.

Theorem 3.2. *Let D be a compact domain in $\mathbb{R}^n$, $\varphi_1, \varphi_2 \in L_2(D)$. If $\forall h \in C^1(D)$ the inclusion*

$$0 \in \left[\int_D \varphi_1(x)h(x)dx; \int_D \varphi_2(x)h(x)dx\right] \tag{3.1}$$

is satisfied, then $0 \in [\varphi_1;\varphi_2] \subset L_2(D)$.

Proof. Following the scheme of the proof of Theorem 3.1 (see [10], Theorem 5.4, p. 116), we represent an arbitrary element $\varphi \in [\varphi_1;\varphi_2]$ in the form $\varphi = (1-t)\varphi_1 + t\varphi_2 = \varphi_1 + t(\varphi_2 - \varphi_1)$, where $0 \le t \le 1$. Let $H = L_2(D)$. First, suppose that (3.1) holds for any $h \in L_2(D)$.

1. Denote by $H^1 = \{\varphi_2 - \varphi_1\}^\perp$. Then for any $h \in H^1$ it follows that $(\varphi_2 - \varphi_1, h) = 0$, i.e. $\int_D \varphi_2 h = \int_D \varphi_1 h$. Suppose that φ_1 is not collinear to $(\varphi_2 - \varphi_1)$. Then there exists $h_0 \in H$ such that φ_1 is not orthogonal to h_0, i.e. $(\varphi_1, h_0) \ne 0$. Therefore, $(\varphi_2 - \varphi_1, h_0) = 0$, $(\varphi_1, h_0) \ne 0$, whence it follows $(\varphi_2, h_0) - (\varphi_1, h_0) = 0$, $(\varphi_1, h_0) \ne 0$. But this is possible if and only if $(\varphi_2, h_0) = (\varphi_1, h_0) \ne 0$. Hence for any $t \in [0;1]$ we get:

$$\int_D \big((1-t)\varphi_1 + t\varphi_2\big)h_0 dx = \int_D \big(\varphi_1 h_0 + t(\varphi_2 - \varphi_1)\big)h_0 dx = \int_D \varphi_1 h_0 dx + t\int_D (\varphi_2 - \varphi_1)h_0 dx \ne 0.$$

Thus, there exists h_0 such that $0 \overline{\in} \int_D [\varphi_1;\varphi_2]h_0 dx$, which contradicts the condition of the theorem. Hence, φ_1 is collinear to $(\varphi_2 - \varphi_1)$, i.e. the whole of segment $[\varphi_1;\varphi_2]$ consists of collinear functions: $[\varphi_1;\varphi_2] = \{\lambda\varphi_3\}_{\lambda_1 \le \lambda \le \lambda_2}$. So, $\int_D [\varphi_1;\varphi_2]hdx = \{\lambda \int_D \varphi_3 hdx\}_{\lambda_1 \le \lambda \le \lambda_2}$. Therefore, the condition $\left(0 \in \int_D [\varphi_1;\varphi_2]hdx \;\forall\, h\right)$ is satisfied if and only if the inclusion

$$0 \in \{\lambda \int_D \varphi_3 hdx\}_{\lambda_1 \le \lambda \le \lambda_2}$$

holds for any $h \in L_2(D)$.

Let us consider two possible cases:

a) $0 \in [\lambda_1; \lambda_2]$. Then $0 \in [\varphi_1; \varphi_2] = \{\lambda\varphi_3\}$.

b) $0 \overline{\in} [\lambda_1; \lambda_2]$. Then $\int\limits_D \varphi_3 h dx = 0 \ \forall h, \in L_2(D)$ whence it follows $(\varphi_3, h) = 0 \ \forall h \in L_2(D) \Longrightarrow$
$\Longrightarrow \varphi_3 = 0 \Longleftrightarrow [\varphi_1; \varphi_2] = \{0\}$.

Thus, the statement of Theorem in this case is proved.

2. Now consider the case in which $h \in C^1(D)$, i.e. $0 \in \int\limits_D [\varphi_1; \varphi_2] h dx$ for any $h \in C^1(D)$. Because $C^1(D)$ is continuously and densely embedded into $L_2(D)$ it easily follows that $0 \in \int\limits_D [\varphi_1; \varphi_2] h dx$ for any $h \in L_2(D)$. □

Recall now the classical Euler–Ostrogradskii equation [4]. For the variational functional

$$\Phi(y) = \int\limits_D f(x, y, \nabla y) dx \quad (f \in C^1(D \times \mathbb{R}^{n+1}), y \in C^1(D), f\ |_{\partial\mathcal{D}} = f_0)$$

the condition $\Phi'(y) = 0$ is equivalent to the equation:

$$\frac{\partial f}{\partial y}(x, y, \nabla y) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial z_i}(x, y, \nabla y) \right) = 0. \tag{3.2}$$

In particular, if Φ attains a local extremum at the point y, then equation (3.2) holds. Theorem 3.2, together with estimate (2.4) for the K-subdifferential of Φ, enables us to generalize condition (3.2) to the case of a C^1*–subsmooth integrand*; the result takes form of an estimate. In the one-dimensional case such a generalization was obtained in the works [10], [12].

Theorem 3.3. *Let D be a compact domain in $\mathbb{R}^n$ with a Lipschitz boundary ∂D,*

$$\Phi(y) = \int\limits_D f(x, y, \nabla y) dx \quad (f \in C^1_{sub}(D \times \mathbb{R}^{n+1}), f\ |_{\partial\mathcal{D}} = 0, y \in C^1(D)). \tag{3.3}$$

Then the condition $0 \in \partial_K \Phi(y)$ is equivalent to the "Euler–Ostrogradskii inclusion"*:*

$$0 \in \left[\underline{\frac{\partial f}{\partial y}}(x, y, \nabla y) - div\left(\underline{\nabla_z f}(x, y, \nabla y)\right);\ \overline{\frac{\partial f}{\partial y}}(x, y, \nabla y) - div\left(\overline{\nabla_z f}(x, y, \nabla y)\right) \right] \tag{3.4}$$

almost everywhere in D, or in the coordinate form:

$$0 \in \left[\underline{\frac{\partial f}{\partial y}}(x, y, \nabla y) - \left(\sum_{i=1}^{n} \underline{\frac{\partial f}{\partial z_i}}(x, y, \nabla y) \cdot \frac{\partial y}{\partial x_i}(x, y, \nabla y) \right); \right.$$
$$\left. \overline{\frac{\partial f}{\partial y}}(x, y, \nabla y) - \left(\sum_{i=1}^{n} \overline{\frac{\partial f}{\partial z_i}}(x, y, \nabla y) \cdot \frac{\partial y}{\partial x_i}(x, y, \nabla y) \right) \right] \tag{3.5}$$

almost everywhere in D. In particular, if Φ attains a local extremum at a point y, then inclusion (3.5) *is satisfied almost everywhere in D.*

Proof. By K–Fermat's Lemma ([9], Theorem 3.7.28, p 103); $0 \in \partial_K \Phi(y)h$ ($\forall\, h \in C^1(D),\ h\,|_{\partial\mathcal{D}} = 0$), that is,

$$0 \in \partial_K \Phi(y)h \subset \Big[\int\limits_D \Big(\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y)h + (\underline{\nabla_z f}(x,y,\nabla y), \nabla h) \Big) dx;$$

$$\int\limits_D \Big(\overline{\frac{\partial f}{\partial y}}(x,y,\nabla y)h + (\overline{\nabla_z f}(x,y,\nabla y), \nabla h) \Big) dx \Big]$$

$$= \Big\{ \int\limits_D \Big[\Big((1-t)\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y) + t \cdot \overline{\frac{\partial f}{\partial y}}(x,y,\nabla y) \Big) \cdot h$$

$$+ \Big((1-t) \cdot (\underline{\nabla_z f}(x,y,\nabla y), \nabla h) + t \cdot (\overline{\nabla_z f}(x,y,\nabla y), \nabla h) \Big) \Big] dx \,\Big|\, 0 \le t \le 1 \Big\}$$

$$=: \Big\{ I_1(t) + I_2(t) \,|\, 0 \le t \le 1 \Big\}. \tag{3.6}$$

Applying to $I_2(t)$ in (3.6) the Green's formula, we obtain:

$$I_2(t) = \left| \begin{array}{c} u = (1-t)\underline{\nabla_z f}(x,y,\nabla y) + t \cdot \overline{\nabla_z f}(x,y,\nabla y); \\ dv = \nabla h dx,\ v = h \\ du = (1-t) \cdot div\Big(\underline{\nabla_z f}(x,y,\nabla y)\Big) + t \cdot div\Big(\overline{\nabla_z f}(x,y,\nabla y)\Big) dx \end{array} \right|$$

$$= \oint_{\partial D} \Big((1-t) \cdot \underline{\nabla_z f}(x,y,\nabla y) + t \cdot \overline{\nabla_z f}(x,y,\nabla y) \Big) \cdot h dx$$

$$- \int\limits_D \Big[(1-t) \cdot div\Big(\underline{\nabla_z f}(x,y,\nabla y)\Big) + t \cdot div\Big(\overline{\nabla_z f}(x,y,\nabla y)\Big) \Big] \cdot h dx. \tag{3.7}$$

Hence, substituting of (3.7) into (3.6) leads to:

$$0 \in \Big\{ \int\limits_D \Big[(1-t) \Big(\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y) - div\Big(\underline{\nabla_z f}(x,y,\nabla y)\Big)$$

$$+ t \cdot \Big(\overline{\frac{\partial f}{\partial y}}(x,y,\nabla y) - div\Big(\overline{\nabla_z f}(x,y,\nabla y)\Big)\Big)\Big] \cdot h dx \Big|\ 0 \le t \le 1 \Big\}$$

$$= \Big[\int\limits_D \Big[\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y) - div\Big(\underline{\nabla_z f}(x,y,\nabla y)\Big) \Big] \cdot h dx;$$

$$\int\limits_D \Big[\overline{\frac{\partial f}{\partial y}}(x,y,\nabla y) - div\Big(\overline{\nabla_z f}(x,y,\nabla y)\Big) \Big] \cdot h dx \Big]. \tag{3.8}$$

By basic Lemma (Theorem 3.2), it follows from (3.8) the inclusion (3.4). □

Any solution of inclusion (3.4) will be called a *subextremal*.

Remark 1. *The Euler–Ostrogradskii inclusion* (3.4) *can be rewritten in the form of the "Euler–Ostrogradskii equation with a parameter":*

$$\left[(1-t)\underline{\frac{\partial f}{\partial y}}(x,y,\nabla y)+t\cdot\overline{\frac{\partial f}{\partial y}}(x,y,\nabla y)\right]$$

$$-div\left[(1-t)\underline{\nabla_z f}(x,y,\nabla y)+t\cdot\overline{\nabla_z f}(x,y,\nabla y)\right]\overset{a.e.}{=}0.$$

A subextremal $y(\cdot)$ is a solution of this equation for some $t\in[0;1]$.

Consider, as an important special case, the case of modulated integrand in Example 1.

Theorem 3.4. *Let*

$$\Phi(y)=\int_D|f(x,y,\nabla y)|dx\quad(f\in C^1(D\times\mathbb{R}^{n+1}),y\in C^1(D),y\mid_{\partial\mathcal{D}}=y_0).\tag{3.9}$$

For functional (3.9) *the Euler–Ostrogradskii inclusion takes the form of* alternative*:*

$$\left[\begin{array}{l} either\ \dfrac{\partial f}{\partial y}(x,y,\nabla y)-\displaystyle\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}(x,y,\nabla y)\right)=0\ (as\ f(x,y,\nabla y)\neq 0);\\ \qquad or\ f(x,y,\nabla y)=0\ (without\ any\ additional\ conditions).\end{array}\right.\tag{3.10}$$

In particular, if $mes(f(x,y,\nabla y)=0)$, we come to the usual Euler–Ostrogradskii equation for y (almost everywhere).

Proof. Denote by $\varphi(x,y,z)=|f(x,y,z)|$. Using the result of Example 1, we get:

$$\partial_K^y\varphi=\left\{\begin{array}{c}-\dfrac{\partial f}{\partial y},\ f(x,y,z)<0;\quad \dfrac{\partial f}{\partial y},\ f(x,y,z)>0\\ (2\lambda-1)\dfrac{\partial f}{\partial y},\ f(x,y,z)=0,\quad 0\le\lambda\le1\end{array}\right\},$$

$$\partial_K^z\varphi=\left\{\begin{array}{ll}-\nabla_z f,\ f(x,y,z)<0; & \nabla_z f,\ f(x,y,z)>0\\ (2\mu-1)\nabla_z f,\ f(x,y,z)=0,\quad 0\le\mu\le1 & \end{array}\right\}.$$

Hence, we find subLagrangian:

$$L_K(\varphi)(y)\overset{\text{a.e.}}{=}\left\{\begin{array}{c}-\left(\dfrac{\partial f}{\partial y}-\displaystyle\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}\right)\right),\ f(x,y,z)<0\quad\left(\dfrac{\partial f}{\partial y}-\displaystyle\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}\right)\right),\ f(x,y,z)>0\\ (2\lambda-1)\dfrac{\partial f}{\partial y}-(2\mu-1)\displaystyle\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}\right),\ f(x,y,z)=0,\quad 0\le\lambda\le1;0\le\mu\le1\end{array}\right\}.$$

Thus, the Euler–Ostrogradskii inclusion takes form:

$$\left[\begin{array}{l} L(f)(y)=\dfrac{\partial f}{\partial y}-\displaystyle\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}\right)=0,\ f(x,y,z)\neq0;\\ \left\{L_{\alpha\beta}(f)(y)=\alpha\cdot\dfrac{\partial f}{\partial y}-\beta\cdot\displaystyle\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial z_i}\right)=0,\ -1\le\alpha,\beta\le1\right\},\quad f(x,y,z)=0.\end{array}\right.$$

In particular, if $\alpha = \beta = 0$, then the equality $L_{00}(f)(y) \equiv 0$ is identically satisfied. Hence the Euler–Ostrogradskii inclusion (3.10) is also identically satisfied. Thus, in the case under consideration the Euler–Ostrogradskii equation reduces to the following alternative:

$$\left[\begin{array}{ll} \text{either} & \dfrac{\partial f}{\partial y} - \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial f}{\partial z_i}\right) = 0, \text{ if } f(x,y,z) \neq 0; \\ \text{or} & f(x,y,z) = 0 \text{ (without any additional conditions).} \end{array}\right. \tag{3.11}$$

□

Consider a concrete example.

Example 3. Let

$$\Phi(y) = \iiint_D \left| \left(\frac{\partial y}{\partial x_1}\right)^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 + \left(\frac{\partial y}{\partial x_3}\right)^2 - y^2 \right| dx_1 dx_2 dx_3. \tag{3.12}$$

Here condition (3.11) takes form:

$$\left[\begin{array}{r} \text{either} \quad \triangle y + y = 0, \text{ as } (y_{x_1})^2 + (y_{x_2})^2 + (y_{x_3})^2 \neq y^2 \\ \text{or} \quad (y_{x_1})^2 + (y_{x_2})^2 + (y_{x_3})^2 = y^2, \text{ as } (y_{x_1})^2 + (y_{x_2})^2 + (y_{x_3})^2 = y^2 \end{array}\right. . \tag{3.13}$$

The first of the equations in (3.13) is the Helmholtz equation with the parameter $c = 1$. The solution of this equation in spherical coordinates, as is well known, has the form:

$$y = A_1 \cos r + A_2 \sin r.$$

The second equation in (3.13) according to the well known classification is an equation of elliptic type and has an analytic solution.

Concluding this section, let us return to the variational problem with a norm in the integrand (see. Example 2).

Theorem 3.5. *Let*

$$\Phi(y) = \int_D f(x, y, ||\nabla y||) dx \quad (f \in C^1(D \times \mathbb{R}^2), y \in C^1(D), y\,|_{\partial \mathcal{D}} = y_0). \tag{3.14}$$

For functional (3.14) *the Euler–Ostrogradskii inclusion takes the form of the following alternative:*

$$\left[\begin{array}{l} \textit{either } \dfrac{\partial f}{\partial y}(x,y,||\nabla y||) - \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(\dfrac{\overline{r}}{r} \cdot \dfrac{\partial f}{\partial z_i}(x,y,||\nabla y||)\right) \ (\textit{as } \nabla y \neq 0); \\ \qquad \textit{or } 0 \in \dfrac{\partial f}{\partial y}(x,y,0) + [-1;1] \cdot \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial f}{\partial z_i}(x,y,0)\right) \ (\textit{as } \nabla y = 0). \end{array}\right.$$

In particular, if $mes(\nabla y = 0) = 0$, *we arrive to the equation*

$$\frac{\partial f}{\partial y}(x,y,||\nabla y||) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\overline{r}}{r} \cdot \frac{\partial f}{\partial z_i}(x,y,||\nabla y||)\right) = 0 \quad (a.\ e.\ in\ D) \tag{3.15}$$

Proof. For simplicity, consider only the case $mes(\nabla y = 0) = 0$. In this case, the application of Green's formula (integration by parts) in (2.26) (Example 2) leads to the equation

$$0 = \int\limits_D \left[\frac{\partial f}{\partial y}(x, y, ||\nabla y||) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\overline{r}}{r} \cdot \frac{\partial f}{\partial z_i}(x, y, ||\nabla y||) \right) \right] hdx$$

$$= \int\limits_D \left[\frac{\partial f}{\partial y}(x, y, ||\nabla y||) - \frac{\overline{r}}{r} \cdot \frac{\partial f}{\partial z}(x, y, ||\nabla y||) \right] hdx$$

for any $h \in C^1(D)$, $h \mid_{\partial \mathcal{D}} = 0$. From here the equation (3.15) follows by a standard way. □

4 The second K - subdifferential of basic variational functional

Recall the classical formula for the second variation [6]. If

$$\Phi(y) = \int\limits_D f(x, y, \nabla y) dx \quad (f \in C^2(D \times \mathbb{R}^{n+1}), y \in C^1(D)), \tag{4.1}$$

then functional (4.1) is twice strongly differentiable everywhere in $C^1(D)$ and, for any $h \in C^1(D)$,

$$\Phi''(y)(h)^2 = \int\limits_D \left[\frac{\partial^2 f}{\partial y^2}(x, y, \nabla y) h^2 + \left(2\nabla_z \left(\frac{\partial f}{\partial y}(x, y, \nabla y) \right) h, \nabla h \right) \right.$$

$$\left. + \nabla_z^2 f(x, y, \nabla y) \cdot (\nabla h)^2 \right] dx \ . \tag{4.2}$$

Here we generalize this condition to the case of a subsmooth integrand from the class C^2_{sub}. In this case, like the case of $\partial_K \Phi$, exact equality (4.2) transforms to an estimate of $\partial^2_K \Phi$. In the one-dimensional case, a similar generalization was obtained in the works [10], [12].

Theorem 4.1. *Consider the variational functional*

$$\Phi(y) = \int\limits_D f(x, y, \nabla y) dx \quad (f \in C^2_{sub}(D \times \mathbb{R}^{n+1}), y \in C^1(D)). \tag{4.3}$$

The functional (4.3) *is twice* K*–subdifferentiable everywhere in* $C^1(D)$*, and the following estimate holds:*

$$\partial^2_K \Phi(y)(h)^2 \subset \left[\int\limits_D \left(\underline{\frac{\partial^2 f}{\partial y^2}}(x, y, \nabla y) h^2 + \left(\underline{\frac{\partial}{\partial y}} \left(\nabla_z f(x, y, \nabla y) \right) h, \nabla h \right) \right) dx; \right.$$

$$\int\limits_D \left(\overline{\frac{\partial^2 f}{\partial y^2}}(x,y,\nabla y)h^2 + \left(\frac{\overline{\partial}}{\partial y}\left(\nabla_z f(x,y,\nabla y)\right)h, \nabla h\right)\right) dx\Big]$$

$$+\Big[\int\limits_D \left(\left(\underline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h, \nabla h\right) + \underline{\nabla_z^2 f}(x,y,\nabla y)\cdot(\nabla h)^2\right) dx;$$

$$\int\limits_D \left(\left(\overline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h, \nabla h\right) + \overline{\nabla_z^2 f}(x,y,\nabla y)\cdot(\nabla h)^2\right) dx\Big]. \tag{4.4}$$

Proof. Since the integrand f is twice K–subdifferentiable, f is differentiable once in the usual sense, i.e. f' exists. Then the variational functional $\Phi(y)$ is also once differentiable in the usual sense, and its differential is as follows

$$\Psi(y)h = \Phi'(y)h = \int\limits_D \left(\frac{\partial f}{\partial y}(x,y,\nabla y)h + (\nabla_z f(x,y,\nabla y), \nabla h)\right) dx. \tag{4.5}$$

Introduce the auxiliary linear operator

$$(Ay)(x) = (x, y(x), \nabla y(x)),\ A:\ C^1(D) \longrightarrow \mathbb{R}^n \times C^1(D) \times C(D,\mathbb{R}^n).$$

Obviously, the operator A is continuous. Now let us introduce the operator of composition

$$\widetilde{A}(y) = B(A(y)) = \left(\frac{\partial f}{\partial y}(A(y)),\ \nabla_z f(A(y))\right) =: \big(B_1(A(y)),\ B_2(A(y))\big),$$

where

$$\widetilde{A}: C^1(D) \to C(D)\times C(D,\mathbb{R}^n)).$$

Let us introduce also the integral operator which is linear in u, v and h:

$$G(u,v)h = \int\limits_D [u(x)h(x) + (v(x), \nabla h(x))]dx, \qquad G: C(D)\times C(D,\mathbb{R}^n) \longrightarrow (C^1(D))^*.$$

Then the variational operator Ψ can be written in the composition form

$$\Psi(y)h = G[B_1(A(y)), B_2(A(y))]h. \tag{4.6}$$

Applying to composition (4.6) the theorem on K–subdifferentiation of composition (see [10], Theorem 3.13, p. 91), we obtain

$$\partial_K\Psi(y)h = \partial_K(G[(B(Ay)])h \subset [\partial_K G(B(Ay))\cdot[\partial_K B(Ay)\cdot\partial_K A(y)]]\,h. \tag{4.7}$$

Now consider individually the components in the right-hand side of (4.7).

1) Because A is a linear continuous operator, it is Frechet differentiable and $A'(y) \equiv A$. Therefore, $\partial_K(Ay)(x) = (x, y(x), \nabla y(x))$.

2) For the operator $B = (B_1, B_2)$, using the theorem on coordinate-wise K–subdifferentiability ([10], Theorem 3.10, p. 89), we obtain:

$$\partial_K B(Ay)h \subset (\partial_K B_1(Ay)h) \times (\partial_K B_2(Ay)h)$$

$$\subset \left[\underline{\frac{\partial}{\partial y}}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h + \underline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y), \nabla h\right);\right.$$

$$\left.\frac{\overline{\partial}}{\partial y}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h + \overline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y), \nabla h\right)\right]$$

$$\left[\underline{\frac{\partial}{\partial y}}(\nabla_z f(x,y,\nabla y))\,h + \underline{\nabla_z}(\nabla_z\,(f(x,y,\nabla y), \nabla h);\right.$$

$$\left.\frac{\overline{\partial}}{\partial y}(\nabla_z f(x,y,\nabla y))\,h + \overline{\nabla_z}\,(\nabla_z f(x,y,\nabla y), \nabla h)\right].$$

3) Because G is a continuous linear operator, it is Frechet differentiable, and $G'(u,v) \equiv G$. From here it follows:

$$\partial_K \Psi(y)h \subset \int\limits_D \left(\left[\underline{\frac{\partial^2 f}{\partial y^2}}(x,y,\nabla y)h + \left(\underline{\frac{\partial}{\partial y}}\,(\nabla_z f(x,y,\nabla y))\,, \nabla h\right);\right.\right.$$

$$\left.\overline{\frac{\partial^2 f}{\partial y^2}}(x,y,\nabla y)h + \left(\frac{\overline{\partial}}{\partial y}\,(\nabla_z f(x,y,\nabla y))\,, \nabla h\right)\right] \cdot h + \left[\underline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h\right.$$

$$\left.\left. + \left(\underline{\nabla_z^2} f(x,y,\nabla y), \nabla h\right);\ \overline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h + \left(\overline{\nabla_z^2 f}(x,y,\nabla y), \nabla h\right)\right]\cdot h'\right)dx$$

$$= \left[\int\limits_D \left(\underline{\frac{\partial^2 f}{\partial y^2}}(x,y,\nabla y)h^2 + \left(\underline{\frac{\partial}{\partial y}}\,(\nabla_z f(x,y,\nabla y))\,h, \nabla h\right)\right)dx;\right.$$

$$\left.\int\limits_D \left(\overline{\frac{\partial^2 f}{\partial y^2}}(x,y,\nabla y)h^2 + \left(\frac{\overline{\partial}}{\partial y}\,(\nabla_z f(x,y,\nabla y))\,h, \nabla h\right)\right)dx\right]$$

$$+ \left[\int\limits_D \left(\left(\underline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h, \nabla h\right) + \left(\underline{\nabla_z^2 f}(x,y,\nabla y), \nabla^2 h\right)\right)dx;\right.$$

$$\left.\int\limits_D \left(\left(\overline{\nabla_z}\left(\frac{\partial f}{\partial y}(x,y,\nabla y)\right)h, \nabla h\right) + \left(\overline{\nabla_z^2 f}(x,y,\nabla y), \nabla^2 h\right)\right)dx\right]. \tag{4.8}$$

□

Here, similarly to estimating $\partial_K \Phi$, we also distinguish between the case of the integrand formed by the external composition of a subsmooth function (now, of the class C^2_{sub}) and the case of a smooth one.

Theorem 4.2. *Let*

$$\Phi(y)=\int_a^b \varphi\left[f(x,y,\nabla y)\right]dx \quad (\varphi\in C^2_{sub}(\mathbb{R}), f\in C^2(D\times\mathbb{R}^{n+1}), y\in C^1(D)).$$

Then Φ is twice K–subdifferentiable everywhere in $C^1(D)$. Moreover, the following estimate takes place (in short):

$$\partial^2_K\Phi(y)(h)^2\subset\int_D \varphi'(f)\cdot\left(\frac{\partial}{\partial y}h+(\nabla_z\ ,\nabla h)\right)^2\cdot f dx$$

$$+\left[\int_D \underline{\varphi''(f)}\cdot\big((f_y)^2h^2+(\nabla_z\cdot f_y h,\nabla h)\big)\,dx;\int_D \overline{\varphi''(f)}\cdot\big((f_y)^2h^2+(\nabla_z\cdot f_y h,\nabla h)\big)\,dx\right]$$

$$+\left[\int_D \underline{\varphi''(f)}\cdot\big((\nabla_z\cdot f_y h,\nabla h)+(\nabla_z f,\nabla h)^2\big)\,dx;\right.$$

$$\left.\int_D \overline{\varphi''(f)}\cdot\big((\nabla_z\cdot f_y h,\nabla h)+(\nabla_z f,\nabla h)^2\big)\,dx\right]. \tag{4.9}$$

Proof. Direct calculations give us:

$$\underline{\varphi(f)}_{y^2}=\underline{\varphi''(f)}\cdot(f_y)^2+\varphi'(f)\cdot f_{y^2};\quad \overline{\varphi(f)}_{y^2}=\overline{\varphi(f)''}\cdot(f_y)^2+\varphi'(f)\cdot f_{y^2};$$

$$\underline{\varphi(f)}_{z^2}=\underline{\varphi''(f)}\cdot(\nabla_z f)^2+\varphi'(f)\cdot\nabla_z^2 f;\quad \overline{\varphi(f)}_{z^2}=\overline{\varphi(f)''}\cdot(\nabla_z f)^2+\varphi'(f)\cdot\nabla_z^2 f;$$

$$\underline{\varphi(f)}_{yz}=\underline{\varphi''(f)}\cdot\nabla_z\cdot f_y+\varphi'(f)\cdot f_y\cdot\nabla_z f;\quad \overline{\varphi(f)}_{yz}=\overline{\varphi''(f)}\cdot\nabla_z\cdot f_y+\varphi'(f)\cdot f_y\cdot\nabla_z f.$$

Substituting of these values in (4.4) leads, after transformation to estimate (4.9). □

Consider, as a concrete example, the integrand of the type $f(x,y,\nabla y)\cdot|f(x,y,\nabla y)|$.

Theorem 4.3. *Let*

$$\Phi(y)=\int_D f(x,y,\nabla y)\cdot|f(x,y,\nabla y)|dx \quad (f\in C^2(D\times\mathbb{R}^{n+1}), y\in C^1(D)).$$

Then the following estimate takes place:

$$\partial^2_K\Phi(y)(h)^2\subset\int_D |f|\cdot\left(\frac{\partial}{\partial y}h+(\nabla_z\ ,\nabla h)\right)^2\cdot f dx+2\int_{(f\neq 0)} sign f\cdot(f_y\cdot h+(\nabla_z f,\nabla h))^2\cdot dx$$

$$+\left[-2\int_{(f=0)}\big(f_{y^2}h^2+(\nabla_z\cdot f_y h,\nabla h)\big)\,dx;\ +2\int_{(f=0)}\big(f_{y^2}h^2+(\nabla_z\cdot f_y h,\nabla h)\big)\,dx\right]$$

$$+\left[-2\int\limits_{(f=0)}\left((\nabla_z\cdot f_y h,\nabla h)+\nabla_z^2 f\cdot(\nabla h)^2\right)dx;\right.$$

$$\left.+2\int\limits_{(f=0)}\left((\nabla_z\cdot f_y h,\nabla h)+\nabla_z^2 f\cdot(\nabla h)^2\right)dx\right]. \qquad (4.10)$$

In particular, if $mes(f(x,y,\nabla y)=0)=0$, *then estimate* (4.10) *transforms to the exact equality:*

$$\partial_K^2\Phi(y)(h)^2=\partial^2\Phi(y)(h)^2=\int\limits_D |f|\cdot\left(\frac{\partial}{\partial y}h+(\nabla_z\ ,\nabla h)\right)^2\cdot f dx$$

$$+2\int\limits_D sign\ f\cdot\left(\frac{\partial f}{\partial y}h+(\nabla_z\ ,\nabla h)\right)^2\cdot f dx.$$

In conclusion, let us give a simple example of application of the last equality.

Example 4. Let

$$\Phi(y)=\int\limits_D |div\nabla y|\cdot(div\nabla y)dx.$$

Here the application of estimate (4.10) leads to the exact equality:

$$\partial_K^2\Phi(y)(h)^2=\partial^2\Phi(y)(h)^2=2\int\limits_{(div\nabla y\neq 0)} sign\ (div\nabla y)\cdot(div\nabla h)^2 dx.$$

In particular, if $mes(div\nabla y=0)=0$, we get:

$$\partial^2\Phi(y)(h)^2=2\int\limits_D sign\ (div\nabla y)\cdot(div\nabla h)^2 dx.$$

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