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A SIMPLE PROOF OF THE BOUNDEDNESS OF BOURGAIN'S CIRCULAR MAXIMAL OPERATOR

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Abstract. Given a set $E = (0, \infty)$, the circular maximal operator $\mathcal{M}$ associated with the parameter set E is defined as the supremum of the circular means of a function when the radii of the circles are in E. Using stationary phase method, we give a simple proof of the $L^p,\ p > 2$ boundedness of Bourgain's circular maximal operator.

1 Introduction

The aim of this paper is to study the boundedness of the circular maximal operator corresponding to $E = (0, \infty)$ from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$. There is a vast literature on maximal and averaging operators over families of lower-dimensional curves of $\mathbb{R}^2$. The main issue turns out to be the curvature: roughly speaking, curved surfaces admit nontrivial maximal estimates, whereas flat surfaces do not. A fundamental and representative positive result is Bourgain's circular maximal operator.

Given a function f, continuous and compactly supported, we consider for each $x \in \mathbb{R}^2$ and $t > 0$, the operator

$$S_t f(x) = \int_{\mathbb{S}^1} f(x - ty) d\sigma(y),$$

where $d\sigma$ is the normalized Lebesgue measure over the unit circle $\mathbb{S}^1$. Then $S_t f(x)$ is the mean value of f over the circle of radius t centered at x and it defines a bounded operator in $L^p(\mathbb{R}^2)$ for $1 \le p \le \infty$. Consider now the circular maximal operator given by

$$\mathcal{M}f(x) = \sup_{t>0} |S_t f(x)|.$$

Then $\mathcal{M}$ defines a bounded operator in $L^p(\mathbb{R}^2)$ if and only if $p > 2$. This result was first proved by Bourgain [1]. Another proof is due to Mockenhaupt, Seeger and Sogge [3]. Bourgain's proof of the circular maximal theorem for $n = 2$ relies more directly on the geometry involved. The relevant geometric information concerns intersections of pairs of thin annuli. (For more details, see [1]). In this paper, using the stationary phase method, we shall give a simple proof of the boundedness of Bourgain's circular maximal operator.

1.1 The dyadic maximal operator

The proof of our main result, as well as many other arguments that involve explicitly (or implicitly) the Fourier transform, makes use of splitting of the dual (frequency) space into dyadic shells. Dyadic decomposition, whose ideas originated in the work of Littlewood and Paley, and others, will now be described in the form most suitable for us.

Let ψ be a nonnegative radial function in $C_c^\infty(\mathbb{R}^2)$ supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ such that $\sum_{j=0}^{\infty} \psi(2^{-j}\xi) = 1$ for $|\xi| \geq 1$.

Define $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \geq 0$, $\phi_0(\xi) = 1 - \sum_{j=0}^{\infty} \psi_j(\xi)$. Denote by σ^j, μ the C^∞ functions given by

$$\widehat{(\sigma^j)}(\xi) = \widehat{(d\sigma)}(\xi)\psi_j(\xi) \quad \text{and} \quad \widehat{\mu}(\xi) = \widehat{(d\sigma)}(\xi)\phi_0(\xi).$$

Let S_t^j and B_t be the operators defined by

$$S_t^j f(x) = \int_{\mathbb{R}^2} f(x - ty)\, \sigma^j(y)\, dy, \quad B_t f(x) = \int_{\mathbb{R}^2} f(x - ty)\, \mu(y)\, dy.$$

Notice that B_t is pointwise majorized by a constant times the Hardy-Littlewood maximal operator. Hence,

$$\mathcal{M}f(x) \leq \sum_{j=0}^{\infty} \sup_t |S_t^j f(x)| + CMf(x).$$

1.2 Angular decomposition

We now discuss the second dyadic decomposition of the frequency-space that is needed for each dyadic operators S_t^j in $\mathbb{R}^2$. To do this, we first choose unit vectors ξ_j^ν, $\nu = 1, \ldots, N(j)$ such that

$$|\xi_j^\nu - \xi_j^{\nu'}| \geq C_0 2^{-\frac{j}{2}}, \nu \neq \nu',$$

for some positive constant C_0 and such that balls of radius $2^{-\frac{j}{2}}$ centered at ξ_j^ν cover $\mathbb{S}^1$. Note that

$$N(j) \approx 2^{\frac{j}{2}}.$$

They give an essentially uniform grid on the unit sphere, with separation $2^{-\frac{j}{2}}$. Let Γ_j^ν denote the corresponding cone in the $\xi-$space whose central direction is ξ_j^ν, i.e.,

$$\Gamma_j^\nu = \{\xi : |\frac{\xi}{|\xi|} - \xi_j^\nu| \leq C.2^{-\frac{j}{2}}\}.$$

Now, we introduce an associated partitions of unity $\mathbb{R}^2 \backslash \{0\}$ that depend on scale j. Specifically, we choose C^∞ functions

$$\chi_\nu, \ \nu = 1, \ldots, N(j) \approx 2^{\frac{j}{2}},$$

satisfying $\sum_\nu \chi_\nu = 1$ and having the following additional properties:

1) χ_ν's are to be homogeneous of degree zero and satisfy the uniform estimates

$$|D^\alpha \chi_\nu(\xi)| \leq C_\alpha 2^{\frac{j|\alpha|}{2}} \text{ for every } \alpha \text{ if } |\xi| = 1,$$

2) $\chi_\nu(\xi_j^\nu) \neq 0$ and χ_ν's are to have the natural support properties, i.e

$$\chi_\nu(\xi) = 0 \text{ if } |\xi| = 1 \text{ and } |\xi - \xi_j^\nu| \geq C2^{-\frac{j}{2}}.$$

Using the homogeneous partitions of unity χ_ν, we make an angular decomposition of the operators by setting

$$\widehat{(\sigma_\nu^j)}(\xi) = \widehat{(d\sigma)}(\xi)\, \psi_j(\xi)\, \chi_\nu(\xi), \tag{1.1}$$

and define the corresponding operators

$$S_t^{j.\nu} f(x) = \int_{\mathbb{R}^2} f(x - ty)\sigma_\nu^j(y)dy.$$

Next, we shall state our main result of this paper.

2 Main results

Theorem 2.1. *Let f be a bounded measurable function on $\mathbb{R}^2$. Then the inequality*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^2)} \leq B_p \|f\|_{L^p(\mathbb{R}^2)}$$

holds whenever $p > 2$.

Proof. We shall prove that

$$\int_{\mathbb{R}^2} \sup_{t>0} |S_t f(x)|^p\, dx \leq B_p \int_{\mathbb{R}^2} |f(x)|^p\, dx, \; p > 2. \tag{2.1}$$

Our proof will consist of three main steps. First we shall decompose each S_t into dyadic operator S_t^j. Then we shall use the method of Littlewood-Paley square function to deduce the general result for $t > 0$ from the inequality where the supremum is only taken over $t \in [1, 2]$. We shall then use the method of stationary phase to expose the behavior of each operator $S_t^{j,\nu}$.

We now turn to the details. To obtain inequality (2.1) by summing a geometric series, it is enough to prove the following: There exists a constant $\epsilon(p) > 0$ such that for $p > 2$,

$$\int_{\mathbb{R}^2} \sup_{1\leq t\leq 2} |S_t^j f(x)|^p\, dx \leq C\, 2^{-j\,\epsilon(p)} \int_{\mathbb{R}^2} |f(x)|^p\, dx. \tag{2.2}$$

By rescaling, inequality (2.2) will be true for $\sup_{2^k \leq t \leq 2^{k+1}}$ with the same constant. To see that (2.2) is enough, we need to use Littlewood-Paley operators L_k, which are defined by

$$\widehat{(L_k f)}(\xi) = \psi(2^{-k}|\xi|)\, \hat{f}(\xi),$$

where ψ's are defined in Section 1. It then must follow that there is an absolute constant C_0 such that, when $t \in [1,2]$, we have

$$S_t^j f(x) \;=\; S_t^j\big(\sum_{|j-k|\leq C_0} L_k f\big)(x).$$

Thus, if (2.2) holds, then a scaling argument will give,

$$\begin{aligned}
\int \sup_{t>0} |S_t^j f(x)|^p\,dx \;&\leq\; \sum_{l=-\infty}^{\infty} \int \sup_{t\in[2^l,2^{l+1}]} |S_t^j\big(\sum_{|k+l-j|\leq C_0} L_k f\big)(x)|^p\,dx \\
&\leq\; C^p\, C_0\, 2^{-j\epsilon_p\, p} \int \sum_{k=-\infty}^{\infty} |L_k f(x)|^p\,dx \\
&\leq\; C^p\, C_0\, 2^{-j\epsilon_p\, p} \int \big(\sum_{k=-\infty}^{\infty} |L_k f|^2\big)^{(\frac{p}{2})}\,dx.
\end{aligned}$$

In the last step we have used the fact that $p > 2$. If we now use the L^p boundedness of Littlewood-Paley square functions, we finish our proof of the claim.

Next we claim that inequality (2.2) in turn would follow from the uniform estimates

$$\int_{\mathbb{R}^2} \sup_{1\leq t\leq 2} |S_t^{j,\nu} f(x)|^p\,dx \leq C 2^{-j[\frac{p}{2}+\epsilon(p)]} \int_{\mathbb{R}^2} |f(x)|^p\,dx. \tag{2.3}$$

To show that (2.3) implies (2.2), using the Hölder inequality for sums, we get,

$$\begin{aligned}
\int_{\mathbb{R}^2} \sup_{1\leq t\leq 2} |S_t^j f(x)|^p\,dx \;&\leq\; 2^{\frac{j(p-1)}{2}} \sum_{\nu} \Big[\int_{\mathbb{R}^2} \sup_{1\leq t\leq 2} |S_t^{j,\nu} f(x)|^p\,dx\Big] \\
&\leq\; 2^{\frac{j(p-1)}{2}} \sum_{\nu} 2^{-j[\frac{p}{2}+\epsilon(p)]} \int_{\mathbb{R}^2} |f(x)|^p\,dx \\
&=\; C 2^{-j\,\epsilon(p)} \int_{\mathbb{R}^2} |f(x)|^p\,dx,
\end{aligned}$$

where we use (2.3) in the second inequality.

To prove (2.3), we shall use a Sobolev embedding to replace $\sup_{1\leq t\leq 2} |S_t^{j,\nu} f(x)|^p$ with

$$\int_1^2 \Big|D_t^\beta S_t^{j,\nu} f(x)\Big|^p\,dt, \quad \beta > \frac{1}{p}.$$

We shall get our result by computing the norm for $\beta = 0$ and $\beta = 1$ and then interpolate between them. Using the Hölder inequality for $\beta = 0$ and $p > 2$, we have

$$\begin{aligned}
&\int_{x\in\mathbb{R}^2}\int_1^2 \left|S_t^{j,\nu} f(x)\right|^p \, dx\, dt \\
= &\int_{x\in\mathbb{R}^2}\int_1^2 \left|\int_{y\in\mathbb{R}^2} f(x-ty)\, \sigma_\nu^j(y)\, dy\right|^p \, dx\, dt \\
= &\int_{x\in\mathbb{R}^2}\int_1^2 \left|t^{-2}\int_y f(x-y)\, \sigma_\nu^j(\frac{y}{t})\, dy\right|^p \, dx\, dt \\
\leq &\int_{x\in\mathbb{R}^2}\int_1^2 \left[t^{-2}\int_y \left(|f(x-y)|^p\, |\sigma_\nu^j(\frac{y}{t})|\right) dy\, \left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)^{p-1}\right] dx\, dt \\
= &\int_1^2 \left[t^{-2}\left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)^{p-1}\int_x\int_y \left[|f(x-y)|^p\, |\sigma_\nu^j(\frac{y}{t})|\right] dy\, dx\right] dt \\
= &\int_1^2 \left[t^{-2}\left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)^{p-1}\int_y \left[|\sigma_\nu^j(\frac{y}{t})|\, \left(\int_x |f(x-y)|^p\, dx\right)\right] dy\right] dt \\
= &\int_1^2 \left[t^{-2}\left[\left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)^{p-1}\left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)\right]\int_x |f(x)|^p\, dx\right] dt \\
\leq &\left[\int_1^2 t^{-2}\left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)^p dt\right]\int_x |f(x)|^p\, dx. \qquad (2.4)
\end{aligned}$$

Now our aim is to estimate the following integral

$$\int_1^2 t^{-2}\left(\int_y |\sigma_\nu^j(\frac{y}{t})|\, dy\right)^{p-1} dt.$$

Using the property of Bessel's function, let us consider the following estimate for $t \in [1,2]$, $|\xi| \in [2^{j-1}, 2^{j+1}]$,

$$\widehat{(d\sigma)}(t\xi) = e^{it|\xi|}\, a(\xi)\, (t|\xi|)^{-\frac{1}{2}},$$

where a is $C^\infty(\mathbb{R}^2 \setminus \{0\})$, homogeneous of degree zero (see [6, 51]).

Now, from (1.1), We get

$$\begin{aligned}
\sigma_\nu^j(\frac{y}{t}) &= \int_\xi e^{i<\xi,y>}\, t^2\, \widehat{(d\sigma)}(t\xi)\, \psi_j(t\xi)\, \chi_\nu(\xi)\, d\xi \\
&= \int_\xi e^{i<\xi,y>}\, t^2\, e^{it|\xi|}\, a(\xi)\, (t|\xi|)^{-\frac{1}{2}}\, \psi_j(t\xi)\, \chi_\nu(\xi)\, d\xi \\
&= 2^{-\frac{j}{2}}\int_{|\xi|=2^{j-1}}^{2^{j+1}} e^{i<\xi,y>}\, t^2\, e^{it|\xi|}\, a(\xi)\, (t|2^{-j}\xi|)^{-\frac{1}{2}}\, \psi(t|\xi|2^{-j})\, \chi_\nu(\xi)\, d\xi \\
&= 2^{-\frac{j}{2}}\, t^{\frac{3}{2}}\, F(y,t), \qquad (2.5)
\end{aligned}$$

where

$$F(y,t) = \int_{|\xi|=2^{j-1}}^{2^{j+1}} e^{i<\xi,y>}\, e^{it|\xi|}\, a'(\xi)\, \psi(t|\xi|2^{-j})\, \chi_\nu(\xi)\, d\xi,$$

and a' is a homogeneous function of degree zero. Since $\psi(0) = 0$, we have

$$\begin{aligned} F(y,t) &= \int_0^t \frac{\partial F}{\partial s} ds = \int_0^t \left[\int_\xi e^{i<\xi,y>} e^{is|\xi|} a'(\xi) (2^{-j}|\xi|) \psi'(s|\xi|2^{-j}) \chi_\nu(\xi) d\xi \right] ds \\ &+ \int_0^t \left[\int_{|\xi|=2^{j-1}}^{2^{j+1}} e^{i<\xi,y>} (i|\xi|) e^{is|\xi|} a'(\xi) \psi(s|\xi|2^{-j}) \chi_\nu(\xi) d\xi \right] ds. \end{aligned}$$

Hence, from (2.5), we have,

$$\begin{aligned} \int_y |\sigma_\nu^j(\frac{y}{t})| dy &= \int_y 2^{-\frac{j}{2}} t^{\frac{3}{2}} |F(y,t)| dy \\ &\leq 2^{-\frac{j}{2}} t^{\frac{3}{2}} (I_1 + I_2). \end{aligned}$$

Using integration by parts and change of variable formula to the following integral we get,

$$\begin{aligned} I_1 &= \int_y \left| \int_0^t \left[\int_\xi e^{i<\xi,y>} e^{is|\xi|} a'(\xi) (2^{-j}|\xi|) \psi'(s|\xi|2^{-j}) \chi_\nu(\xi) d\xi \right] ds \right| dy \\ &= \int_y 2^{\frac{3j}{2}} \left| \int_0^t \left[\int_\xi e^{i2^j<\xi,y>} e^{is2^j|\xi|} a'(\xi) \psi'(s|\xi|) \chi_\nu(\xi) d\xi \right] ds \right| dy \\ &= \int_y 2^{\frac{3j}{2}} \left| \int_\xi e^{i2^j<\xi,y>} a'(\xi) \chi_\nu(\xi) \left(\int_{s=0}^{t|\xi|} e^{is2^j} \psi'(s) \frac{ds}{|\xi|} \right) d\xi \right| dy \\ &\leq 2^{\frac{3j}{2}} \int_y \left| \frac{1}{(1+|2^j y|^2)^N} \right| dy \\ &= 2^{\frac{3j}{2}} 2^{-2j} \int_y \frac{1}{(1+|y|^2)^N} dy \leq C 2^{-\frac{j}{2}} \text{ if } N > \frac{3}{2}. \end{aligned} \tag{2.6}$$

where $a'(\xi)$ is a homogeneous function of degree zero in ξ.

To simplify the writing of the estimates for I_2, we set $e_1 = \xi_j^\nu$, and the corresponding partitions of unity $\chi_{e_1}(\xi) = \chi_\nu(\xi)$. Now we choose axes in the ξ space so that ξ_1 is in the direction of e_1 and ξ_2 is perpendicular to e_1.

Therefore, in our coordinates, we have

$$2^j \leq |\xi| \leq 2^{j+1}, \text{ and } \quad |\xi'| \leq c 2^{\frac{j}{2}}.$$

Let, $L = I - 2^{2j}(\frac{\partial}{\partial \xi_1})^2 - 2^j {\nabla_{\xi'}}^2$ and $y = (y_1, y_2)$ and $\xi = (\xi_1, \xi_2)$. Next with the help of integration by parts formula (see Stein, Chapter 9, [5]) and change of variable formula to the following integral we get,

$$\begin{aligned}
I_2 &= \int_y \left| \int_0^t \left[\int_\xi e^{i<\xi,y>} \, (i|\xi|) \, e^{is|\xi|} \, a'(\xi) \, \psi(s|\xi|2^{-j}) \, \chi_{e_1}(\xi) \, d\xi \right] ds \right| \, dy \\
&= 2^j \int_y \left| \int_0^t \left[\int_\xi e^{i<\xi,y>} \, e^{is|\xi|} \, (i2^{-j}|\xi|) \, a'(\xi) \, \psi(s|\xi|2^{-j}) \, \chi_{e_1}(\xi) \, d\xi \right] ds \right| \, dy \\
&= 2^j \int_y \left| \int_0^t \left[\int_\xi e^{i[<\xi,y>+s\xi_1]} \, \left[e^{is[|\xi|-\xi_1]} \, a'(\xi) \, \psi(s|\xi|2^{-j}) \, \chi_{e_1}(\xi) \right] \, d\xi \right] ds \right| \, dy \\
&\leq \int_y \int_0^2 \left| \frac{2^j}{\{1+2^j|y_1+s|+2^{\frac{j}{2}}|y_2|\}^{2N}} \right| \, ds \, dy \\
&\leq C \, 2^j \, 2^{-\frac{3j}{2}} \int_y \left| \frac{1}{\{1+|y_1|+|y_2|\}^{2N}} \right| \, dy \, \leq \, C \, 2^{-\frac{j}{2}}, \text{ if } N > \frac{3}{2}. \qquad (2.7)
\end{aligned}$$

Therefore, using (2.5), (2.6) and (2.7), we get

$$\int_1^2 t^{-2} \left(\int_y |\sigma_\nu^j(\frac{y}{t})| \, dy \right)^p dt \, \leq \, C \, 2^{-jp}. \qquad (2.8)$$

Hence, using (2.8) in (2.4) we get,

$$\int_{\mathbb{R}^2} \int_1^2 |S_t^{j,\nu} f(x)|^p \, dxdt \, \leq \, C \, 2^{-jp} \int_{\mathbb{R}^2} |f(x)|^p \, dx.$$

For $\beta = 1$, we have $D_t^1 S_t^{j,\nu}$ which is essentially 2^j times an operator similar to $S_t^{j,\nu}$, so that the above estimate appears multiplied by 2^{jp}. By interpolation, we finally get

$$\begin{aligned}
\int_{\mathbb{R}^2} \int_1^2 |D_t^\beta S_t^{j,\nu} f(x)|^p \, dx \, dt \; &\leq \; C 2^{-jp} \times 2^{jp\beta} \int_{\mathbb{R}^2} |f(x)|^p \, dx \\
&= \; 2^{j[-p+p\beta]} \int_{\mathbb{R}^2} |f(x)|^p \, dx.
\end{aligned}$$

Inequality (2.3) follows by taking β such that $-p + \frac{p}{2} + p\beta \, < \, -\epsilon(p)$, since $p > 2$. Hence the theorem. □

Remark 1. Similarly, in the higher dimension case, the spherical maximal operator is given by

$$\mathcal{M}f(x) = \sup_{t>0} |S_t f(x)|,$$

where $S_t f(x)$ is the mean value of f over the sphere of radius t centered at x. Then $\mathcal{M}$ defines a bounded operator in $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$ with $n > 1$.

This result was first proved by Stein [7, 8], for $n \geq 3$. Stein's proof of the spherical maximal theorem for $n \geq 3$, exploit curvature via the decay of the Fourier transform of the surface measure on the sphere. In the case of the sphere, the Fourier transform decays like $|\xi|^{-\frac{n-1}{2}}$ at infinity. The decay estimates are weaker for surfaces with flat directions, which is reflected in the range of exponents in maximal and averaging estimates.

In fact, using the above two dimensional techniques, we can proof the boundedness of Stein's spherical maximal operator. Our proof of this result is based on the approach in [4]. Here, we shall use the method of Littlewood-Paley square function for $p \geq 2$ and in the case $p < 2$, we use an argument due to M. Christ which can be seen in [2] to deduce the general result for $t > 0$ from the inequality where the supremum is only taken over $t \in [1, 2]$.

Now, using the property of Bessel's function , let us consider the following estimate for $t \in [1, 2],\ |\xi| \in [2^{j-1}, 2^{j+1}]$,

$$\widehat{(d\sigma)}(t\xi) = e^{it|\xi|}\, a(\xi)\, (t|\xi|)^{-\frac{n-1}{2}}, \tag{2.9}$$

where a is $C^\infty(\mathbb{R}^n \setminus \{0\})$- function homogeneous of degree zero. We only then use the above estimate (2.9) in the proof of Theorem 2.1, we shall get

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \,\leq\, B_p\, \|f\|_{L^p(\mathbb{R}^n)},$$

whenever $p > \frac{n}{n-1}$. Here, we omit the proof.

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